Abstract

If $\mu$ is an infinite cardinal, $\mathcal{H}$ a system of $\mu$-element sets with $|A \cap B| \leq r$ (for $r$ finite) for $A \neq B \in \mathcal{H}$, then $\mathcal{H}$ has a minimal covering. This is false if we only assume $|A \cap B| < \omega$ for $A \neq B \in \mathcal{H}$. GCH is consistent with the statement that if $\tau < \mu$ are infinite cardinals, $\mu$ is regular, $\mathcal{H}$ is a system of $\mu$-element sets, $|A \cap B| < \tau$ for $A \neq B \in \mathcal{H}$, then $\mathcal{H}$ has a minimal covering set. GCH is also consistent with the existence of a set system as above for $\tau = \omega < \mu = \omega_1$ with no minimal covering set.

A covering set of a hypergraph is a set of vertices which intersects each hyperedge. One can immediately see that any system consisting of finitely many finite sets has a minimal covering set, i.e., a covering set no proper subset of which is covering: take the underlying set and remove its elements one by one as long as it is possible while keeping it covering. This argument does not work, in fact, the statement is no longer true for infinite systems of finite and infinite sets (see a counterexample in Lemma 4). It is relatively easy to show that any system of finite sets has a minimal covering set (see Theorem 5).

These raise the interesting problem when an infinite set system possesses a minimal covering set.

Dominic van der Zypen asked the following question (see [1], [9]):

If $r < \omega$ and $\mathcal{H}$ is a set system with the property that $|A \cap B| \leq r$ for $A \neq B \in \mathcal{H}$, does there exist a minimal coverings set for $\mathcal{H}$?

In 1937, E. W. Miller showed that if $\mathcal{H}$ is a system of infinite sets with $|A \cap B| \leq r$ ($A \neq B \in \mathcal{H}$) for some $r < \omega$, then $\mathcal{H}$ is 2-chromatic, that is, it is possible to color the underlying set with 0 and 1 such that each set in $\mathcal{H}$
gets both colors. This was investigated and generalized by Erdős and Hajnal in [2]. We use some of their techniques concerning the above results in order to investigate and partially solve van Zypen’s question.

We answer the above question in the affirmative for uniform systems, or even for systems of finite and infinite sets in which all infinite members have the same cardinality (Corollary 9.). In the proof we use a result of J. Klimó, see [5].

We show that a minimal covering set may not exist if the condition $|A \cap B| \leq r$ is relaxed to $|A \cap B|$ finite, i.e., in the case of almost disjoint set systems (Theorem 15). Perhaps the simplest type of almost disjoint systems is the following: let $S \subseteq \omega_1$ be a set of limit ordinals and let $\mathcal{H} = \{H_\xi : \xi \in S\}$ be a set system, where $H_\xi$ is a cofinal subset of $\xi$ of order type $\omega$. Now $\mathcal{H}$ has a minimal covering set if $S$ is nonstationary (Lemma 13), or $S$ is arbitrary, if $\text{MA}_{\omega_1}$ holds (Theorem 14), but consistently there is a system $\mathcal{H}$ for which there is no minimal covering set (Lemma 12).

If, however, $\mathcal{H}$ is a system of $\mu$-tuples and the condition on intersections is changed to $|A \cap B| < \tau$ for some $\tau < \mu$ then there is a minimal covering set assuming that GCH and $\Box_\lambda$ hold for $\lambda$ singular, $\text{cf}(\lambda) \leq \tau$ (Theorem 16). For the opposite direction, if the existence of a supercompact cardinal is consistent, then GCH is consistent with the existence of a system $\mathcal{H} \subseteq [\omega_{\omega+1}]^{\omega_1}$ with $|A \cap B| < \omega$ for $A \neq B \in \mathcal{H}$ and $\mathcal{H}$ has no minimal covering set (Theorem 17).

The original problem raised by van der Zypen, that is, when no condition on the size of the sets is imposed, remains open.

**Notation. Definitions.** We use the notions and definitions of axiomatic set theory. In particular, each ordinal is a von Neumann ordinal, each cardinal is identified with the least ordinal of that cardinality. $\kappa$ is the cardinality of the set of reals. If $f$ is a function, $A$ a set, then $f[A] = \{f(x) : x \in A\}$. A cardinal $\kappa$ is $\tau$-inaccessible if $\mu^\tau < \kappa$ holds for all $\mu < \kappa$. $\clubsuit$ is the following statement: there is a sequence $\langle A_\alpha : \alpha < \omega_1, \text{limit} \rangle$ such that $\text{tp}(A_\alpha) = \omega A_\alpha$ is cofinal in $\alpha$ and if $A \subseteq \omega_1$ is uncountable, then $A_\alpha \subseteq A$ for some $\alpha < \omega_1$. Both this statement and its negation are consistent with the axioms of set theory. For Martin’s axiom, see [3].

If $(A, <)$ is an ordered set, then $\text{tp}(A, <)$ or just $\text{tp}(A)$ denotes its order type. If $A, B$ are subsets of the same ordered set, then $A < B$ denotes that $x < y$ holds for any $x \in A$, $y \in B$. If $A$ or $B$ is a singleton, we write $a < B$ instead of $\{a\} < B$, etc. If $S$ is a set, $\kappa$ a cardinal, then $[S]^\kappa = \{x \subseteq S : x \leq \kappa\}$. 


\[ |x| = \kappa, \ [S]^{<\kappa} = \{ x \subseteq S : |x| < \kappa \}, \ [S]^{\leq\kappa} = \{ x \subseteq S : |x| \leq \kappa \}. \]

A system \( \mathcal{H} \subseteq [S]^\mu \) is almost disjoint if \( |A \cap B| < \mu \) for \( A \neq B \in \mathcal{H} \).

If \( \mathcal{H} \subseteq \mathcal{P}(V) - \{\emptyset\} \) is a set system, a \( M \subseteq V \) is a covering set if \( M \cap H \neq \emptyset \) for every \( H \in \mathcal{H} \). \( M \) is a minimal covering set if it is covering, but no proper subset of it is.

If \( (V, X) \) is a graph, \( v \in V \), then \( st(v) = \{ w \in V : \{v, w\} \in X \} \) and \( ST(X) = \{st(v) : v \in V \} \) is the star system of \( X \).

