§0. Introduction. It was J. E. Baumgartner who in [1] proved that when a weakly compact cardinal is Lévy-collapsed to $\omega_2$ the new $\omega_2$ inherits some of the large cardinal properties; e.g. if $S$ is a stationary set of $\omega$-limits in $\omega_2$ then for some $\alpha < \omega_2$, $S \cap \alpha$ is stationary in $\alpha$. Later S. Shelah extended this to the following theorem: if a supercompact cardinal $\kappa$ is Lévy-collapsed to $\omega_2$, then in the resulting model the following holds: if $S \subseteq \lambda$ is a stationary set of $\omega$-limits and $\text{cf}(\lambda) \geq \omega_2$ then there is an $\alpha < \lambda$ such that $S \cap \alpha$ is stationary in $\alpha$, i.e. stationary reflection holds for countable cofinality (see [1] and [3]). These theorems are important prototypes of small cardinal compactness theorems; many applications and generalizations can be found in the literature. One might think that these results are true for sets with an uncountable cofinality $\mu$ as well, i.e. when an appropriate large cardinal is collapsed to $\mu^+$. Though this is true for Baumgartner’s theorem, there remains a problem with Shelah’s result. The point is that the lemma stating that a stationary set of $\omega$-limits remains stationary after forcing with an $\omega_1$-closed partial order may be false in the case of $\mu$-limits in a cardinal of the form $\lambda^+$ with $\text{cf}(\lambda) < \mu$, as was shown in [8] by Shelah. The problem has recently been solved by Baumgartner, who observed that if a universal box-sequence on the class of those ordinals with cofinality $\leq \mu$ exists, the lemma still holds, and a universal box-sequence of the above type can be added without destroying supercompact cardinals beyond $\mu$. Here we give an alternative proof using a weaker box-type sequence on the critical cardinals; it still suffices for the lemma, and is much more easily introduced by forcing. It is worth mentioning that there is an implicit proof in [6] for the following statement: If $\sigma_1 < \sigma_2$ are strongly compact, and first $\sigma_1$ is collapsed to $\omega_1$ and then $\sigma_2$ is collapsed to $\omega_3$, then stationary reflection holds for $\omega_1$-limits.

Our set theory notation is standard. Notice that $P(X)$ denotes the power set of $X$, $[X]^\mu = \{a \subseteq X : |a| = \mu\}$, and $[X]^{<\mu} = \{a \subseteq X : |a| < \mu\}$. In §1 we introduce our version of box-sequence and in §2 we prove the consistency of stationary reflection for uncountable cofinality. In §3 we add these box-sequences to the ground model by forcing, and in §4 we show that supercompact cardinals survive this forcing.
§1. Weak box-sequences.

Definition 1. Assume that $\mu < \tau$ are regular cardinals. $A(\mu, \tau)$ abbreviates the following statement: There exists a sequence $\langle X(\alpha): \alpha < \tau, \alpha \text{ limit, } \text{cf}(\alpha) < \mu \rangle$, with the following properties:

(1.1) $X(\alpha) \subseteq P(\alpha)$, and $|X(\alpha)| < \tau$.
(1.2) If $H \in X(\alpha)$, then $H$ is closed and unbounded in $\alpha$ ($\text{cf}(\alpha) = \omega$ is allowed).
(1.3) If $H \in X(\alpha)$ and $\text{cf}(\alpha) < \mu$ then $\text{tp}(H) < \mu$.
(1.4) If $\text{cf}(\alpha) = \mu$ then $X(\alpha)$ is nonempty.
(1.5) If $H \in X(\alpha)$ and $\beta$ is a limit point of $H$ then $H \subseteq \beta$.

Notice that from (1.3) and (1.5) it follows that if $\text{cf}(\alpha) = \tau$ then every set in $X(\alpha)$ has order-type $\mu$.

