On the limit superior of analytic sets

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0. Introduction

In [5] LACZKOVICH proved that if the sets $A^0, A^1, \ldots$ are Borel sets of reals and for every infinite set $H$ of indices $\limsup_{j \in H} A^j$ is uncountable, then there exists an infinite set $H$ of indices for which even the intersection $\bigcap_{j \in H} A^j$ is uncountable, as well. He showed that, in the presence of the continuum hypothesis, this statement is not true for arbitrary sets.

In this paper we extend the result on Borel sets to analytic sets, and show that under a special set theoretic assumption (Martin’s axiom, to be specific) it is true for any sequence of sets. From this latter result it is also possible to deduce the statement on analytic sets. Some other independence results are also discussed.

One of the aims of this paper is to show that forcing can be used to prove non-independence results. This observation is not new. In fact, there exist, up to date, quite a few proofs of this type, see, e.g. [1, 2, 8]. One can even say, that in certain cases the proof via forcing is simpler than the proof by “classical” means (but, of course, only for forcing experts).

The usual case is, however, that not much model theory is involved; the innocent reader can understand everything except the proof of the existence of certain models. In this paper also, everything except the proof of Theorem 3 can be understood without a knowledge of forcing, all the necessary tools will be given. For those, who are interested in an introduction to forcing, we recommend [3].

1. Notation, preliminaries

In this section we give the basic definitions and notations and prove some preparatory statements for Theorem 1.

We adopt some of the conventions of modern set theory, e.g. $\omega$ denotes the set of non-negative integers and the cardinality of it, as well. $\omega^*$ is the set of infinite sequences of natural numbers, $^\omega \omega$ stands for the set of sequences of natural numbers of

Received August 5, 1982; in revised form December 28, 1983.
length $n$. Accordingly, $2$ consists of the length $n$ zero-one sequences. For sequences, $s \subseteq t$ denotes that $s$ is an initial segment of $t$. If $z \in \omega$ and $n \in \omega$, $z \mid n$ is the initial segment of $z$ with length $n$.

Whenever $A$ is a set, $[A]^{\omega}$ denotes $\{S \subseteq A : |S| = \omega\}$. For $A, B \in [\omega]^{\omega}$, $A \subseteq^* B$ denotes that $A - B$ is finite (that is, "$A$ is almost contained in $B$ "). The following easy observation will be used throughout the paper: if $A_0 \supseteq A_1 \supseteq \ldots$ then there exists an $A \in [\omega]^{\omega}$ with $A \subseteq^* A_i$ ($i = 0, 1, \ldots$).

Assume that we are given a sequence $A_j$ ($j = 0, 1, \ldots$) of analytic sets in a complete, separable, metric space, $X$. For $N \in [\omega]^{\omega}$ put

$$(1.1) \limsup_{j \in N} A_j = \{x : x \in A_j \text{ for infinitely many } j \in N\}.$$ 

Definition. If $N \in [\omega]^{\omega}$, $Y \subseteq X$, $Y$ is said to be good with respect to $N$ if for every $H \in [N]^{\omega}$, $Y \cap \limsup_{j \in H} A_j$ is uncountable.

By the classical theory of analytic sets, we can exhibit $A^j$ as the result of Suslin operation (see [4], §39):

$$(1.2) A^j = \bigcup_{z \in \omega^\omega} \bigcap_{n \in \omega} F^j_z \mid n$$

where each $F^j_z \mid n$ is a closed set with $\text{diam} (F^j_z \mid n) < 1/n$ ("diam" stands for diameter), and the sequence is decreasing, i.e. if $n > m$, $F^j_z \mid n \subseteq F^j_z \mid m$. Let us define for every $s \in \omega$:

$$(1.3) A^j_s = \bigcup_{z \in \omega^\omega} \bigcap_{k \in \omega} F^j_z \mid k.$$ 

In the following we take some general observations on the concept of goodness.

Lemma 1. If $Y = \bigcup_{i \in \omega} Y_i$ is good with respect to $H$, then there is an $i \in \omega$ and an $H' \in [H]^{\omega}$ such that $Y_i$ is good with respect to $H'$.