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First we prove that we can discard of singletons.

**Lemma 1.** Let \( \mathcal{H} \subseteq \mathcal{P}(S) - \{\emptyset\} \) be a set system, \( R = \{ x \in S : \{x\} \in \mathcal{H} \} \), \( \mathcal{H}_0 = \{ H \in \mathcal{H} : H \cap R \neq \emptyset \} \). If \( T \) is a minimal covering set for \( \mathcal{H} - \mathcal{H}_0 \) then \( T \cup R \) is a minimal covering set for \( \mathcal{H} \).

**Proof.** \( T \subseteq S - R \) as otherwise the points outside \( S - R \) could be removed. As \( R \) covers \( \mathcal{H}_0 \) and \( T \) covers \( \mathcal{H} - \mathcal{H}_0 \), \( T \cup R \) covers \( \mathcal{H} \).

For minimality, pick \( x \in T \cup R \). If \( x \in R \), then \( T \cup R - \{x\} \) does not cover \( \{x\} \in \mathcal{H} \). If \( x \in T \), \( T - \{x\} \) does not cover some \( A \in \mathcal{H} - \mathcal{H}_0 \), but then, as \( A \cap R = \emptyset \), neither does \( R \cup T - \{x\} \).

**Lemma 2.** Assume that \( \mathcal{H} \subseteq \mathcal{P}(S) \), \( T \subseteq S \) is a covering set for \( \mathcal{H} \). If \( U \) is a minimal covering set for \( \mathcal{H}' = \{ A \cap T : A \in \mathcal{H} \} \), then \( U \) is a minimal covering set for \( \mathcal{H} \).

**Proof.** \( U \subseteq T \) as otherwise the points in \( S - T \) could be removed. \( U \) is clearly covering for \( \mathcal{H} \). For minimality, pick \( x \in U \). There is \( A \in \mathcal{H} \) such that \( (U - \{x\}) \cap (A \cap T) = \emptyset \). As \( U \subseteq T \), we have \( (U - \{x\}) \cap A = \emptyset \).

**Theorem 3.** If \((V, X)\) (\(|V| > 1\)) is a graph with no isolated vertices, then \( ST(X) \) has a minimal covering set. In other words, \( V \) can be covered by vertex-disjoint edges and stars.

**Proof.** Using Zorn’s lemma, we select a maximal system of independent edges, \( E = \{e_i : i \in I\} \). Set \( e_i = \{a_i, b_i\} \) and \( W = \bigcup E \). By maximality, \( V - W \) is independent. As \( X \) has no isolated vertex, each \( v \in V - W \) is joined to some \( e_i \), we add \( v \) to one of them, with one edge joining it. Now \( V \) is covered by the disjoint union of the structures of the following types:

1. an edge \( e_i \);
2. an edge \( e_i \), plus some vertices joined to \( a_i \), but none to \( b_i \);
3. an edge \( e_i \), plus some vertices joined to \( b_i \), but none to \( a_i \);
(4) an edge $e_i$, plus some vertices joined to $a_i$, plus some vertices joined to $b_i$.

The structures in (1) are edges. The structures in (2) and (3) are stars. The structures in (4) are double stars, each can be turned into two stars by removing the middle edge $e_i$.

We next give a simple example of a set system with no minimal covering set.

**Lemma 4.** If $\mathcal{H} = \{H_0, H_1, \ldots\}$ with $H_0 \supseteq H_1 \supseteq \ldots$ and $\bigcap \{H_n; n < \omega\} = \emptyset$, then $\mathcal{H}$ has no minimal covering set.

**Proof.** Let $M$ be a covering set, $x \in M$ arbitrary. There is $n < \omega$ such that $x \notin H_n$. As $M$ is covering, there is $y \in M \cap H_n$. For every $i < \omega$, if $x \in H_i$, then $y \in H_i$, so $x$ can be removed from $M$.

**Theorem 5.** If $\mathcal{H}$ consists of finite sets then there is a minimal covering set for $\mathcal{H}$.

**Proof.** Let $S$ be the underlying set of $\mathcal{H}$. We define the following partially ordered set $(P, \leq)$. $p \in P$ if $p \subseteq S$ and $p \cap H \neq \emptyset$ for $H \in \mathcal{H}$, i.e., $p$ is a covering set. Set $q \leq p$ iff $p \subseteq q$. $P \neq \emptyset$ as $\bigcup \mathcal{H} \subseteq P$.

**Claim.** Each chain has an upper bound in $(P, \leq)$.

**Proof.** Let $L \subseteq P$ be a chain. We claim that $p = \bigcap L$ is an element of $P$ (and so an upper bound of $L$). Indirectly, assume that $p \cup A = \emptyset$ for some $A \in \mathcal{H}$. Then for every $x \in A$ there is $p_x \in L$ such that $x \notin p_x$. As $L$ is a chain, among the finitely many elements $\{p_x : x \in A\}$ one is largest, say $p_x$. Then $p_x \cap A = \emptyset$, a contradiction. Using the Claim, we can apply Zorn’s lemma and obtain a maximal element $p$ of $(P, \leq)$. Then $p$ is covering by definition and for $x \in p$, $p - \{x\}$ is not covering, as $p$ is maximal. Consequently $p$ is a minimal covering set.

**Lemma 6.** If $\mathcal{H} \subseteq \mathcal{P}(V)$ is a set system and $T \subseteq V$ is such that $0 < |H \cap T| < \omega$ for $H \in \mathcal{H}$ then $\mathcal{H}$ has a minimal covering set.

**Proof.** By Lemma 2 and Theorem 5.

The following is the main result of the paper.

**Theorem 7.** Let $S$ be a set and $\mu$ an infinite cardinal, $\mathcal{H} \subseteq [S]^\mu$. Consider the following statements.

1. For some $r < \omega$, $|A \cap B| \leq r$ for all $A \neq B$ in $\mathcal{H}$.
(2) \( \mathcal{H} \) has a well ordering \(<\) such that \(|H \cap \bigcup \{H' : H' < H\}| < \mu\) for each \( H \in \mathcal{H} \).

(3) \( \mathcal{H} \) has a well ordering such that \( H \not\subseteq \bigcup \{H' : H' < H\} \) for \( H \in \mathcal{H} \).