Lemma 1. For every $\xi < \tau$, a sequence $\langle X(\alpha): \alpha < \tau, \alpha \text{ limit, } \text{cf}(\alpha) < \mu \rangle$ as described in (1.1)-(1.5) exists.

Proof. Let $T = \{ \alpha < \xi: \text{cf}(\alpha) = \mu \}$. For every $\alpha \in T$ let $H_\alpha$ be a closed unbounded set in $\alpha$ of type $\mu$. Put $X(\alpha) = \{ H_\alpha \}$ for $\alpha < \xi$, $\text{cf}(\alpha) = \mu$, and $X(\beta) = \{ H_\alpha \cap \beta: \alpha < \xi, \text{cf}(\alpha) = \beta, \beta < \alpha \text{ is a limit point of } H_\alpha \}$ for $\beta < \xi$, $\text{cf}(\beta) < \mu$. Clearly $|X(\beta)| < |\xi| < \tau$.

Lemma 2 (GCH). $A(\mu, \tau)$ holds unless $\tau = \lambda^+$ with $\text{cf}(\lambda) < \mu$.

Proof. As $\omega^+ < \tau$ holds for $\xi < \tau$, put all clubs in $\alpha$ with order-type less than $\xi$ into $X(\alpha)$ if $\text{cf}(\alpha) < \mu$, and put one club in $\alpha$ with order-type $\mu$ into $X(\alpha)$ if $\text{cf}(\alpha) = \mu$.

Lemma 3. Assume that $A(\mu, \tau)$ holds, $S \subseteq \{ \xi < \tau: \text{cf}(\xi) = \mu \}$ is a stationary set, and $P$ is a $\mu^+$-closed partial order. Then $S$ remains stationary after forcing with $P$.

Proof. Assume that $1 \equiv p \langle C: C \subseteq \mathcal{S} \cup \mathcal{S} = \varnothing \rangle$. By induction on $\alpha < \tau$ choose $\gamma_\alpha < \tau$ and $\{ p_\alpha: H \in X(\alpha) \}$ such that

(1.6) $p_\alpha \| \langle \gamma_\alpha, \gamma_{\alpha+1} \rangle \neq \emptyset$,
(1.7) if $\beta$ is a limit point of $H$, then $p_\alpha \leq p_{H \cap \beta}$, and
(1.8) $\gamma_\alpha > \gamma_\beta (\alpha < \beta)$.

This can obviously be done, using $A(\mu, \tau)$ and the $\mu^+$-closure property of $P$ (in (1.7)). Put $D = \{ \xi: \alpha < \xi \text{ then } \gamma_\alpha < \xi \}$, a closed unbounded set in $\tau$. As $S$ is stationary, there is a point $\delta$ in $\tau \cap D$. If $H$ is in $X(\delta)$ and $H = \{ h(\xi): \xi < \mu \}$, by (1.7), $p_{H \cap h(\xi)} (\xi < \mu, \text{ limit})$ is a decreasing sequence, and, by (1.6), $p_\alpha$ forces $\delta \in C$, a contradiction.

§2. Stationary reflection for $\mu$. In this section we prove that if $A(\mu, \tau)$ holds for every regular $\tau > \mu$ and $\kappa > \mu$ is a strongly compact cardinal, then, if $\kappa$ is Lévy-collapsed to $\mu^+$, in the resulting model the following holds: If $\text{cf}(\tau) \geq \mu^+$ and $S \subseteq \{ \xi < \tau: \text{cf}(\xi) = \mu \}$ is stationary then there exists an $\alpha < \tau$ with $S \cap \alpha$ stationary in $\alpha$. The case $\text{cf}(\tau) = \mu^+$ is exactly the same as in [1]. To get the other case we first need a lemma.

Lemma 4. If $\kappa > \mu$ is strongly compact, $\lambda$ satisfies $\text{cf}(\lambda) > \kappa$ and $S \subseteq \{ \xi < \lambda: \text{cf}(\xi) = \mu \}$ is stationary then there exists an $\alpha < \lambda$ with $\text{cf}(\alpha) < \kappa$ such that $S \cap \alpha$ is stationary in $\alpha$.