Proof. If $Y_0$ is not good with respect to $H$, there is, by definition an $H_0 \in [H]^{\omega}$ with $|Y_0 \cap \limsup_{j \in H} A^j| \leq \omega$. If $Y_1$ is not good with respect to $H_0$, there is, similarly, an $H_1 \in [H_0]^{\omega}$ with $|Y_1 \cap \limsup_{j \in H_1} A^j| \leq \omega$. Continuing this process we get, assuming the lemma is false, a decreasing sequence $H \supseteq H_0 \supseteq \ldots \supseteq H_t \supseteq \ldots$ from $[\omega]^{\omega}$ such that $Y_t \cap \limsup_{j \in H_t} A^j$ is countable. Choose a set $H' \in [H]^{\omega}$ with $H' \subseteq^* H_t$ ($t = 0, 1, \ldots$). In that case clearly $|Y_t \cap \limsup_{j \in H'} A^j| \leq \omega$, so $|Y \cap \limsup_{j \in H'} A^j| \leq \omega$ is impossible, as $Y$ is good with respect to $H$.

Lemma 2. If $P, A \subseteq X$, $P$ is closed and $P \cap A$ is good with respect to $H \in [\omega]^{\omega}$, then there exists a point $x \in P$ and an $H' \in [H]^{\omega}$ such that for every neighborhood $G$ of $x$ $P \cap G \cap A$ is good with respect to $H'$. 

Proof. Exhibit $P$ as $\bigcup_{i \in \omega} P_i$ where $\operatorname{diam}(P_i) < 1$, $P_i$ closed ($i \in \omega$). By Lemma 1, there are $i_0 \in \omega$ and $H_0 \in [H]^\omega$ such that $A \cap P_{i_0}$ is good with respect to $H_0$. Put $P_{i_0} = \bigcup_{i \in \omega} P_{i_0}^i$ with $P_{i_0}^i$ closed, $\operatorname{diam}(P_{i_0}^i) < 1/2$. Again, by Lemma 1, there are $i_1 \in \omega$ and $H_1 \in [H]^\omega$ such that $A \cap P_{i_0}^i$ is good with respect to $H_1$. Iterating this step, we eventually get a sequence of closed sets, $P \supseteq P_{i_0} \supseteq P_{i_0}^{i_1} \supseteq \ldots$ with diameter converging to 0, and $A \cap P_{i_0}^{i_1} \supseteq \ldots$ is good with respect to $H_t$ where $H = H_0 \supseteq \ldots$ and $H_t \in [\omega]^{\omega}$. As $X$ is complete and $P$ is closed, there is a unique point $x \in \bigcap_{i \in \omega} P_{i_0}^{i_1 \ldots i_t}$. If we choose $H' \in [H]^{\omega}$ with $H' \subseteq H_t$ ($t \in \omega$), each $A \cap P_{i_0}^{i_1 \ldots i_t}$ will be good with respect to $H'$. As for every neighborhood $G$ of $x$, there is a $i \in \omega$ with $P_{i_0}^{i_1 \ldots i_t} \subseteq G$, we are done.

Lemma 3. Assume that $P, A \subseteq X$, $P$ is closed, $H \in [\omega]^\omega$ and $\epsilon > 0$. If $P \cap A$ is good with respect to $H$, then there are closed sets $P_0, P_1 \subseteq P$ and an $H' \in [H]^{\omega}$ such that $\operatorname{diam}(P_0), \operatorname{diam}(P_1) < \epsilon$, $g(P_0, P_1) > 0$ and $A \cap P_0, A \cap P_1$ are both good with respect to $H'$. ($g(P_0, P_1)$ denotes the distance between $P_0$ and $P_1$.)

Proof. By Lemma 2 there is a point $x \in P$ and a set $H_0 \in [\omega]^{\omega}$ such that for every neighborhood $G$ of $x$, $P \cap G \cap A$ is good with respect to $H$. Put $Q_i = \{y \in P : g(x, y) \leq \frac{\epsilon}{t+1}\}$. As $P = \{x\} \cup \bigcup_{t \in \omega} Q_t$ and clearly one-point sets cannot be good, by Lemma 1 there is a $t \in \omega$ and $H_t \in [H_0]^{\omega}$ such that $A \cap Q_t$ is good with respect to $H_t$. As $Q_t$ can be covered by countably many closed sets of diameter less than $\epsilon$, using Lemma 1 again we can find a closed $P_0 \subseteq Q_t$ and $H_2 \in [H_t]^{\omega}$ such that $\operatorname{diam}(P_0) < \epsilon$ and $A \cap P_0$ is good with respect to $H_2$. If we choose $P_1 = P \cap \{y : g(x, y) \leq \frac{\epsilon}{2(t+1)}\}$, $A \cap P_1$ will be good with respect to $H_2$, even with respect to $H_0$, and obviously $\operatorname{diam}(P_1) < \epsilon$, $g(P_0, P_1) = \frac{\epsilon}{2(t+1)} > 0$.