(4) \( S \) has a well ordering \(<\) such that for each \( H \in \mathcal{H} \), \( \max(H) \) exists and \( \max(H) \neq \max(H') \) (\( H \neq H' \in \mathcal{H} \)).

(5) \( S \) has a well order \(<\) such that for each \( H \in \mathcal{H} \), \( \max(H) \) exists.

(6) \( \mathcal{H} \) has a minimal covering set.

Then (1) \( \Rightarrow \) (2) \( \Rightarrow \) (3) \( \equiv \) (4) \( \Rightarrow \) (5) \( \Rightarrow \) (6).

**Proof.** (1) \( \Rightarrow \) (2): Assume that \( r < \omega \) and \( \mu \geq \omega \) is an infinite cardinal, \( S \) is some set and \( \mathcal{H} \subseteq [S]^{\omega} \) is such that \(|A \cap B| \leq r\) for \( A \neq B \in \mathcal{H} \).

If \( A \subseteq S \), \( \mathcal{F} \subseteq \mathcal{H} \), then we call the pair \((A, \mathcal{F})\) closed if

(a) if \( H \in \mathcal{F} \), then \( H \subseteq A \), and

(b) if \( H \in \mathcal{H} - \mathcal{F} \), then \(|H \cap A| < \mu\).

**Claim 1.** \((\emptyset, \emptyset)\) and \((S, \mathcal{H})\) are closed.

**Proof.** Straightforward. \( \square \)

If \((A, \mathcal{F}), (A', \mathcal{F}')\) are closed, we set \((A, \mathcal{F}) \leq (A', \mathcal{F}')\) if \( A \subseteq A' \) and \( \mathcal{F} \subseteq \mathcal{F}' \) hold.

**Claim 2.** If \((A, \mathcal{F}) \leq (A', \mathcal{F}')\), then \(|\mathcal{F}' - \mathcal{F}| \leq |A' - A| + \omega\).

**Proof.** For each \( H \in \mathcal{F} - \mathcal{F} \) we have \(|H \cap (A' - A)| = \mu\), therefore we can pick a subset \( s(H) \in [H \cap (A' - A)]^{\omega+1} \). If \( H \neq H' \), \( s(H) \neq s(H') \) as \(|H \cap H'| \leq r\) by our assumption. Consequently, \( s \) injects \( \mathcal{F}' - \mathcal{F} \) into \([A' - A]^{\omega+1}\), whose cardinality is at most \(|A' - A| + \omega\). \( \square \)

**Claim 3.** If \((A, \mathcal{F}) \leq (A', \mathcal{F}'), \kappa = |\mathcal{F}' - \mathcal{F}| > \mu\), then there exist \( A_\alpha, \mathcal{F}_\alpha \) (\( \alpha \leq \kappa \)) such that

(a) \( A_0 = A, \mathcal{F}_0 = \mathcal{F}, A_\kappa = A', \mathcal{F}_\kappa = \mathcal{F}'\);

(b) \( \{A_\alpha : \alpha \leq \kappa\} \) and \( \{\mathcal{F}_\alpha : \alpha \leq \kappa\} \) are increasing, continuous;

(c) \( (A_\alpha, \mathcal{F}_\alpha) \) is closed (\( \alpha \leq \kappa \)), \( (A_\alpha, \mathcal{F}_\alpha) \leq (A_{\alpha + 1}, \mathcal{F}_{\alpha + 1})\);

(d) \(|A_{\alpha + 1} - A_\alpha| < \kappa\).

**Proof.** Enumerate \( \mathcal{F}' - \mathcal{F} \) as \( \{H_\alpha : \alpha < \kappa\} \). For \( \alpha < \kappa \) let \( (A_\alpha, \mathcal{F}_\alpha) \) be the smallest pair such that

(o) \( A \subseteq A_\alpha, \mathcal{F} \subseteq \mathcal{F}_\alpha\),

(\beta) \( H_\beta \in \mathcal{F}_\alpha (\beta < \alpha)\),

(\gamma) if \( H \in \mathcal{F}_\alpha - \mathcal{F} \), then \( H \subseteq A_\alpha\),

(\delta) if \( H \in \mathcal{F}' - \mathcal{F}, |H \cap (A_\alpha - A)| \geq r + 1\), then \( H \in \mathcal{F}_\alpha\).
A Skolem-type argument gives that each \((A_\alpha, F_\alpha)\) uniquely exists and is closed, and \((A_\alpha, F_\alpha) \leq (A_{\alpha+1}, F_{\alpha+1})\) for \(\alpha < \kappa\). (b) follows from the way the sequences are defined. The inequality \(|A_\alpha - A|, |F_\alpha - F| \leq |\alpha| + \mu\) is immediate as \(A_\alpha - A, F_\alpha - F\) are obtained in at most \(|\alpha| + \mu\) steps.

In order to show that \((A_\alpha, F_\alpha)\) is closed, assume that \(H \in F_\alpha\). If \(H \in F\), then \(H \subseteq A\) as \((A, F)\) is closed. If \(H \in F_\alpha - F\), then \(H \subseteq A_\alpha\) by \((\gamma)\) above.

Finally, assume that \(H \in H - F_\alpha\). Then we have \(|H - A| < \mu\) as \((A, F)\) is closed and \(|H \cap (A_\alpha - A)| \leq r\) by \((\delta)\) above. Adding them up one has \(|A_\alpha \cap H| < \mu\) and so \((A_\alpha, F_\alpha)\) is closed.

Claim 4. If \((A, F) \leq (A', F')\) and < is a well ordering of \(F\) satisfying (2) of the Theorem, then < can be end extended to a well ordering <' of \(F'\) satisfying (2) of the Theorem.

Proof. By induction on \(\kappa = |F' - F|\).

Case 1. \(\kappa \leq \mu\).

Enumerate \(F' - F\) as \(\{H_\alpha : \alpha < \kappa\}\). Set \(H_\alpha <' H_\beta\) if \(\alpha < \beta\) and, of course, \(H <' H_\alpha\) for \(H \in F\).