Proof. If $\text{cf}(\lambda) = \kappa$, then, as $\kappa$ is measurable, we can use the ultrapower.

If $\text{cf}(\lambda) > \kappa$, take an elementary embedding associated with the ultrapower, $j: V \rightarrow M$. If is well known and easy to see that $j'' \lambda < j(\lambda)$. As the embedding is monotone and continuous at $\mu$-limits, $j(S) \cap j'' \lambda$ is stationary at $j'' \lambda$, so in $M$ it is true that there is an $\alpha < j(\lambda)$ where $j(S)$ is stationary; so $S$ is stationary at an $\alpha < \lambda$. If $\text{cf}(\alpha) \geq \kappa$
we can repeat the above argument, so there is such an \( \alpha \) with \( \text{cf}(\alpha) < \kappa \). Notice that \( \text{cf}(\alpha) \leq \mu \) is impossible here.

**Theorem 1.** Assume that \( \kappa > \mu \) is strongly compact, and \( A(\mu, \tau) \) holds for every \( \tau < \kappa \). If \( \kappa \) is Lévy-collapsed to \( \mu^{++} \) then in the resulting model stationary reflection holds for sets with cofinality \( \mu \).

**Proof.** Let \( P \) be the collapsing partial order, and assume that \( S \subseteq \{ \xi < \lambda : \text{cf}(\xi) = \mu \} \) is stationary in \( V^P \), \( \text{cf}(\lambda) \geq \kappa \). The \( \text{cf}(\lambda) = \kappa \) case is completely analogous to the one described in [1]. If \( \text{cf}(\lambda) > \kappa \), as \( |P| = \kappa \), \( S \) contains a set \( S' \in V \) which is stationary; therefore \( S' \) is stationary in \( V \) as well, so by Lemma 6 there is an \( \alpha < \lambda, \mu < \text{cf}(\alpha) < \kappa \), such that \( S' \cap \alpha \) is stationary in \( \alpha \) (in \( V \)). As \( A(\mu, \text{cf}(\alpha)) \) holds, \( S' \cap \alpha \) remains stationary in \( V^P \) by Lemma 3.

**§3. The consistency of** \( A(\mu, \tau) \). In this section we define a partial order (actually a class forcing) which simultaneously adds a weak box-sequence to every cardinal. From now on GCH will be assumed. First we give a partial order for one \( A(\mu, \tau) \).

**Definition 2.** Assume \( \tau \geq \mu \) are regular cardinals, \( p \) is an element of \( Q_\tau \) if and only if \( p \) is a function, \( \text{Dom}(p) = \alpha + 1 < \tau \), \( \{ p(\xi) : \xi < \alpha \} \) satisfies the following conditions:

(3.1) \( p(\xi) \subseteq P(\xi) \) and \( |p(\xi)| < \tau \).
(3.2) If \( H \in p(\xi) \), then \( H \) is closed and unbounded in \( \xi \) (\( \text{cf}(\xi) = \omega \) is again allowed).
(3.3) If \( H \in p(\xi) \) and \( \text{cf}(\xi) < \mu \), then \( \text{tp}(H) \leq \mu \).
(3.4) If \( \text{cf}(\xi) = \mu \) then \( p(\xi) \neq \emptyset \).
(3.5) If \( H \in p(\xi) \) and \( \gamma \) is a limit point of \( H \), then \( H \cap \gamma \in p(\gamma) \).

For \( p, q \in Q_\tau \) define \( p \leq q \) if and only if \( q \) is an initial segment of \( p \).