2. Proof of Theorem 1

Theorem 1. If $A^j$ is analytic for $j = 0, 1, 2, \ldots$, and for every $H \in [\omega]^{\omega}$, $\lim \sup_{j \in H} A^j$ is uncountable, then there exists an $H \in [\omega]^{\omega}$ such that $\bigcap_{j \in H} A^j$ is uncountable.

Proof. Assume that $A^0 = X$. We are going to give an inductive construction. At the $n$th step we will choose an index $j_n \in \omega$, $2^n$ perfect sets, $P_s(s \in \omega^2), s \in \omega^2$, finite sequences $t(k, s) \in \omega^\omega (k = n, s \in \omega^2)$ and a set $H_n \in [\omega]^{\omega}$ satisfying the following
stipulations:

(2.1) \[ j_n > j_{n-1}, \ H_n \subseteq \{H_{n-1}\}^\omega, \ j_n \notin H_{n-1}, \]

(2.2) \[ P_{s_0}, P_{s_1} \subseteq P_s, \ \phi(P_{s_0}, P_{s_1}) > 0 \ (s \in \omega^{n-2}), \]

(2.3) \[ \text{diam} (P_s) < \frac{1}{n} \ (s \in \omega^2), \]

(2.4) \[ P_s \cap A^{I_0}_{(i,s_0)} \cap \ldots \cap A^{I_n}_{(i,s_n)} \]

is good with respect to \( H_n, \ (s \in \omega^2), \)

(2.5) \[ P_s \subseteq F^{I_0}_{(i,s_0)} \cap \ldots \cap F^{I_n}_{(i,s_n)} \ (s \in \omega^2), \]

(2.6) \[ t(k, s) \subseteq t(k, s_0), \ t(k, s_1) \ (s \in \omega^{n-2}, k \leq n-1). \]

First we show that the theorem follows from (2.1–6). Conditions (2.2) and (2.3) guarantee that the "fusion" of our perfect sets

\[ P = \bigcap_{n \in \omega} \left( \bigcup_{s \in \omega^2} P_s \right) \]

is a perfect set, therefore uncountable. If \( x \in P_s \) by (2.2) for every \( n \) there exists a unique \( s_n \) with \( x \in P_{s_n} \), moreover, \( s_0 \subseteq s_1 \subseteq \ldots \). By (2.5), if \( n \geq i \), \( x \in F^{I_n}_{(i,s_n)} \) and by (2.6), \( t(i,s_i) \subseteq t(i,s_{i+1}) \subseteq \ldots \) hold. From this, we get \( x \in A^{I_n}_{(i,s_n)} \subseteq A^{I_i} \), that is \( P \subseteq \bigcap_{i \in \omega} A^{I_i} \) holds.

We have only to choose the objects described by (2.1-6) on induction by \( n \). To start, choose \( j_0 = 0, \ P_0 = X, \ H_0 = \omega \). As by the assumption of the theorem, \( X \) is good with respect to \( \omega \), this choice together with \( t(0, 0) = \emptyset \) satisfies (2.1-6).

Assume that \( n \in \omega \) and \( j_k (k \leq n), \ P_s (s \in \omega^2, k \equiv n), t(k, s) (k \leq l \leq n, s \in \omega^2), \ H_k (k \equiv n) \) have already been chosen. First we prove that there exists a \( j \in H_n, j \neq j_n \) together with an \( H'_i \subset [H_n]^\omega \) satisfying

(2.7)

if \( s \in \omega^2 \) then \( P_s \cap A^{I_0}_{(i,s_0)} \cap \ldots \cap A^{I_n}_{(i,s_n)} \cap A^j \) is good with respect to \( H'_n \).

