A Ramsey statement for infinite groups

Péter Komjáth

November 1, 2016

Abstract

If \( \kappa \) is a cardinal, \( n < \omega \), then there exists an Abelian group \( G \) such that if \( F : G \to \kappa \), then there exist distinct elements \( a_{i, \alpha} \in G \) (\( 1 \leq i \leq n, \alpha < \kappa \)), and a color \( \tau < \kappa \) such that if \( 1 \leq i_0 < \cdots < i_r \leq n, \alpha_1, \ldots, \alpha_r < \kappa \), then \( F(a_{i_1, \alpha_1} + \cdots + a_{i_r, \alpha_r}) = \tau \).

One of the important results in Ramsey theory is Hindman’s theorem: if the set of natural numbers is colored with finitely many colors, then there are infinitely many natural numbers so that all finite sums of them get the same color. Various recent results showed that the natural uncountable generalizations of this theorem are false ([3], [4], [5]). For example, every Abelian group can be colored with countably many colors, so that no color class contains infinitely many elements with all their subsums. (A similar result was proved in [1].)

In this note we prove results in the positive direction. First we show that for every finite \( n \) and infinite cardinal \( \kappa \) there is an Abelian group \( G \) such that for every coloring of \( G \) with \( \kappa \) colors there are \( n \) elements so that the same color class contains all subsums of them. Then we extend this to establish the existence of an Abelian group \( G \) such that for every coloring with \( \kappa \) colors there are distinct elements \( \{a_{i, \alpha} : 1 \leq i \leq n, \alpha < \kappa \} \) such that all sums of the form \( a_{i_1, \alpha_1} + \cdots + a_{i_r, \alpha_r} \) (\( i_1, i_2, \ldots, i_r \) different) are distinct and in the same color class.

Notation. Definitions. We use the notions and definitions of axiomatic set theory. In particular, each ordinal is a von Neumann ordinal, each cardinal is identified with the least ordinal of that cardinality. If \( \kappa \) is an infinite cardinal, then \( \kappa^+ \) is its successor cardinal. Further, \( \exp_0 \kappa = \kappa \) and then
by induction $\exp_{n+1} \kappa = 2^{\exp_n \kappa}$. If $A$ and $B$ are sets, then their symmetric difference, $A \triangle B$ is $(A - B) \cup (B - A)$. A simple inductive argument shows that $A_1 \triangle \cdots \triangle A_n$ consists of those points contained in an odd number of $A_1, \ldots, A_n$. If $(A, <)$ is an ordered set, then $\text{tp}(A, <)$ or just $\text{tp}(A)$ denotes its order type. If $A$, $B$ are subsets of the same ordered set, then $A < B$ denotes that $x < y$ holds for any $x \in A$, $y \in B$. If $S$ is a set, $\kappa$ a cardinal, then $[S]^{\kappa} = \{ x \subseteq S : |x| = \kappa \}$, $[S]^{<\kappa} = \{ x \subseteq S : |x| < \kappa \}$.

We will frequently invoke the Erdős–Rado theorem: $(\exp_n \kappa)^+ \to (\kappa^+)^{n+1}_\kappa$. In words: if $\lambda = (\exp_n \kappa)^+$ and $F : [\lambda]^{n+1} \to \kappa$, then there is a set $A \in [\lambda]^\kappa$, such that if $a, b \in [A]^{n+1}$, then $F(a) = F(b)$, i.e., $A$ is homogeneous for $F$ (cf [2]).

Acknowledgment. My thanks go to a fast, intelligent, and helpful referee.

**Lemma 1.** If $|S| = 2^n$, then there are subsets $A_1, \ldots, A_n \subseteq S$ such that $|A_1 \triangle \cdots \triangle A_n| = 2^{n-1}$ ($1 \leq i_1 < \cdots < i_r < n, \tau \geq 1$) and $|A_i - \bigcup \{ A_j : j \neq i \}| = 1$ for $1 \leq i \leq n$.