In order to show (2) for \(H_\alpha\), observe that

\[
H_\alpha \cap (\{H : H <' H_\alpha\}) \subseteq \left( H_\alpha \cap \bigcup F \right) \cup \bigcup \{H_\alpha \cap H_\beta : \beta < \alpha\},
\]

and here \(|H_\alpha \cap \bigcup F| \leq |H_\alpha \cap A| < \mu\) as \((A, F)\) is closed, while \(|\bigcup \{H_\alpha \cap H_\beta : \beta < \alpha\}\) is the union of \(|\alpha|\) sets, each of size \(\leq r\), so it also has size \(< \mu\).

Case 2. \(\kappa > \mu\).

Let \(\{(A_\alpha, F_\alpha) : \alpha \leq \kappa\}\) be a sequence of closed sets given by Claim 3. By recursion on \(\alpha < \kappa\) we define a well ordering <\(\alpha\) of \(F_\alpha\) as follows. \(<_0 = <\). If \(\alpha\) is limit, then <\(\alpha\) = \(\bigcup\{<\beta : \beta < \alpha\}\). Let <\(\alpha+1\) be a well ordering of \(F_{\alpha+1}\) end-extending the well ordering <\(\alpha\) of \(F_\alpha\) satisfying (2). Finally, set <\('\) = <\(\kappa\). Notice that the union of an end extending sequence of well orders is a well order, so the argument cannot break down in a limit step.

By applying Claim 4 for \((A, F) = (\emptyset, \emptyset)\) and \((A', F') = (S, H)\) we are finished.

(2) \(\rightarrow\) (3) Obvious.

(3) \(\rightarrow\) (4) By (3) \(H\) can be enumerated as \(H = \{H_\alpha : \alpha < \varphi\}\) such that \(H_\alpha = H_\alpha - \bigcup \{H_\beta : \beta < \alpha\}\) is nonempty for \(\alpha < \varphi\). Pick \(y_\alpha \in H_\alpha\) \((\alpha < \varphi)\).
Let < be a well ordering of $S$ such that $H_0^* < \cdots < H_\alpha^*$, i.e., each $H_\alpha^*$ is an interval in $(S,<)$ with $y_\alpha$ as its maximal element. Then < is as required.

(4) $\rightarrow$ (3) Let < be a well ordering of $S$ as in (4). Define the ordering $<$ by $H < H'$ iff $\max(H) < \max(H')$. It is easy to see that < is as required.

(4) $\rightarrow$ (5) Obvious.

(5) $\rightarrow$ (6) (Klimó, [5]) Let < be a well order of $S$ as in (5). We construct $M \subseteq S$ by recursively determining if $x \in M$. Assume we have reached $x \in S$ and $\{y \in M : y < x\}$ is already determined. We add $x$ to $M$ iff there is $H \in \mathcal{H}$ such that $\max(H) = x$ and $(H - \{x\})$ is disjoint from $\{y \in M : y < x\}$.

Claim 5. $M$ is covering.

Proof. Let $H \in \mathcal{H}$ be arbitrary, $x = \max(H)$. If $(H - \{x\}) \cap M \neq \emptyset$, then surely $H \cap M \neq \emptyset$. If, however, $(H - \{x\}) \cap M = \emptyset$, then by the construction $x \in M$, so again $H \cap M \neq \emptyset$ holds.

Claim 6. If $x \in M$, then $M' = M - \{x\}$ is not covering.

Proof. Indeed, as $x \in M$, there is some $H \in \mathcal{H}$ with $\max(H) = x$ and $H \cap \{y < x : y \in M\} = \emptyset$, so $H \cap M' = \emptyset$.

We are finished with the proofs of the implications.

Corollary 8. If $S$ is a set, $r < \omega$, $\mathcal{H} \subseteq [S]^\mu$ is a system such that $|A \cap B| \leq r$ for $A \neq B \in \mathcal{H}$, then $\mathcal{H}$ has a minimal covering set.

First Proof. By (1) $\rightarrow$ (6) of Theorem 4.

Second Proof. By Theorem 8(b) in [2], there is a set $T \subseteq S$ such that $0 < |T \cap A| < \omega$ for $A \in \mathcal{H}$, which, by Lemma 3, implies the result.

Corollary 9. If $S$ is a set, $\mu$ an infinite cardinal, $r < \omega$, $\mathcal{H} \subseteq [S]^\omega \cup [S]^\mu$ is a system such that $|A \cap B| \leq r$ for $A \neq B \in \mathcal{H}$, then $\mathcal{H}$ has a minimal covering set.

Proof. Set $\mathcal{H}' = \{A \in \mathcal{H} : |A| = \mu\}$. By Theorem 7 (5) there is a well ordering $<$ of $S$ such that each $A \in \mathcal{H}'$ has a largest element under $<$. As each finite set has a largest element under $<$, this holds for each member of $\mathcal{H}$, as well. Applying (5)$\rightarrow$(6) we obtain that $\mathcal{H}$ has a minimal covering set.

Theorem 10. If $r < \omega$, $\mathcal{H} \subseteq \mathcal{P}(S)$ is a set system such that $|A \cap B| \leq r$ for $A \neq B \in \mathcal{H}$, $T \subseteq S$ is such that $1 \leq |A \cap T| \leq \omega$ for $A \in \mathcal{H}$, then $\mathcal{H}$ has a minimal covering set.

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Proof. Set $\mathcal{H} = \{H_i : i \in I\}$, $H'_i = H_i \cap T$ ($i \in I$). As $|H_i \cap H_j| \leq r$ ($i \neq j$), by Corollary 9, there is a minimal covering set $U \subseteq T$ for $\{H'_i : i \in I\}$. $U$ is a minimal covering set for $\mathcal{H}$: if $i \in I$, then $\emptyset \neq U \cap H'_i \subseteq T \cap H_i$ and if $x \in T$, there is $i \in I$ such that $H'_i \cap (T - \{x\}) = \emptyset$ and so $H_i \cap (T - \{x\}) = \emptyset$.

Lemma 11. There is a nonuniform system $\mathcal{H}$ on some set $S$ with $|A \cap B| \leq 1$ ($A \neq B \in \mathcal{H}$) which has no set $T \subseteq S$ with $1 \leq |T \cap A| < \omega$ ($A \in \mathcal{H}$).

Proof. Set $S = \omega \times \omega_1$, and $\mathcal{H} = \{H_i : i < \omega\} \cup \{Z_\xi : \xi < \omega_1\}$ where $H_i = \{i\} \times \omega_1$ and $Z_\xi = \omega \times \{\xi\}$. This is ST($K_{\aleph_0, \aleph_1}$) from Theorem 3.