Assume the contrary. Then we can inductively select \( k_0 < k_1 < \ldots \) and \( H_n = G_0 \supseteq G_1 \supseteq \ldots \) with \( k_m \in G_m \subset [\omega]^\omega \) such that for every \( m \in \omega \) there is an \( s_m \in \omega^2 \) with

(2.8) \[ |P_{s_m} \cap A^{I_0}_{(i,s_m)} \cap \ldots \cap A^{I_n}_{(i,s_m)} \cap A^{I_m} \cap \limsup_{t \in G_{m+1}} A^t| \equiv \omega. \]

As there are only \( 2^n \) possibilities for \( s_m \), there is an \( s \in \omega^2 \) with \( \Gamma = \{k_m : s_m = s\} \) infinite. Clearly, \( \Gamma \subseteq \omega \) holds for every \( m \in \omega \). From this, we get

(2.9) \[ |P_s \cap A^{I_0}_{(i,s_0)} \cap \ldots \cap A^{I_n}_{(i,s_n)} \cap (\bigcup_{m \in \Gamma} A^m) \cap \limsup_{m \in \Gamma} A^m| \equiv \omega \]

but this is impossible, as \( \Gamma \subseteq H_n \) and (2.4) holds.
So a number \( j \) satisfying (2.7) and \( j \in J_0 \), \( j \in H_0 \) exists. This will be our choice for \( j_{n+1} \). With \( 2^n \) applications of Lemma 3 we can find perfect sets \( \overline{P}_s0, \overline{P}_s1 \subseteq P_s \) with \( \text{diam} (\overline{P}_s0), \text{diam} (\overline{P}_s1) \leq \frac{1}{n+1}, \ t(\overline{P}_s0, \overline{P}_s1) > 0 \), and a set \( H_n' \in [H_n]'^\omega \) such that

\[
\overline{P}_s \cap A_{i(0, s)}^j \cap \ldots \cap A_{i(n, s)}^j \cap A_{i(n+1, s)}^j
\]

is good with respect to \( H_n'' \) if \( s \in n^2, \ i=0, 1 \).

The set

\[
A_{i(0, s)}^j \cap \ldots \cap A_{i(n, s)}^j \cap A_{i(n+1, s)}^j
\]

is covered by

\[
\bigcup_{z_0} \ldots \bigcup_{z_{n+1}} (A_{i(0, s)}^j \cap \ldots \cap A_{i(n+1, s)}^j)
\]

where the union is extended over the sequences with \( z_0, \ldots, z_{n+1} \in \omega^{n+1} \) and \( z_0 \supseteq t(0, s), \ldots, z_n \supseteq t(n, s) \). As this is a countable union, one of these sets will be good by Lemma 1, and \( 2^{n+1} \) applications of Lemma 1 actually give an \( H_n \) witnessing this fact simultaneously for all \( s \in n^2, \ i=0, 1 \). Choose \( t(0, s), \ldots, t(n+1, s) \) as the sequence \( z_0, \ldots, z_{n+1} \) corresponding to \( s \). We are almost done, only the choice of the sets of form \( \overline{P}_s \) must be corrected. Put

\[
Q_{sl} = \overline{P}_s \cap A_{i(0, s)}^j \cap \ldots \cap A_{i(n+1, s)}^j.
\]

These sets still satisfy (2.4) as \( Q_{sl} \) and \( \overline{P}_s \) have the same intersection with

\[
A_{i(0, s)}^j \cap \ldots \cap A_{i(n+1, s)}^j.
\]

\( Q_{sl} \), a closed set of cardinality continuum (as it is good) is the union of a perfect set and a countable set. Call the perfect constituent as \( P_{sl} \). As the removal of a countable set cannot change the status of goodness, \( P_{sl} \) is still good with respect to \( H_n \) which therefore can be chosen as \( H_{n+1} \), and all (2.1-6) will be satisfied.

### 3. Independence

Assume that \( A^j \subseteq R \) (\( j \in \omega \)). Define, for \( x \in R \), \( D_x = \{ j \in \omega : x \in A^j \} \).

**Lemma 4.** If \( \lim \sup_{j \in \omega} A^j = \omega \) for every \( H \in [\omega]^{\omega} \), then there exists an uncountable \( S \subseteq R \) with \( \{ D_x : x \in S \} \) centered, i.e., \( |D_{x_0} \cap \ldots \cap D_{x_n}| = \omega \), if \( x_0, \ldots, x_n \in S \).