**Proof.** A well known fact is that there exists an independent family $\{A_1, \ldots, A_n\}$ of $n$ subsets of $S$, i.e., that $|A_i^{(1)} \cap \cdots \cap A_i^{(r)}| = |S|/2^r$ for $i_1 < \cdots < i_r \in \{1, 2, \ldots, n\}$ and $\varepsilon(i) = 0$ or $1$, where $A_1^1 = A$, $A_1^n = S - A$. One can, for example, assume that $S$ is the set of functions $f : \{1, 2, \ldots, n\} \to \{0, 1\}$ and $A_i = \{ f \in S : f(i) = 1 \}$.

The second property is immediate.

For the first property, assume that $1 \leq i_1 < \cdots < i_r \leq n$. The set $S$ decomposes into the $2^r$ sets of the form $A_i^{(1)} \cap \cdots \cap A_i^{(r)} (\varepsilon(i) \in \{0, 1\})$. By our construction, all these sets are of size $2^{n-r}$. $A_i \triangle \cdots \triangle A_i$ is the union of those for which $\varepsilon(1) + \cdots + \varepsilon(r)$ is odd, i.e., half of them. Consequently, $|A_i \triangle \cdots \triangle A_i| = 2^{n-r} \cdot 2^{n-r} = 2^{n-1}$.

**Theorem 1.** If $\kappa$ is a cardinal, $n < \omega$, then there exists an Abelian group $G$ such that if $F : G \to \kappa$, then there exist distinct elements $a_1, \ldots, a_n \in G$, and $\tau < \kappa$ such that if $1 \leq i_1 < \cdots < i_r \leq n$, then $F(a_i + \cdots + a_i) = \tau$.

**Proof.** Set

$$\lambda = (\exp_{2^{n-1}} \kappa)^+.$$

Our group $G$ is $([\lambda]^{<\omega}, \triangle)$, i.e., the set of finite subsets of $\lambda$ with the symmetric difference as addition.

Assume that $F : G \to \kappa$ is a coloring. By the Erdős–Rado theorem and the choice of $\lambda$ there is a set $S \in [\lambda]^{2^n}$ homogeneous for the restriction of
Proof. If 1 \leq i < \cdots < i_r \leq n then 
B = A_i \triangle \cdots \triangle A_{i_r} \in [\lambda]^{2n-1}, so F(B) = \tau and we are finished. \ □

Theorem 2. If 2 \leq n < \omega, \ \kappa is a cardinal, then there is an Abelian group 
G, such that if F : G \to \kappa, then there exist distinct elements a_{i,\alpha} \in G 
(1 \leq i \leq n, \ \alpha < \kappa) and a color \ \tau < \kappa, such that if 1 \leq r \leq n, 1 \leq i_1 < i_2 < 
\cdots < i_r \leq n, \ \alpha_1, \ldots, \alpha_r < \kappa, then 
\[ F(a_{i_1,\alpha_1} + a_{i_2,\alpha_2} + \cdots + a_{i_r,\alpha_r}) = \tau. \]

Proof. Set N = 2^{n-1}, \ \lambda = (\exp_{N-1} \kappa)^+, \ \mu = (\exp_N \lambda)^+.

Our group G is ([\mu \times \lambda]^\omega, \triangle). Assume that F : G \to \kappa.

Claim 2.1. There are \gamma_1 < \gamma_2 < \cdots < \gamma_{2N} < \mu and F^* : [\lambda]^N \to \kappa such that 
if 1 \leq i_1 < i_2 < \cdots < i_N \leq 2N, \ \xi_1 < \xi_2 < \cdots < \xi_N < \lambda, then 
\[ F(\langle \gamma_{i_1}, \xi_1 \rangle, \ldots, \langle \gamma_{i_N}, \xi_N \rangle) = F^*(\xi_1, \ldots, \xi_N). \]

Proof. If \gamma_1 < \gamma_2 < \cdots < \gamma_N, then color \{\gamma_1, \ldots, \gamma_N\} with the function 
\[ \langle \xi_1, \ldots, \xi_N \rangle \mapsto F(\langle \gamma_{i_1}, \xi_1 \rangle, \ldots, \langle \gamma_{i_N}, \xi_N \rangle). \]