**Lemma 5.** \( Q_\tau \) is \( \tau \)-Baire, i.e., the intersection of less than \( \tau \) open dense sets is dense.

**Proof.** Assume that \( \tau' < \tau \) and \( \{ D_\xi : \xi < \tau' \} \) are dense, open sets. Choose a sequence \( \{ X(\alpha) : \alpha < \tau', \lim \} \) as described in (1.1)-(1.5), existing by Lemma 1. We are going to define a decreasing sequence of conditions \( \{ p_\xi : \xi < \tau' \} \) such that the union \( p_\tau \) satisfies the conclusion: it is in the intersection of our open-dense sets. We will also require that if \( \text{Dom}(p_\xi) = \tau + 1 \), then the sequence \( \{ r(\xi) : \xi < \tau' \} \) must be closed.

Choose \( p_0 \in Q_\tau \) arbitrarily. If \( p_\xi \in Q_\tau \) is already defined, choose \( p_{\xi + 1} < p_\xi \) with \( p_{\xi + 1} \in D_\xi \). If \( \xi \) is limit, choose \( p_\xi < p_\xi (\xi < \xi) \) with \( \text{Dom}(p_\xi) = r(\xi) + 1 \), \( r(\xi) = \sup \{ r(\zeta) : \zeta < \xi \} \) and \( p_\xi (r(\xi)) = \{ r''H : H \in X(\xi) \} \). This uniquely defines \( p_\xi \). It is an element of \( Q_\tau \), for if \( K \in p_\xi (r(\xi)) \) and \( \gamma \) is a limit point of \( K \), then there is an \( H \in X(\xi) \) such that \( K = r''H \), and there is a limit point \( \delta \) of \( H \) such that \( \gamma = r(\delta) \) and \( K \cap \gamma = r''(H \cap \delta) \in p_{r(\delta)} (r(\delta)) = p_r (r(\delta)) \). Therefore the meshing property of \( p_\xi \) is proved.

**Lemma 6.** a) \( Q_\tau \) is \( \mu \)-cosed.

b) \( V^{Q_\tau} \models A(\mu, \tau) \).

**Proof.** a) is trivial.

b) \( \mu \) and \( \tau \) are still regular in \( V^{Q_\tau} \), as by Lemma 5 no new sequence of length < \( \tau \) is introduced in the forcing. As by Lemma 1 for every \( \alpha < \tau \) the set \( D_\alpha = \{ p \in Q_\tau : \text{Dom}(p) > \alpha \} \) is dense, a generic set gives \( A(\mu, \tau) \).

In the next part we define a partial order which adds \( A(\mu, \tau) \) for all \( \tau \) we need it.

**Definition 3.** If \( \alpha, \beta \) are ordinals, \( p \in P^P \) if and only if \( p \) is a function, \( \text{Dom}(p) \subseteq \{ \text{max}(\alpha, \mu) : \beta \} \cap \{ \text{cf}(\tau) < \mu \} \), \( p(\xi) \in Q_\xi \) for \( \xi \in \text{Dom}(p) \). \( p \leq q \) if and only if...
$p(\xi) \leq q(\xi)$ holds for every $\xi \in \text{Dom}(p)$. $P^\alpha_\infty = \bigcup \{P^{\beta_\alpha}_\xi: \beta_\alpha\}$, i.e. the class forcing with set supports. $P_\alpha = P^{\alpha}_\alpha$, and $P = P^0_\infty$.

**Lemma 7.** $P^\beta_\alpha \oplus P^\beta_\gamma \cong P^\beta_\gamma$.

**Proof.** Straightforward.

**Lemma 8.** If $\text{cf}(\beta) \geq \mu$ then $V^{P^\beta_\gamma} \models P^\beta_\gamma$ is $\beta^{+\omega}$-Baire (with $P^\beta_\gamma$ defined within $V$).

**Proof.** First we prove that if $1 \Vdash \forall \gamma \in P^\beta_\gamma \text{ is open, dense}$

then there exists a $q \in P^\beta_\gamma$ such that $1 \Vdash \forall \gamma \in P^\beta_\gamma \exists \tilde{q} \in \bar{D}$.