**Proof.** Assume contrary. Choose a maximal centered family \( \{ D_x : x \in S \} \) with \( S \), therefore, countable. By diagonalization we get an \( H \in [\omega]^{\omega} \) such that \( H \subseteq \bigcup_{x_0, \ldots, x_n \in S} D_{x_0} \cap \ldots \cap D_{x_n} \). We claim that \( \lim \sup_{j \in H} A^j \subseteq S \) which gives the
desired contradiction. To prove this, let \( x \in R - S \). As \( S \) is maximal, there are \( x_0, \ldots, x_n \in S \) such that \( D_x \cap D_{x_0} \cap \ldots \cap D_{x_n} \) is finite, so \( D_x \cap H \) is finite, too, that is, \( \forall \xi \in \limsup A^j \).

Concerning \( MA_{\omega_1} \) (Martin’s axiom) see \([3, 6]\).

**Theorem 2.** Assume \( MA_{\omega_1} \). If \( A^j \subseteq R \) \((j=0, 1, \ldots)\) and \( \limsup_{j \in H} A^j \) is countable for every \( H \in [\omega]^{\omega} \), then there exists an \( H \in [\omega]^{\omega} \) such that \( \bigcap_{j \in H} A^j \) is uncountable.

**Proof.** By Lemma 4, we can choose an uncountable centered \( \{D_x; x \in S\} \). A lemma of SOLOVAY (see \([6]\)) states that, under \( MA_{\omega_1} \), there is a \( D \in [\omega]^{\omega} \) such that \( D \subseteq^* D_x \) \((x \in S)\). As an uncountable set is not the union of countably many countable sets, there is an uncountable \( S' \subseteq S \) and a \( k \in \omega \) such that if \( x \in S' \), then \( D - D_x \subseteq \{0, \ldots, k\} \). But this gives \( \forall \xi \in \limsup A \) \( S' \).

**Theorem 3.** Assume that \( M \) is a countable, transitive model of the Zermelo—Fraenkel set theory satisfying the generalized continuum hypothesis. If \( \kappa \) is (in \( M \)) a cardinal with uncountable cofinality, there exists a cardinal and cofinality preserving generic extension, \( M[G] \), of \( M \) blowing the continuum up to \( \kappa \), and in \( M[G] \) the following is true: there is a sequence \( A^j \) \((j \in \omega)\) such that for every \( H \in [\omega]^{\omega} \) both \( \limsup_{j \in H} A^j = \kappa \) and \( |\bigcap_{j \in H} A^j| \leq \omega \) hold.

**Proof.** We will adjoin \( \kappa \) Cohen-reals to \( M \) by side-by-side forcing. The applied notion of forcing \( P \) is the set of functions from a finite subset of \( \kappa \times \omega \) into \( \{0, 1\} \). For \( p, q \in P \) define \( p \equiv q \) if and only if \( q \) is a subfunction of \( p \). If \( G \subseteq P \) is a generic filter, define \( r_\xi: \omega \to \{0, 1\} \) for \( \xi < \kappa \) as follows: \( r_\xi(i) = j \) if and only if there is a \( p \in G \) with \( p(\xi, i) = j \). The genericity of \( G \) can be used to show that this gives \( \kappa \) different total functions from \( \omega \) to \( \{0, 1\} \). A standard argument gives that the continuum in \( M[G] \) is \( \kappa \), so \( R \) can be enumerated as \( \{s_\xi; \xi < \kappa\} \). Define \( A^j = \{s_\xi; r_\xi(j) = 1\} \). For \( A \subseteq \kappa \) let \( P_A \) denote the set of partial functions from \( A \times \omega \) into \( \{0, 1\} \), i.e. a subset of \( P \). If \( H \in [\omega]^{\omega} \) is in \( M[G] \), we can find a countable set \( A \) in \( M \) such that \( H \in M[G \cap P_A] \) and \( M[G] = M[G \cap P_A][G \cap P_{\kappa - A}] \) a double generic extension. If we show that, for \( \xi \in \kappa - A \), \( s_\xi \in \limsup_{j \in H} A^j \), \( s_\xi \in \bigcap_{j \in H} A^j \), we are done. Work in \( M[G \cap P_A] \). We have to show that for any \( p \in P_{\kappa - A} \) and \( n \in \omega \) there is an extension \( q \equiv p \) forcing \( r_\xi(j) = 0 \) for an appropriate \( l \in H, j > n \), and similarly, there is an (other) \( q \) forcing \( r_\xi(j) = 1 \) for a \( j \in H, j > n \). As these conditions are equivalent to \( q(\xi, j) = 0 \) and \( q(\xi, j) = 1 \), respectively, and \( p \) is defined only at finitely many points, we can find such conditions.