This is a coloring of [\mu]^N with 2^\lambda colors. By the Erdős–Rado theorem 
\[ \mu = (\exp_{N-1} 2^\lambda)^+ \to (2N)^2, \]
and this gives a homogeneous set as required. \ □

As \lambda = (\exp_{N-1} \kappa)^+, we can apply again the Erdős–Rado theorem and 
get \[ U \subseteq \lambda, \ \tau < \kappa, \ \text{tp}(U) = \kappa^+ \] such that if \( u \in [U]^N \), then \( F^*(u) = \tau. \)

Choose the subsets \( A_1, \ldots, A_n \) of \{1, \ldots, 2N\} as in Lemma 1. Clearly, 
\( |A_i| = N \) \( (i < n) \). Let 1 \leq x_i \leq 2N be such that 
\[ A_i - \bigcup\{A_j : j \neq i\} = \{x_i\} \quad (1 \leq i \leq n). \]

Choose subsets \( U_1 < U_2 < \cdots < U_{2N} \) of \( U \) such that if there is no 
1 \leq i \leq n such that \( j = x_i \), then \( |U_j| = 1 \), otherwise \( |U_j| = \kappa \). Accordingly, 
\( U_j = \{y_j\} \) or \( U_j = \{y_{j,\alpha} : \alpha < \kappa\} \).
Define
\[ s_{i,\alpha} = \{ \langle \gamma_j, y_j \rangle : j \in A_i - \{ x_i \} \} \cup \{ \langle \gamma_{x_i}, y_{x_i,\alpha} \rangle \}. \]

**Claim 2.2.** If \( 1 \leq i_1 < \cdots < i_r \leq n \), \( \alpha_1, \ldots, \alpha_r < \kappa \), then
\[ F(s_{i_1,\alpha_1} \triangle \cdots \triangle s_{i_r,\alpha_r}) = \tau. \]

**Proof.** Set \( s = s_{i_1,\alpha_1} \triangle \cdots \triangle s_{i_r,\alpha_r} \). We have to show that \( F(s) = \tau \).

Projection to the first coordinate maps the sets \( s_{i_1,\alpha_1}, \ldots, s_{i_r,\alpha_r} \) to the sets \( A_{i_1}, \ldots, A_{i_r} \subseteq \{ \gamma_1, \ldots, \gamma_{2N} \} \).

If \( \gamma_j \) is contained in more than one \( A_i \), then \( j \neq x_i \) (\( i = i_1, \ldots, i_r \)). Thus the point in the sets with first coordinate \( \gamma_j \) is \( \langle \gamma_j, y_j \rangle \) and \( \langle \gamma_j, y_j \rangle \in s \) iff \( j \in A_{i_1} \triangle \cdots \triangle A_{i_r} \).

If \( \gamma_j \) is contained in exactly one \( A_i \), then the element of \( s \) which projects to \( \gamma_j \) is either \( \langle \gamma_j, y_j \rangle \) or \( \langle \gamma_j, y_j, \alpha_{i_h} \rangle \) where \( j = x_{i_h} \).

By \( |A_{i_1} \triangle \cdots \triangle A_{i_r-1}| = N \) and the choice of \( y_j, y_j, \gamma_j \) we have
\[ s = \{ \langle \gamma_{j_1}, \xi_1 \rangle, \langle \gamma_{j_2}, \xi_2 \rangle, \ldots, \langle \gamma_{j_N}, \xi_N \rangle \} \]
where \( j_1 < \cdots < j_N \), \( \xi_1 < \cdots < \xi_N \), and \( \{ \xi_1, \ldots, \xi_N \} \subseteq U \).

By the choice of \( F^* \) and \( U \) we have \( F(s) = F^*(\{ \xi_1, \ldots, \xi_N \}) = \tau. \)

By Claim 2.2, we are finished.

**References**


Péter Komjáth
Institute of Mathematics
Eötvös University
Budapest, Pázmány P. s. 1/C
1117, Hungary
e-mail: kope@cs.elte.hu