As $H_i \cap H_j = Z_\xi \cap Z_\eta = \emptyset$ ($i \neq j$, $\xi \neq \eta$) and $H_i \cap Z_\xi = \{i, \xi\}$, we have that $|A \cap B| \leq 1$ for $A \neq B \in \mathcal{H}$.

Assume now that $T \subseteq S$ satisfies $1 \leq |A \cap T| < \omega$ for each $A \in \mathcal{H}$. Then for each $\xi < \omega_1$ there are $i(\xi) < \omega$ such that $\langle i(\xi), \xi \rangle \in T$ (as $T \cap Z_\xi \neq \emptyset$). There is $R \in [\omega_1]^{\omega_1}$ with $i(\xi) = i$ for $\xi \in R$, and then $|T \cap H_i| = \aleph_1$, a contradiction.

Next we investigate systems of countably infinite sets with pairwise finite intersection but with no fixed bound on the size of the intersection.

Lemma 12. If $\langle A_\alpha : \alpha < \omega_1, \text{limit} \rangle$ is a ♠-sequence, then $\mathcal{H} = \{A_\alpha : \alpha < \omega_1, \text{limit} \}$ is almost disjoint and has no minimal covering set.

Proof. It is obvious that $\mathcal{H}$ is almost disjoint.

For the other statement, let $M \subseteq \omega_1$ be a covering set. If $|\omega_1 - M| = \aleph_1$, then by the ♠ property, some $A_\alpha$ has $A_\alpha \subseteq \omega_1 - M$, so $M$ is not covering. Therefore, $[\gamma, \omega_1] \subseteq M$ for some $\gamma < \omega_1$. Pick $\beta$ with $\gamma < \beta < \omega_1$, $M' = M - \{\beta\}$. If $\alpha \leq \beta$, then $A_\alpha \cap M' = A_\alpha \cap M \neq \emptyset$. If $\beta < \alpha < \omega_1$, then $M'$ contains a cofinal part of $A_\alpha$, so $M' \cap A_\alpha \neq \emptyset$. This gives that $M'$ is covering and so $M$ is not minimal.

Lemma 13. Let $S \subseteq \omega_1$ be a nonstationary set of limit ordinals, $H_\xi \subseteq \xi$ a cofinal set of type $\omega$ for $\xi \in S$, then $\mathcal{H} = \{H_\xi : \xi \in S\}$ has a minimal covering set.

Proof. As $S$ is nonstationary, there is a closed unbounded set $C$ with $0 \in C$, $C \cap S = \emptyset$. Let $C = \{\gamma^\alpha : \alpha < \omega_1\}$ be the increasing enumeration of $C$. Enumerate $\{\gamma^\alpha < \xi < \gamma^{\alpha+1} : \xi \in S\}$ as $\{\delta^\alpha_n : n < \omega\}$.

Let $<$ be the following well ordering of $\mathcal{H}$. If $H_\xi, H_\eta \in \mathcal{H}$, $\xi = \delta^\alpha_n$, $\delta^\beta_m$, then $H_\xi < H_\eta$ if either $\alpha < \beta$ or $\alpha = \beta$ and $n < m$. Clearly, $<$ satisfies (b) of Theorem 7, which implies that $\mathcal{H}$ has a minimal covering set. □
Theorem 14. (MA$_\omega$) If $\mathcal{H} = \{H_\xi : \xi \in S\}$ where $S \subseteq \{\xi < \omega_1 : \text{limit}\}$, $H_\xi$ is a cofinal subset of $\xi$ of type $\omega$, then $\mathcal{H}$ has a minimal covering set.

Proof. Let $\mathcal{H} = \{H_\xi : \xi \in S\}$ be as in the Theorem.

We define the following poset $(P, \leq)$. $(s, a) \in P$ if $s \in [S]^{<\omega}$, $a \in [\omega_1]^{<\omega}$, and $H_\xi \cap a \neq \emptyset$ ($\xi \in s$). $(s', a') \leq (s, a)$ iff $s' \supseteq s$, $a' \supseteq a$, and $(a' - a) \cap H_\xi = \emptyset$ ($\xi \in s$).

Claim 1. $(P, \leq)$ is a notion of forcing.

Proof. As $\leq$ is clearly transitive.

Claim 2. $(P, \leq)$ is ccc.

Proof. Assume that $p_\alpha \in P$ ($\alpha < \omega_1$). By appropriately thinning, we can assume that $p_\alpha = (s \cup s_\alpha, a \cup a_\alpha)$ where $s < s_0 < \cdots < s_\alpha < \cdots$ and $a < a_0 < \cdots < a_\alpha < \cdots$ and further, if $\xi \in s_\alpha$, then $H_\xi < a_\beta$ ($\alpha < \beta$).

In order to find $\alpha < \beta$ such that $p_\alpha = (s \cup s_\alpha, a \cup a_\alpha)$ and $p_\beta = (s \cup s_\beta, a \cup a_\beta)$ are compatible we need to ensure

(a) $H_\xi \cap s_\beta = \emptyset$ ($\xi \in a_\alpha$) and
(b) $H_\xi \cap s_\alpha = \emptyset$ ($\xi \in a_\beta$).

By our above assumptions, (a) holds for any $\alpha < \beta$, so we have to concentrate on (b). If $\xi \in s_{\omega+1}$, then $\sup\{s_n : n < \omega\} < \xi$, therefore $a_\alpha \cap H_\xi = \emptyset$ for $n < \omega$ sufficiently large. Consequently, if $n < \omega$ is enough large, then (b) holds.

Claim 3. $D_\xi = \{(s, a) : \xi \in s\}$ is dense in $(P, \leq)$ ($\xi \in S$).

Proof. Assume that $(s, a) \in P$, $\xi \notin s$. Pick $x \in H_\xi - \bigcup\{H_\eta : \eta \in s\}$ (possible, as $\mathcal{H}$ is almost disjoint). Then $(s', a') \leq (s, a)$ and $(s', a') \in D_\xi$, where $s' = s \cup \{\xi\}$, $a' = a \cup \{x\}$.

By MA$_\omega$, there is a filter $G \subseteq P$ with $G \cap D_\xi \neq \emptyset$ for $\xi \in S$.