**Theorem 4.** If the axiom of constructibility holds, there is a sequence of coanalytic sets \( A^j \) \((j=0, 1, 2, \ldots)\) such that, for \( H \in [\omega]^{\omega} \), \( \limsup_{j \in H} A^j > \omega \), \( |\bigcap_{j \in H} A^j| \leq \omega \).
Proof. By the axiom of constructibility there exists a Lusin set $T \subseteq \mathbb{R}$, i.e. an uncountable set such that every nowhere dense subset of $T$ is countable. Moreover, $T$ is a continuous image of a coanalytic set: $T = f(S)$. By the Novikoff—Kond6—Addison uniformization theorem (see [4], §39) we can even assume that $f$ is one-to-one on $S$. For $j \in \omega$ let $B_j$ denote the set of reals with 1 as $j$-th binary digit after the "decimal" point, $A^j = S \cap f^{-1}(B_j)$. As, for $H \in [\omega]^{<\omega}$, $\bigcap_{j \in H} B_j$ is nowhere dense and the complement of $\limsup_{j \in H} B_j$ is of first category, the Lusin property of $T$ gives $\bigcap_{j \in H} A^j = \bigcap_{j \in H} f^{-1}(B_j) \subseteq \bigcap_{j \in \omega} B_j \subseteq \omega$ and $\limsup_{j \in H} A^j = \limsup_{j \in H} B_j \geq \omega$.

4. Metamathematical deduction of Theorem 1

In this section we prove Theorem 1 from Theorem 2. The proof is by forcing. For the sake of simplicity we treat the case of analytic sets on the real line, the general case is analogous. By the Gödel compactness theorem it is enough to show that Theorem 1 holds in every model of (any finite part of) the Zermelo—Fraenkel set theory. A further reduction is available by the Löwenheim—Skolem theorem: we can restrict to countable models.

First we prove a lemma which is the heart of the usual absoluteness proofs. A partially ordered set $(P, \leq)$ is called well-founded if and only if every non-empty subset has a minimum element, in other words, it satisfies the minimum condition. As the axiom of choice is assumed throughout, this is equivalent to: there is no infinite decreasing sequence. Both the next lemma and its proof are well-known.

Lemma 5. Assume that $M \subseteq N$ are $\varepsilon$-models of set theory with the same ordinals, $(P, \equiv)$ is a partially ordered set in $M$, then $(P, \equiv)$ is well-founded in $M$ if and only if $(P, \equiv)$ is wellfounded in $N$.

Proof. If $(P, \equiv)$ is ill-founded in $M$, there is an infinite decreasing sequence which is an element of $M$, therefore it is also an element of $N$.

If $(P, \equiv)$ is well-founded in $M$, the axiom of choice enables us to define by transfinite recursion a monotone function $f$ from $(P, \equiv)$ into a set of ordinals, i.e. if $x < y$ in $P$ then $f(x) < f(y)$ (by the usual ordering of ordinals). As this $f$ is in $M \subseteq N$, $(P, \equiv)$ cannot contain an infinite decreasing sequence in $N$ either.

From now on, assume that $M$ is a countable, transitive model of Zermelo—Fraenkel set theory, $M[G]$ is a forcing extension of it. For any closed set $F$ of reals in $M$ we correspond another closed set $F$ in $M[G]$ as follows: put $F = R - \bigcup G_i$ where the sets $G_i$ are rational, open intervals, and define $F$ as $R - \bigcup G_i$ in $M[G]$. This latter set is still closed, and, though essentially the "same" as $F$, may contain many of the new reals of $M[G]$. 
Lemma 6. (a) If \( x, F \in M \), \( F \) closed, \( x \) real, then \( x \in F \) if and only if \( x \in F' \).
(b) If \( F_1, \ldots, F_n \) are closed sets in \( M \), \( F_1 \cap \ldots \cap F_n = \emptyset \) in \( M \) if and only if \( F_1 \cap \ldots \cap F_n = \emptyset \) in \( M[G] \).

Proof. (a) Straightforward.
(b) Assume \( F_i = R - \bigcup G_i \). If in one of the models \( \cap F_i = \emptyset \), for every closed rational interval \([p, q]\), \([p, q]\) is covered by \( \bigcup_i G_i \), therefore covered by finitely many of them, which must be true in the other model, so \([p, q] \cap \cap F_i = \emptyset \).

