A note on uncountable chordal graphs

Péter Komjáth

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Abstract

We show that if $X$ is a chordal graph containing no clique of size $\mu$ ($\mu$ an infinite cardinal) then the chromatic (even coloring) number of $X$ is at most $\mu$. The same conclusion holds if the condition ‘is chordal’ is replaced by ‘contains no induced $C_4$ (or $K_{k,k}$ for $k$ finite)’.

In [9] Wagon asked if the following holds. If $X$ is a chordal graph with $K_\omega \not\subseteq X$ then $\text{Chr}(X) \leq \omega$. This was proved by Halin in [5].

Here we improve this result. We show that if $\mu$ is an infinite cardinal, $X$ is a chordal graph such that $K_\mu \not\subseteq X$, then $\text{Chr}(X) \leq \mu$, in fact even the coloring number of $X$ is at most $\mu$. With a different argument we show the same result if only an induced $C_4$ (or any induced $K_{k,k}$) is excluded.

An example of Galvin’s shows that for every infinite cardinal $\mu$ there is an interval graph $X$ containing no $K_{\mu^+}$ but $\text{Chr}(X) = \mu^+$, that is, $X$ is not perfect.

Notation. Definitions. We use the notation and definitions of axiomatic set theory. In particular, ordinals are von Neumann ordinals, and each cardinal is identified with the least ordinal of that cardinality. For the notions of regular cardinals, closed unbounded and stationary sets, the reader can consult [4]. A graph is an arbitrary set of unordered pairs of some set $V$, the set of vertices. If $v \in V$, then $N(v)$ is the set of neighbors of $v$, i.e., $N(v) = \{w : \{v, w\} \in X\}$. If $\kappa$ is a cardinal, then $K_\kappa$ is the complete graph on $\kappa$ vertices. If $a, b$ are cardinals, then $K_{a,b}$ is the complete bipartite graph with bipartition classes of cardinality $a$, $b$, respectively. $C_n$ denotes the circuit of length $n$. A graph is chordal if it does not contain an induced $C_n$ for $n \geq 4$. 
Chr(X) is the chromatic number of X, the least (finite or infinite) cardinal \( \mu \) such that there is a mapping \( f : V \to \mu \) which is a good coloring, that is, if \( \{x, y\} \in X \) then \( f(x) \neq f(y) \). The coloring number of a graph X, in short, Col(X), is the least cardinal \( \mu \) such that the vertex set of X has a well ordering in which each vertex is joined to \( < \mu \) smaller vertices. This notion was introduced by Erdős and Hajnal in [1]. The inequality \( \text{Chr}(X) \leq \text{Col}(X) \) can be established by a straightforward transfinite recursion.

Fodor proved in [2] that if V is a set, \( f(v) \subseteq V \) for every \( v \in V \) such that \( |f(v)| < \mu \), then V is the union of \( \mu \) free subsets, i.e., sets W with \( x \notin f(y) \) for \( x \neq y \in W \). Fodor’s theorem can naturally be reformulated in graph theory terms as follows. If \( X \) is a directed graph such that the outdegree of each vertex is less than \( \mu \), then \( \text{Chr}(X) \leq \mu \). The proof of this theorem (see e.g., in [4]) gives in fact the stronger statement that \( \text{Col}(X) \leq \mu \).

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Lemma 1. (Fulkerson–Gross, [3]) A finite graph \( (V, X) \) is chordal iff there is an ordering \( < \) of \( V \) such that \( \{w < v : \{w, v\} \in X\} \) is a clique for each \( v \in V \).

Theorem 2. If \( (V, X) \) is a \( K_\mu \)-free chordal graph, then \( \text{Chr}(X) \leq \mu \).

Proof. By Lemma 1 each finite subgraph \( (W, X|W) \) has an ordering such that \( \{w < v : \{w, v\} \in X|W\} \) is a clique for every \( v \in W \). Compactness gives that \( (V, X) \) has such an ordering, too. (Indeed, we can encode every total ordering of a set W by a choice of one of \( <, =, > \) for each pair \( (x, y) \in W \times W \). Endow \( \{<, =, >\}^{V(G) \times V(G)} \) with the product topology. Similar as in the proof of the de Bruijn–Erdős, we can use the compactness of T to prove the existence of a suitable total ordering. Alternatively, one can refer to Rado’s selection principle [6].)