Set $A = \bigcup\{a : (s, a) \in G\}$.

Claim 4. $0 < |A \cap H_\xi| < \omega$ for $\xi \in S$.

Proof. There is $(s, a) \in G \cap D_\xi$, hence $A \cap H_\xi \neq \emptyset$. Further, if $(s', a') \in G$, then there is $(s'', a'') \in G$ with $(s'', a'') \leq (s', a')$, $(s, a)$, and so $a' \cap H_\xi \subseteq a'' \cap H_\xi = a \cap H_\xi$, therefore $A \cap H_\xi = a \cap H_\xi$ which is finite.

By Lemma 6 and Claim 4 we can conclude the proof of the Theorem.

For general almost disjoint system there is always a counterexample.
Theorem 15. There is an almost disjoint system of countable sets with no minimal covering set.

Proof. We use a construction of [6] which is quickly described as follows. Set $A = \{ A \in [\mathbb{R}]^{\aleph_0} : |A| = c \}$, here $\overline{A}$ denotes the topological closure of $A$. Clearly, $|A| = c$ and each $B \subseteq \mathbb{R}$ with $|B| = c$ contains a member of $A$. Pick a limit point $r(A)$ of $A$ such that $r(A) \neq r(B)$ ($A \neq B$) and let $H(A) \subseteq A$ be a sequence converging to $r(A)$.

We claim that the family $H = \{ H(A) : A \in A \}$ is as required.

First, if $A \neq A'$, then $H(A) \cap H(A')$ is finite, as $H(A)$ is a sequence converging to $r(A)$, $H(A')$ is one converging to $r(A')$, and $r(A) \neq r(A')$.

Second, assume indirectly that $M \subseteq \mathbb{R}$ is a minimal covering set for $H$. Set $B = \mathbb{R} - M$. Now $B$ is countable (as otherwise $H \subseteq B$ holds for some $H \in H$).

As $M$ is a minimal covering set, for every $x \in M$ there is $H_x \in H$ such that $x \in H_x$ and $H_x - \{ x \} \subseteq B$. In other words, $H_x \cap B = H_x - \{ x \}$, $H_x \cap M = \{ x \}$. This implies that $H_x \neq H_y$ for $x \neq y$. As there are continuum many elements $x \in M$, there are continuum many sequences $H_x - \{ x \} \subseteq B$ necessarily converging to different reals, i.e., $|B| = c$, a contradiction.

Our last topic is when a system of $\mu$ element sets is considered with pairwise intersection of size $< \tau$ where $\tau < \mu$ are infinite cardinals. Here we utilize the technique of [7]. The main difference of the problem treated in [7] and the one considered here is that there one could shrink all hyperedges of the system to a subset of the same size with no harm, while that operation can make harm to the property investigated here (the existence of a minimal covering set).

Theorem 16. Assume GCH. Let $\mu$ be an uncountable regular cardinal, $\omega \leq \tau < \mu$, and assume $\Box_\lambda$ for each singular cardinal $\lambda$ with $\text{cf}(\lambda) \leq \tau$. Then, if $H \subseteq [S]^{\omega}$, $|A \cap B| < \tau$ for $A \neq B \in H$, then there is a minimal covering set for $H$.

Proof. By Theorem 7, it suffices to show that $H$ has a well ordering with $|H \cap \bigcup \{ H' \in H : H' < H \}| < \mu$ for every $H \in H$. This we prove by induction on $|H|$.

Case 1. $|H| \leq \mu$.

Enumerate $H$ as $H = \{ H_\alpha : \alpha < \varphi \}$ for some $\varphi \leq \mu$. The ordering $H_\beta < H_\alpha$ iff $\beta < \alpha$ is as required: $|H_\alpha \cap \bigcup \{ H_\beta : \beta < \alpha \}| \leq |\alpha| \tau < \mu$ for $\alpha < \varphi$. 

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Case 2. $\kappa = |\mathcal{H}| > \mu$ is $\tau$-inaccessible, i.e., $\lambda < \kappa$ implies $\lambda^+ < \kappa$.

Claim 1. If $S' \subseteq S$, $\omega \leq |S'| < |S|$, then there is $S' \subseteq S'' \subseteq S$, $|S''| \leq |S'|^\tau \mu$, such that if $H \in \mathcal{H}$, $H \not\subseteq S''$, then $|H \cap S''| < \tau$.

Proof. Recursively on $H$.

Otherwise, if $\alpha < \kappa$.

Claim 2.

Using Claim 1, we construct an increasing, continuous decomposition of $S$ as $S = \bigcup\{S_\alpha : \alpha < \text{cf}(\kappa)\}$, such that $S_0 = \emptyset$, $|S_\alpha|^\tau < |S|$ and if $\alpha = \beta + 1$, $H \not\subseteq S_\alpha$, then $|H \cap S_\alpha| < \tau$.

Claim 2. If $\alpha$ is limit, $H \not\subseteq S_\alpha$, then $|H \cap S_\alpha| < \mu$.

Proof. Otherwise, $|H \cap S_\alpha| = \mu$. Let $\alpha' \leq \alpha$ be minimal such that $|H \cap S_{\alpha'}| = \mu$. Then $\alpha'$ is limit with $\text{cf}(\alpha') = \mu$, consequently there is $\beta < \alpha'$ with $|H \cap S_\beta| \geq \tau$ and then $H \subseteq S_{\beta+1} \subseteq S' \subseteq S_\alpha$, a contradiction.

Define $\mathcal{H}_\alpha = \{H \in \mathcal{H} : H \subseteq S_\alpha\}$ for $\alpha < \text{cf}(\kappa)$. Notice that $\mathcal{H}_0 = \emptyset$.

Further, if $\alpha < \text{cf}(\kappa)$ is limit, then $\mathcal{H}_\alpha = \bigcup\{\mathcal{H}_\beta : \beta < \alpha\}$: assume that $H \in \mathcal{H}_\alpha$ but $H \not\in \mathcal{H}_\beta$ for $\beta < \alpha$. Then $|H \cap S_{\beta+1}| < \tau$ for $\beta < \alpha$ and then $H \subseteq S_\alpha$ is only possible if $\tau = \mu = \text{cf}(\alpha)$, but these contradict our assumptions.