It is easy to see that $|P^\beta_\gamma| \leq \beta$. By Lemma 2, we can find a sequence $\langle X(\gamma): \gamma < \beta^+, \gamma \text{ limit}\rangle$ witnessing $A(\mu, \beta^+)$. Define, by recursion on $\xi < \beta^+$, a condition $\langle p_\xi, q_\xi \rangle \in P^\beta_\alpha \oplus P^\beta_\gamma$ such that $p_\xi$ is incompatible with $\{p_\xi: \zeta < \xi\}$, $q_\xi < q_\xi$ ($\zeta < \xi$), $p_\xi \Vdash \exists \tilde{q}_\xi \in D$ and for $\xi$ limit, if $\bigcup \{q_\xi: \zeta < \xi\}$ is different from the empty condition at $\varepsilon \in (\beta, \gamma)$ and $r_\xi(\varepsilon) = \bigcup \{\text{Dom}(q_\xi(\varepsilon)): \zeta < \xi\}$, put

$q_\xi(\varepsilon)(r_\xi(\varepsilon)) = \{t(H): H \in X(\xi)\},$

where $t(H) = \{r_\xi(\varepsilon): \zeta \in H\}$.

That $q_\xi$ is an element of $P^\beta_\gamma$ can be proved exactly as in Lemma 5. Notice that if $\varepsilon \in \text{Dom}(q_\xi)$ then $\varepsilon \geq \beta^{+\omega+1}$ as $\text{cf}(\beta) \geq \mu$. As $P^\beta_\alpha$ has the $\beta^+$-chain condition, this process terminates at an ordinal $\xi_0 < \beta^+$, which gives $1 \Vdash \exists \xi_0 < \beta^+$.

Assume that $1 \Vdash \forall \gamma \in P^\gamma_\beta$ is open, dense” for $\eta < \beta^+ n$ ($n < \omega$). Again, fix an $A(\mu, \beta^{+n+1})$-sequence $\langle X(\gamma): \gamma < \beta^{+n+1}, \gamma \text{ limit}\rangle$. Define, for $\eta < \beta^+ n$, by recursion the conditions $q_\eta$ as follows: If $q_\eta$ is defined, put $q_\eta < q_\eta \exists \exists \eta_1 \in D_\eta$. For a limit $\eta < \beta^{+n}$, if $\bigcup \{q_\xi: \zeta < \eta\}$ is different from the empty condition at $\varepsilon \in (\beta, \gamma)$ (again, $\varepsilon \geq \beta^{+\omega+1}$), and

$r_\eta(\varepsilon) = \bigcup \{\text{Dom}(g_\xi(\varepsilon)): \zeta < \eta\},$

put

$q_\eta(\varepsilon)(r_\eta(\varepsilon)) = \{t(H): H \in X(\eta)\}$

with $t(H) = \{r_\xi(\varepsilon): \zeta \in H\}$. As above, $q_{\beta^{+n}} \in P^\beta_\gamma$, and $1 \Vdash q_{\beta^{+n}} \in D_\eta$ for $\eta < \beta^+ n$.

**Lemma 9.** If $\text{cf}(\beta) \geq \mu$ then every subset of $\beta^{+n}$ in $V^P$ is already in $V^{P^\beta_\gamma}$ ($n < \omega$).

**Proof.** This immediately follows from the previous lemma.

Notice that this lemma also gives that though $P$ is a proper class, $V^P$ is a model of ZFC.

**Lemma 10.** $V^P \models \text{GCH}$.

**Proof.** If $\beta \leq \mu$ then $P$ does not force a new subset of $\beta$. If $\text{cf}(\beta) \geq \mu$ then by Lemma 9, $2^{\beta}$ in $V^P$ is at most $\beta^+ = 2^{\beta}$. If $\text{cf}(\beta) < \mu$ then by Lemma 8 no new set of size $\text{cf}(\beta)$ is added at all; therefore $\beta^{\text{cf}(\beta)}$ remains $\beta^+$, so, as for $\tau < \beta$, $\text{cf}(\tau) \geq \mu$, $2^\tau = \tau^+$ holds, $2^\beta = \beta^+$ is still true.

