1 Introduction

In this paper, the graphs we consider are finite, undirected and have no parallel edges, but they may have loops. A graph parameter is a real valued function defined on graphs, invariant under isomorphisms.

For two graphs $F$ and $G$, let $\text{hom}(F,G)$ denote the number of homomorphisms $F \rightarrow G$, that is, adjacency preserving maps $V(F) \rightarrow V(G)$. The definition can be extended to weighted graphs (when the nodes and edges of $G$ have real weights). In [1] graph parameters of the form $\text{hom}(\cdot, G)$ were characterized, where $G$ is a weighted graph. Several variants of this result have been obtained, characterizing graph parameters $\text{hom}(\cdot, G)$ where all nodeweights of $G$ are 1 [3], such graph parameters defined on multigraphs [2] etc. These characterizations involve certain infinite matrices, called connection matrices, which are required to be positive semidefinite and to satisfy a condition on their rank.

The goal of this paper is to study the dual question, and characterize graph parameters of the form $\text{hom}(F, \cdot)$, where $F$ is an (unweighted) graph. It turns out that reversing the arrows in the category of graphs gives the right hints for the condition, and the characterization involves the dually defined connection matrices.

For two graphs $G$ and $H$, the product $G \times H$ is the graph with node set $V(G) \times V(H)$, two nodes $(u, v)$ and $(u', v')$ being adjacent if and only if $uu' \in E(G)$ and $vv' \in E(H)$. Then

$$\text{hom}(F, G \times H) = \text{hom}(F, G)\text{hom}(F, H).$$

(1)

An $S$-colored graph is a pair $(G, \phi)$, where $G$ is a graph and $\phi : V(G) \rightarrow$
S, where S is a finite set. We call \((G, \phi)\) colored if it is S-colored for some S. We call \(\phi(v)\) the color of \(v\).

The product \((G, \phi) \times (H, \psi)\) of two colored graphs is the colored graph \((J, \vartheta)\), where \(J\) is the subgraph of \(G \times H\) induced by the set of nodes \((u, v)\) with \(\phi(u) = \psi(v)\), and where \(\vartheta(u, v) := \phi(u) (= \psi(v))\).

For two colored graphs \((F, \phi)\) and \((G, \psi)\), a homomorphism \(h : V(F) \rightarrow V(G)\) is color-preserving if \(\phi = \psi h\). Let \(\text{hom}_c((F, \phi), (G, \psi))\) denote the number of color-preserving homomorphisms \(F \rightarrow G\).

It is easy to see that for any three colored graphs \(F, G\) and \(H\),

\[\text{hom}_c(F, G \times H) = \text{hom}_c(F, G) \text{hom}_c(F, H).\] (Equation (1) is the special case where all nodes have color 1.) Moreover, if both \(G\) and \(H\) are \(S\)-colored, then for any uncolored graph \(F\),

\[\text{hom}(F, G \times H) = \sum_{\phi : V(F) \rightarrow S} \text{hom}_c((F, \phi), G) \text{hom}_c((F, \phi), H).\] (Equation (2) implies that for any three colored graphs \(F, G\) and \(H\),

\[\text{hom}_c(F, G \times H) = \text{hom}_c(F, G) \text{hom}_c(F, H).\]

Here \(\text{hom}(F, (G, \phi)) := \text{hom}(F, G)\) for any colored graph \((G, \phi)\). More generally, we extend any graph parameter \(f\) to colored graphs by defining \(f(G, \phi) := f(G)\) for any colored graph \((G, \phi)\).

For every graph parameter \(f\) and \(k \geq 1\), we define an (infinite) matrix \(N(f, k)\) as follows. The rows and columns are indexed by \([k]\)-colored graphs (where \([k] = \{1, \ldots, k\}\)), and the entry in row \(G\) and column \(H\) (where \(G\) and \(H\) are \([k]\)-colored graphs) is \(f(G \times H)\).