Now $\mathcal{H}$ can be partitioned as $\mathcal{H} = \bigcup\{\mathcal{H}_{\alpha+1} - \mathcal{H}_\alpha : \alpha < \text{cf}(\kappa)\}$ and as $|\mathcal{H}_{\alpha+1} - \mathcal{H}_\alpha| \leq |\mathcal{H}_{\alpha+1}| \leq |S_{\alpha+1}|^\tau < \kappa$, each $\mathcal{H}_{\alpha+1} - \mathcal{H}_\alpha$ has a well ordering $<_\alpha$ as required. Define the well ordering $<_\mathcal{H}$ of $\mathcal{H}$ as follows: $H < H'$ for $H \in \mathcal{H}_\beta$, $H' \in \mathcal{H}_\alpha$ iff either $\beta < \alpha$ or $\beta = \alpha$ and $H <_\alpha H'$. This is good as for $H \in \mathcal{H}_{\alpha+1} - \mathcal{H}_\alpha$ we have $|H \cap \bigcup\mathcal{H}_\alpha| \leq |H \cap S_\alpha| < \mu$ by Claim 2.

Case 3. $|\mathcal{H}| = \lambda^+$ where $\lambda$ is singular and $\text{cf}(\lambda) \leq \tau$.

We assume that $S = \lambda^+$, i.e., $\mathcal{H} \subseteq [\lambda^+]^\mu$.

Fix an increasing sequence $\langle \lambda_\xi : \xi < \text{cf}(\lambda) \rangle$ such that $\sup\{\lambda_\xi : \xi < \text{cf}(\lambda)\} = \lambda$.

By the $\square_\lambda$ principle we have a sequence $\langle C_\alpha : \alpha < \lambda^+, \text{ limit}\rangle$ such that

1. $C_\alpha$ is closed, unbounded in $\alpha$;
2. $\text{tp}(C_\alpha) < \lambda$;
3. if $\beta \in C_\alpha$, then $C_\beta = C_\alpha \cap \beta$.
Let the increasing enumeration of $C_\alpha$ be $C_\alpha = \{ \gamma^\alpha_\xi : \xi < \text{tp}(C_\alpha) \}_<$, we can assume that $\gamma^\alpha_0 = 0$.

Fix an increasing decomposition $[\beta, \alpha) = \bigcup\{T(\beta, \alpha, \xi) : \xi < \text{cf}(\lambda)\}$ with $|T(\beta, \alpha, \xi)| \leq \lambda_\xi$ for $\beta < \alpha < \lambda^+$. Define

$$T(\alpha, \xi) = \bigcup\{T(\gamma^\alpha_\eta, \gamma^\alpha_{\eta+1}, \xi) : \eta < \text{tp}(C_\alpha)\}.$$ 

Notice that $\alpha = \bigcup\{T(\alpha, \xi) : \xi < \text{cf}(\lambda)\}$, an increasing union, and $|T(\alpha, \xi)| \leq \lambda_\xi \text{tp}(C_\alpha) < \lambda$.

Claim 3. If $\beta \in C'_\alpha$, $\xi < \text{cf}(\lambda)$, then $T(\beta, \xi) = T(\alpha, \xi) \cap \beta$.

Proof. As $C_\beta = C_\alpha \cap \beta$. \square

Claim 4. $|[T(\alpha, \xi)]^\tau| < \lambda$ $(\xi < \text{cf}(\lambda), \alpha < \lambda^+)$. 

Proof. As $|T(\alpha, \xi)| < \lambda$ and $\lambda$ is strong limit singular, therefore $\tau$-inaccessible. \square

Assume now that $\alpha < \lambda^+$ and $\mathcal{H}' = \{ H \in \mathcal{H} : H \subseteq \alpha \}$.

Claim 5. $|\mathcal{H}'| \leq \lambda$.

Proof. Decompose $\alpha$ as $\alpha = \bigcup\{A_\xi : \xi < \text{cf}(\lambda)\}$, $|A_\xi| = \lambda_\xi$. If $H \subseteq \alpha$, then $|H \cap A_\xi| = \mu$ for some $\xi < \text{cf}(\lambda)$, and as $|H \cap H' < \tau$ for $H \neq H' \in \mathcal{H}'$, there are at most $\lambda_\xi^\alpha < \lambda$ such $H \in \mathcal{H}'$ for a given $\xi$. Altogether, this gives $|\mathcal{H}'| \leq \sum\{\lambda_\xi^\alpha : \xi < \text{cf}(\lambda)\} = \lambda$. \square

If $\beta < \lambda^+$, $B \subseteq T(\beta, \xi)$ for some $\xi < \text{cf}(\lambda)$ then we call $B$ low in $\beta$. The number of low subsets $B \subseteq \beta$ with $|B| = \tau$ is $\lambda$ by Claim 4. For each such $B$ there can be at most one $H \in \mathcal{H}$ with $B \subseteq H$, so we can devise a function $g : \lambda^+ \to \lambda^+$ such that if $B \subseteq \beta$, $|B| = \tau$ is low in $\beta$, $B \subseteq H \in \mathcal{H}$, then $H \subseteq g(\beta)$.

Set $D = \{ \delta < \lambda^+ : \beta < \delta \to g(\beta) < \delta \}$, a closed, unbounded set in $\lambda^+$.

Claim 6. If $\delta \in D$, $H \in \mathcal{H}$, $H \nsubseteq \delta$, then $|H \cap \delta| < \mu$.