**Lemma 11.** Under forcing with $P$ no cardinal collapses, and $A(\mu, \tau)$ holds for every regular $\tau$.

**Proof.** If $\tau$ is regular in $V$ but $\text{cf}(\tau)$ becomes $\beta < \tau$, then $\text{cf}(\beta) = \beta$ in $V$; so,
as $P = P_{\beta} \oplus P_{\beta}$ by Lemma 8, the length-$\beta$ sequence cofinal in $\tau$ is in $P_{\beta}$ but $P_{\beta}$ is $\beta^+-c.c.$, so $\tau \geq \beta^+$ remains regular. The other claim is clear.

§4. Supercompactness remains.

**Theorem 2.** Assume GCH. If $\kappa > \mu$ is supercompact, it remains supercompact after forcing with $P$.

**Proof.** It is enough to prove that if $\lambda > \kappa$, $\text{cf}(\lambda) < \mu$, there is (in $V^P$) a supercompact ultrafilter on $[\lambda^+]^{<\kappa}$. There is a supercompact ultrafilter on $[\lambda^+]^{<\kappa}$ in $V$; this gives an elementary embedding $i: V \to M$ with $M$ closed under $\lambda^+$-sequences.

$P$ splits as $P_{\lambda^+} \oplus P_{\lambda^+}$. We need to show that there exists a supercompact ultrafilter in $V^P = V^{(P_{\lambda^+} \oplus P_{\lambda^+})}$. But forcing with the second partial order cannot introduce it, so we have to show that if $G \subseteq P_{\lambda^+}$ is generic then in $V[G]$ there is an appropriate ultrafilter. In $M$, $i(P)$ is defined by the same iteration as $P$ in $V$; in particular, $i(P) = P_{\lambda^+} \oplus \bar{P}$, where $\bar{P}$ is not $P_{\lambda^+}$, though $\bar{P}$ is (in $V$) $\lambda^+-$Baire, by the closure property of $M$. As in the usual proofs (see [4] and [7]), we need to show that there exists a master condition $s \in \bar{P}$ such that if $H \subseteq \bar{P}$ is $V[G] - \bar{P}$-generic and $s \in H$ then there is an extension of $i, j: V[G] \to M[G][H]$ in $V[G][H]$. By the above remark the corresponding ultrafilter is in $V[G]$, and we are done.

To do this, as usual we have to show that if $G \subseteq P_{\lambda^+}$ is generic, then the conditions $\{i(p)(\lambda^+, \infty): p \in G\}$ have a common lower bound in $\bar{P}$. This set is an element of $M[G]$, by a well-known lemma (see [4] and [7]). Therefore, if these elements do not have a common lower bound, then there exists a coordinate $\alpha > \lambda^+$ such that the $i(p)(\alpha)$ $(p \in G)$ have no common extension. By the definition of $Q^\alpha$ this is only possible if the union of them (a function) has a domain with cofinality $\mu$. So, there is a collection (in $V[G]$) $\{p_\xi: \xi < \mu\}$ such that $i(p_\xi)(\alpha)$ is cofinal in $\{i(p)(\alpha): p \in G\}$. As no new $\mu$-sequences are introduced (Lemma 8), $\langle p_\xi: \xi < \mu \rangle \in V$. By the truth lemma, there is a $p \in G$ with $p \models \langle p_\xi: \xi < \mu \rangle \subseteq G$, and this is only possible if $p \preceq p_\xi (\xi < \mu)$, i.e., $p$ is a lower bound for $\{p_\xi: \xi < \mu\}$ and so is $i(p)(\alpha)$ for $i(p_\xi)(\alpha)$ $(\xi < \mu)$.

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**References**


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