For an analytic set \( A \in M \), choose a decomposition \( A = \bigcup_{z \in \omega} F_{z} \), as described in section 1, then put \( \bar{A} = \bigcup_{z \in \omega} F_{z} \) in \( M[G] \), so not only the sets \( F_z \) are changed to \( F_z \), but the possibilities in \( \omega \), too.

Lemma 7. If \( A \) is analytic, \( x \) real in \( M \), then \( x \in A \) if and only if \( x \in \bar{A} \).

Proof. Let us define \( T = \{ s \in \bigcup_{n \in \omega} : x \in F_s \} \), and order \( T \) as follows: \( s' < s \) if and only if \( s' \supseteq s \). By Lemma 6(a) the \((T, \equiv)\) defined in \( M \) is the same as the \((T, \equiv)\) defined in \( M[G] \). As \( x \in A \) is equivalent to: \((T, \equiv)\) is ill-founded, we are done by Lemma 5.

Proof of Theorem 1. Assume that we are given a sequence \((A^j : j \in \omega)\) of analytic sets in \( M \). If \( x \) is a real number in \( M \), define \( D_x = \{ j \in \omega : x \in A^j \} \). If (in \( M \)) there is no uncountable centered subset, there exists a countable \( \lim \sup \) by Lemma 4 and we are done. In the opposite case, let \( M[G] \) be a forcing extension satisfying \( MA_\alpha \) (see [3, 6]). By Theorem 2, there exists a set \( H \in [\omega]^\omega \) in \( M[G] \) such that \( \bigcap_j A_j \) is uncountable. We have to prove that in \( M \) there is a set \( H \in [\omega]^\omega \) with \( \bigcap_j A_j \) uncountable, too. This, like Lemma 7 will be done with the help of a partially ordered set \((P, \equiv)\), which is ill-founded if and only if a set \( H \in [\omega]^\omega \) described above exists. Also, we have to guarantee that the \((P, \equiv)\) defined from \((A^j : j \in \omega)\) in \( M \) and the \((P, \equiv)\) defined from \((\bar{A}^j : j \in \omega)\) in \( M[G] \) are the same, and thus Lemma 5 is applicable.

Assume that \( A^j = \bigcup_{z \in \omega} \bigcap_{n \in \omega} F^j_{z,n} \). The elements of \( P \) will be of the form
\[
(4.1) \quad p = \langle N; n_0, \ldots, n_{k-1}; t(s, i) : i < k, s \in B \rangle
\]
where \( N \) is a natural number, \( n_0 < n_1 < \ldots < n_{k-1} \) are also elements of \( \omega \), \( t(s, i) \in \omega \), moreover
\[
(4.2) \quad F^j_{(s_0, 0)} \cap F^j_{(s_1, 0)} = \emptyset \quad \text{if} \quad s_0 \neq s_1,
\]
and
\[
(4.3) \quad \bigcap_{i = k} F^j_{(s, 0)} \neq \emptyset \quad \text{for a fixed} \quad s \in B.
\]
By Lemma 6, this definition gives the same result both in $M$ and $M[G]$. The order on $P$ is defined as follows: if $p' = \langle N'; n'_0, \ldots, n'_{k-1}; t'(s, i): i < k', s \in k2 \rangle$ is another element, $p' \leq p$ holds if and only if the following conditions are satisfied: $p' = p$ or $N' \supseteq N$, $k' > k$, $n'_i = n_i (i < k)$, $t'(s', i) \supseteq t(s, i)$ if and only if $s' \supseteq s$. Once again, the order is the same in $M$ and in $M[G]$.

Assume first that $p_0 > p_1 > \ldots$ is an infinite decreasing sequence in $(P, \equiv)$. The $n$-parts will immediately produce an index-sequence $n_0 < n_1 < \ldots$; moreover, we will get $F^n_{i(s, o)} (s \in \bigcup_{k \equiv b_k} k2)$ such that, if $z \in n_0$, then

$$\bigcap_{i < k} F^n_{i(s, i)}$$

so, by compactness,

$$\bigcap_{i \in \omega} \bigcap_{k \leq \omega} F^n_{i(s, i)} \neq \emptyset$$

and its unique element, $x_z$ will be in $\bigcap_{i < \omega} A^n$. Also, by (4.2) and (4.4), $x_z \neq z_z$ whenever $z \neq z'$, so $\bigcap_{i < \omega} A^n$ is, in fact, uncountable.