Consequently, \( (V, X) \) has an ordering such that \( |\{w < v : \{w, v\} \in X\}| < \mu \) holds for each \( v \in V \). If we direct each edge \( \{x, y\} \) from y to x when \( x < y \), then this means that X can be oriented such that each outdegree is less than \( \mu \). By Fodor’s Theorem then \( \text{Chr}(X) \leq \text{Col}(X) \leq \mu \).

We slightly extend the above result.

We need the following theorem.

Lemma 3. (Shelah, [8]) Let X be a graph with coloring number \( > \mu \). Then X has an induced subgraph H with coloring number \( > \mu \) and \( |V(H)| \) regular.
Theorem 4. If $\mu$ is an infinite cardinal, $2 \leq k < \omega$, $X$ is a graph with no induced $K_{k,k}$, $K_\mu \not\subseteq X$, then $\Col(X) \leq \mu$.

Proof. Assume indirectly that $\Col(X) > \mu$. By passing to induced subgraphs, we can assume that $\Col(Y) \leq \mu$ holds for every induced subgraph $Y$ of smaller cardinality. By Lemma 3, we can assume that $V(X) = \kappa$ for some regular cardinal $\kappa$. Next we show that

$$S = \{ \alpha < \kappa : \exists \beta(\alpha) \geq \alpha, |N(\beta(\alpha)) \cap \alpha| \geq \mu \}$$

is stationary.

Suppose for a contradiction that $S$ is nonstationary, then its complement contains a club $C$. Let $\{\alpha_i : i < \kappa\}$ be the well order of $C$ induced by that of $\kappa$. We can as well assume that $\alpha_0 = 0$. We define a well ordering of $\kappa$ by recursion on $i$. Assume that we already defined a partial well order $\sigma_j$ of $\{\alpha : \alpha < \sigma_j\}$ for $j < i$. If $i$ is a limit we take for $\sigma_i$ the union of all the $\sigma_j$ for $j < i$. Otherwise, $i$ has a predecessor $j = i - 1$. By the minimality hypothesis on $\kappa$ we have that $\Col(X|_{[\alpha_j, \alpha_i)}) \leq \mu$. Let $<^*$ be a well order witnessing this. We end-extend $\sigma_j$ with $[\alpha_j, \alpha_i)$ ordered by $<^*$. The union of all well orders $\sigma_i$ witnesses that $\Col(X) \leq \mu$, a contradiction.

Next we are going to define the set $C \subseteq \kappa$ as follows. $\gamma \in C$ if the following holds: whenever $y_0, \ldots, y_{k-1} < \gamma$, $|N(y_0) \cap \cdots \cap N(y_{k-1})| < \kappa$, then $\sup(N(y_0) \cap \cdots \cap N(y_{k-1})) < \gamma$. It is easy to see that $C$ is closed, unbounded in $\kappa$. As $S$ is stationary, we can pick $\delta \in C \cap S$. By the definition of $S$, there are $x \geq \delta$ and different $y_\xi < \delta$ such that $\{y_\xi, x\} \in X (\xi < \mu)$. By assumption $K_\mu \not\subseteq X$, therefore by the Erdős–Dushnik–Miller theorem (i.e., a graph on $\mu$ has either a complete $\mu$ or an independent $k$) there are $\xi_0 < \cdots < \xi_{k-1} < \mu$ such that $\{y_{\xi_0}, \ldots, y_{\xi_{k-1}}\}$ is independent in $X$.

The set $U = N(y_{\xi_0}) \cap \cdots \cap N(y_{\xi_{k-1}})$ must have cardinality $\kappa$, as otherwise we had $\sup(U) < \kappa$ and so $\sup(U) < \delta$, as $\delta \in C$, but this contradicts $\delta \leq x \in U$. Applying again the Erdős–Dushnik–Miller theorem to $U$, we find $\delta < u_0 < \cdots < u_{k-1}$ in $D$ such that $\{u_0, \ldots, u_{k-1}\}$ is independent and then $\{y_{\xi_0}, \ldots, y_{\xi_{k-1}}, u_0, \ldots, u_{k-1}\}$ gives an induced copy of $K_{k,k}$.

References


Péter Komjáth
Institute of Mathematics
Eötvös University
Budapest, Pázmány P. s. 1/C
1117, Hungary
e-mail: kope@cs.elte.hu