1 Correction

Martin Dyer points out that the proof of Theorems 6.36–6.40 only works if we allow loops. This gap is in the proof of Case 1 on page 103, where the case \( i = j \) must also be considered to reach the conclusion.

But the theorems remain true without the loops, which follows from the results of the preceding section (6.3. Contractors and connectors). To see this, let us consider three algebras: \( Q_k^{\text{loop}} \), consisting of \( k \)-labeled quantum graphs whose constituents are looped multigraphs; \( Q_k^{\text{multi}} \), consisting of \( k \)-labeled quantum graphs whose constituents are unlooped multigraphs; and \( Q_k^{\text{simp}} \), consisting of \( k \)-labeled quantum graphs whose constituents are simple graphs with nonadjacent labeled nodes. Clearly

\[
Q_k^{\text{loop}} \supseteq Q_k^{\text{multi}} \supseteq Q_k^{\text{simp}},
\]

which implies that

\[
Q_k^{\text{loop}}/H \supseteq Q_k^{\text{multi}}/H \supseteq Q_k^{\text{simp}}/H.
\]

(One needs to verify that if \( x \equiv 0 \pmod{\text{hom}(., H)} \) in \( Q_k^{\text{simp}} \) for some \( x \in Q_k^{\text{simp}} \), then this relation holds in \( Q_k^{\text{loop}} \) as well. This follows since in either algebra, \( x \equiv 0 \pmod{\text{hom}(., H)} \) is equivalent to \( t([x], H) = 0 \).)

We claim that equality holds in this last line. It suffices to show that for any \( k \)-labeled looped multigraph \( F \) we can construct a congruent \( k \)-labeled simple quantum graph \( F' \) whose labeled nodes are nonadjacent in every constituent. Let \( z \) be a contractor for \( H \), and let \( w \) be the 1-labeled quantum graph obtained from the product \( K_2^*z \) by unlabeling node 2. Let us replace every loop in \( F \) by \( w \) (attached at its labeled node, which is then unlabeled unless it was labeled in \( F \)). Since \( w \) has no loops by construction, we get a quantum graph \( F'' \in Q_k^{\text{multi}} \). Replacing every edge in every constituent by a simple connector, we get a quantum graph \( F' \in Q_k^{\text{simp}} \). It is clear that

\[
F \equiv F'' \equiv F' \pmod{\text{hom}(., H)},
\]
which proves that
\[ Q_k^{\text{loop}}/H = Q_k^{\text{multi}}/H = Q_k^{\text{simp}}/H. \] (1)

Now the proof in Section 6.4.1 yields that if \( H \) is a twin-free weighted graph, then the dimension of \( Q_k^{\text{loop}}/H \) is the number of orbits of the automorphism group of \( H \) on the ordered \( k \)-tuples of nodes. By (1), we get the same for the other two algebras above.

In terms of connection matrices, we can again consider three kinds: connection matrices \( M^{\text{loop}}(k, \text{hom}(., H)) \) for looped multigraphs, the submatrices \( M^{\text{multi}}(k, \text{hom}(., H)) \) for unlooped multigraphs, and the submatrices \( M^{\text{simp}}(k, \text{hom}(., H)) \) of these for simple graphs with nonadjacent labeled nodes. The constructions above imply that these three matrices have the same rank for every \( k \).

## 2 Connection with complexity problems

The constructions and arguments concerning graph algebras are quite similar to those in the complexity theory of graph homomorphism problems. In fact, the constructions of connectors and contractors are quite similar in spirit to the constructions of “gadgets” in the complexity literature, and the graph algebras themselves seem to be related to basic techniques in complexity theory like interpolation (see e.g. [1]). It should be very interesting to explore these connections. (I am grateful to Martin Dyer for these remarks.)

### References