Proof. Assume indirectly, that $|H \cap \delta| = \mu$. Let $\delta' \leq \delta$ be minimal such that $|H \cap \delta'| = \mu$, i.e., the supremum of the first $\mu$ elements of $H$. Clearly, $\text{cf}(\delta') = \mu$. As $\text{cf}(\mu) > \text{cf}(\lambda)$, there is a $B \subseteq H \cap \delta'$ which is low in $\delta'$. Pick $B' \subseteq B$, $\text{tp}(B') = \tau$. Pick $B' < \beta \in C'_\delta$, then by Claim 3 $B'$ is low in $\beta$, and so $H \subseteq g(\beta) < \delta$, a contradiction. \square

We now conclude as in Case 2: enumerate $D$ as $\{ \delta_\alpha : \alpha < \lambda^+ \}$ with $\delta_0 = 0$. Set $\mathcal{H}_\alpha = \{ H \in \mathcal{H} : H \subseteq \delta_\alpha \}$. Then each $\mathcal{H}_\alpha$ has $|\mathcal{H}_\alpha| < \lambda^+$ and $\mathcal{H}$ is
partitioned as $\mathcal{H} = \bigcup \{ \mathcal{H}_{\alpha+1} - \mathcal{H}_\alpha : \alpha < \lambda^+ \}$. By the induction hypothesis, each $\mathcal{H}_{\alpha+1} - \mathcal{H}_\alpha$ has a well ordering $<_\alpha$ as in (b), which gives a well ordering $<_\alpha$ of $\mathcal{H}$ as follows: $H < H'$ for $H \in \mathcal{H}_\beta, H' \in \mathcal{H}_\alpha$ if either $\beta < \alpha$ or $\beta = \alpha$ and $H <_\alpha H'$. It is easy to see that $<_\alpha$ is as required. □

**Theorem 17.** If the existence of a supercompact cardinal is consistent, then it is consistent that there is a family $\mathcal{H} \subseteq [\omega_{\omega+1}]^{<\omega}$ with pairwise finite intersection such that $\mathcal{H}$ has no minimal covering set.

**Proof.** It is consistent that GCH holds and there is a stationary set $S \subseteq \{ \alpha < \omega_{\omega+1} : \text{cf}(\alpha) = \omega_1 \}$ and there are $\{ H_\alpha : \alpha \in S \}$ such that $H_\alpha$ is a set of order type $\omega_1$, cofinal in $\alpha$ ($\alpha \in S$) and $|H_\alpha \cap H_\beta| < \omega$ ($\alpha \neq \beta \in S$) (see [4]).

By [8] there is a $\diamond$-sequence $\{ S_\alpha : \alpha \in S \}$. Set $S^* = \{ \alpha \in S : \text{tp}(S_\alpha) = \alpha \}$. Notice that if $X \subseteq \omega_{\omega+1}$, $|X| = \omega_{\omega+1}$, then $\{ \alpha \in S^* : S_\alpha = X \cap \alpha \}$ is stationary.

Let the increasing enumeration of $S_\alpha$ be $S_\alpha = \{ x^\alpha(\xi) : \xi < \alpha \}$ and that of $H_\alpha$ be $H_\alpha = \{ y^\alpha_\xi : \xi < \omega_1 \}$. If $\alpha \in S^*$, set $H^*_\alpha = \{ x^\alpha(\xi) : \xi < \omega_1 \} \subseteq S_\alpha$.

Define the set $S' \subseteq S^*$ by recursively determining if $\alpha \in S'$: we set $\alpha \in S'$ iff $\alpha \in S^*$ and $|H^*_\alpha \cap H^*_\beta| < \omega$ for all $\beta \in \alpha \cap S'$.

**Claim 1.** If $X \subseteq \omega_{\omega+1}$, $|X| = \omega_{\omega+1}$, then $\{ \alpha \in S' : S_\alpha = X \cap \alpha \}$ is stationary.

**Proof.** Assume indirectly that $X \subseteq \omega_{\omega+1}$, $|X| = \omega_{\omega+1}$ and

$$\{ \alpha \in S' : S_\alpha = X \cap \alpha \}$$

is nonstationary. As $\{ \alpha \in S' : S_\alpha = X \cap \alpha \}$ is stationary, there is a stationary set $\mathcal{S} \subseteq S^*$ such that if $\alpha \in \mathcal{S}$, then $S_\alpha = X \cap \alpha$ and there is $\beta(\alpha) < \alpha$ such that $|H^*_\alpha \cap H^*_\beta(\alpha)| \geq \omega$.

By repeatedly shrinking, we can find a stationary set $S'' \subseteq \mathcal{S}$ such that for $\alpha \in S''$ we have

(a) $\beta(\alpha) = \beta$ (by Fodor’s theorem),

(b) $\{ \xi < \omega_1 : x^\alpha(\xi) \in H^*_\beta \} = Z$ for some $Z \in [\omega_1]^{<\omega}$, (as $2^{\aleph_0} < \aleph_{\omega+1}$), and

(c) $x^\alpha(\xi) = x^\beta(\eta(\xi)) = z_\xi$ for $\xi \in Z$ and some $\eta(\xi)$ (as $2^{\aleph_0} < \aleph_{\omega+1}$, i.e., there are $2^{\aleph_0} = \aleph_1$ possibilities for the sequence $\langle \eta(\xi) : \xi \in Z \rangle$).

But then $\eta_\xi^\alpha = \text{tp}(X \cap z_\xi)$ for $\xi \in Z$, $\alpha \in S''$, so $y_\xi^\alpha = y_\xi^\alpha$ holds for $\alpha, \alpha' \in S''$, that is, $|H_\alpha \cap H_{\alpha'}| \geq \omega$, a contradiction. □

**Claim 2.** The system $\{ H^*_\alpha : \alpha \in S' \}$ has no minimal covering set.

**Proof.** Let $M \subseteq \omega_{\omega+1}$ be a minimal covering set.
Subclaim. \(|\omega_{\omega+1} - M| < \aleph_{\omega+1}\).

Proof. Otherwise \(X = \omega_{\omega+1} - M\) is cofinal in \(\omega_{\omega+1}\) and by Claim 1, \(H'_\alpha \subseteq X\) for some \(\alpha \in S',\) so \(M\) is not covering. □

By the Subclaim \(|\gamma, \omega_{\omega+1}| \subseteq M\) for some \(\gamma < \omega_{\omega+1}\). Pick \(\beta\) with \(\gamma < \beta < \omega_{\omega+1}\), \(M' = M - \{\beta\}\). If \(\alpha \leq \beta\), then \(H'_\alpha \cap M' = H'_\alpha \cap M \neq \emptyset\). If \(\beta < \alpha < \omega_{\omega+1}\), then \(M'\) contains a cofinal part of \(H'_\alpha\), so \(M' \cap H'_\alpha \neq \emptyset\). This gives that \(M'\) is covering and so \(M\) is not minimal. □

By Claim 2, the proof of the Theorem concluded. □

References


[9] Dominic van der Zypen: Minimal covering sets in families of sets intersecting in at most 1 point, https://mathoverflow.net/questions/277856