Assume that, in order to prove the other direction, there is an infinite index-sequence $n_0 < n_1 < \ldots$ with $T = \bigcap_{i \in \omega} A^n$ uncountable. We will give a descending sequence with elements $p_k = \langle N_k, n_0, \ldots, n_{k-1}; t(s, i): i < k, s \in k2 \rangle$ with $t(s, i) \in \not\subseteq k, s \in k2$, such that (4.2) holds and, instead of (4.3)

$$T_s = \bigcap_{i < k} A^n_{i(s, o)} \bigcap_{i \geq k} A^n$$

is uncountable for $s \in k2$.

Assume that $p_k$ is defined, that is, $N_k, t(s, i)$ are decided; we shall give $p_{k+1}$. For $s \in k2$, as $T_s$ is uncountable, there are two uncountable subsets, $U_{s0}$ and $U_{s1}$ with $q(U_{s0}, U_{s1}) > 0$. Choose $N_{k+1} = N$ so large that

$$\frac{1}{N} < \frac{1}{4} q(U_{s0}, U_{s1}) \quad \text{for} \quad s \in k2,$$

and

$$\frac{1}{N} < \frac{1}{4} q(F^n_{i(s, o)}, F^n_{i(s, o)}) \quad \text{for} \quad i < k, \quad s_0 \neq s_1.$$

As $U_{s0}$ is covered by

$$\bigcup_{z_0 \in k, s_0 \in (s, 0)} \bigcup_{z_1 \in k, s_1 \in (s, 1)} [A^n_{z_0} \bigcap \ldots \bigcap A^n_{z_k} \bigcup U_{s0}],$$

one of the summands, say

$$A^n_{i(s, o)} \bigcap \ldots \bigcap A^n_{i(s, k)} \bigcup U_{s0},$$

is uncountable; the definition for $t(s_1, i)$ is similar ($i \equiv k$).
Having done this, all entries in $p_{k+1}$ are defined. (4.8) gives (4.5) for $s_0, s_1$: we have to check (4.2). Assume that $F_{t(s_0, 0)} \cap F_{t(s_1, 0)}$ is non-empty. If $s_0 = s_0, s_1 = s_1$ for an appropriate $s$ and $i<k$, then

$$\theta(F_{t(s_0, 0)}^{u_i}, F_{t(s_1, 0)}^{u_i}) \equiv \theta(U_{s_0}, U_{s_1}) - [\text{diam}(F_{t(s_0, 0)}^{u_i}) + \text{diam}(F_{t(s_1, 0)}^{u_i})] \equiv \frac{2}{n} > 0.$$  

If $i<k$ and $s_0|k \neq s_1|k$, by (4.7)

$$\theta(F_{t(s_0, 0)}^{u_i}, F_{t(s_1, 0)}^{u_i}) \equiv \frac{4}{N} > 0.$$  

If $i=k>0$,

$$\theta(F_{t(s_0, 0)}^{u_i}, F_{t(s_1, 0)}^{u_i}) \equiv \theta(F_{t(s_0, 0)}^{u_i}, F_{t(s_1, 0)}^{u_i}) - [\text{diam}(F_{t(s_0, 0)}^{u_i}) + \text{diam}(F_{t(s_1, 0)}^{u_i})] \equiv \frac{2}{N} > 0,$$

as (4.3) holds. If $i=k=0$ there is nothing to prove as there is only $2^k=1$ sequence, so (4.2) is trivial.

**References**


О верхнем пределе аналитических множеств

II. КОМЬЯТ

Теорема М. Лакковича утверждает, что если $A_0, A_1, ...$—бореллевские множества на вещественной прямой и верхний предел $\limsup_{j \in H} A_j$ несчетен для любого бесконечного множества индексов $H$, то для некоторого бесконечного множества индексов несчетным будет пересечение $\bigcap_{i \in H} A_i$. Он доказал также, что в предположении континуум-гипотезы для произвольных множеств вещественных чисел это неверно.
В настоящей работе этот результат, относящийся к борелевским множествам, переносится на аналитические множества; доказано, что если принять аксиому Мартина, то он справедлив для любых множеств. Из этого последнего результата мы выводим альтернативное доказательство для случая аналитических множеств. Обсуждаются также и некоторые другие результаты типа независимости.

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