Equation (3) implies that for any graph \(F\) and any \(k \geq 1\), the matrix \(N(f, k)\) belonging to \(f = \text{hom}(F, \cdot)\) is positive semidefinite and has rank at most \(|S|^{\|V(F)\|}\). Trivially, it satisfies \(f(\bar{K}_1) = \text{hom}(F, \bar{K}_1) = 1\) (where \(\bar{K}_1\) is a single node with a loop). The main result of this paper is that these properties characterize such graph parameters:

**Theorem 1** Let \(f\) be a graph parameter and \(n \geq 1\). Then \(f = \text{hom}(F, \cdot)\) for some graph \(F\) with at most \(n\) nodes if and only if \(f(\bar{K}_1) = 1\), and for each \(k \geq 1\), the matrix \(N(f, k)\) is positive semidefinite and has rank at most \(k^n\).

The proof will require a development of an algebraic machinery, similar to the one used in [1] (but the details are different). It is important to realize that the conditions imply that

\[f(G \times H) = f(G) f(H)\] (Equation (4) is the special case where all nodes have color 1.)
for any uncolored graphs $G$ and $H$, since $N(f, 1)$ is positive semidefinite and has rank at most 1, and since $f(K_i) = 1$.

## 2 The algebras $\mathcal{A}$ and $\mathcal{A}_S$

The colored graphs form a semigroup under multiplication $\times$. Let $\mathcal{G}$ denote its semigroup algebra (the elements of $\mathcal{G}$ are formal linear combinations of colored graphs with real coefficients, also called *quantum colored graphs*). Let $\mathcal{G}_S$ denote the semigroup algebra of the semigroup of $S$-colored graphs. For each $S$, the complete graphs $K_S$ on $S$ (with a loop at each node) is the unit element of $\mathcal{G}_S$.

By the positive semidefiniteness of $N(f, k)$, the function

$$\langle G, H \rangle := f(G \times H)$$

defines a semidefinite (but not necessarily definite) inner product on $\mathcal{G}$. The set

$$I := \{ g \in \mathcal{G} \mid \langle g, g \rangle = 0 \} = \{ g \in \mathcal{G} \mid \langle g, x \rangle = 0 \text{ for all } x \in \mathcal{G} \}$$

is an ideal in $\mathcal{G}$, and hence the quotient $\mathcal{A} = \mathcal{G}/I$ is a commutative algebra with (definite) inner product. We denote multiplication in $\mathcal{A}$ by concatenation.

It is easy to check that

$$I \cap \mathcal{G}_S = \{ g \in \mathcal{G}_S \mid \langle g, x \rangle = 0 \text{ for all } x \in \mathcal{G}_S \}$$

is an ideal in $\mathcal{G}_S$, and hence the quotient $\mathcal{A}_S = \mathcal{G}_S/(I \cap \mathcal{G}_S)$ is also a commutative algebra with a (definite) inner product. This algebra can be identified with $\mathcal{G}_S/I$ in a natural way. The dimension of $\mathcal{A}_S$ is equal to the rank of the matrix $N(f, |S|)$, hence it is at most $|S|^n$ (in particular, finite).

Note that $1_S = K_S + I$ is the unit element of $\mathcal{A}_S$ and that $\mathcal{A}_S$ is an ideal in $\mathcal{A}$. Moreover, $\mathcal{A}_S \subseteq \mathcal{A}_T$ if $S \subseteq T$. In fact, the stronger relation $\mathcal{A}_S \cap \mathcal{A}_T = \mathcal{A}_S \mathcal{A}_T = \mathcal{A}_{S \cap T}$ holds. To see this, we show that

$$\mathcal{A}_S \cap \mathcal{A}_T \subseteq \mathcal{A}_S \mathcal{A}_T \subseteq \mathcal{A}_{S \cap T} \subseteq \mathcal{A}_S \cap \mathcal{A}_T.$$  

Indeed, if $x \in \mathcal{A}_S \cap \mathcal{A}_T$ then $x = x1_T \in \mathcal{A}_S \mathcal{A}_T$, which proves the first inclusion. If $g \in \mathcal{G}_S$ and $h \in \mathcal{G}_T$, then $(g + I)(h + I) = gh + I \in \mathcal{G}_{S \cap T}$, which proves the second. The third inclusion is trivial.
As the inner product \( \langle \cdot, \cdot \rangle \) satisfies \( \langle xy, z \rangle = \langle x, yz \rangle \) for all \( x, y, z \in \mathcal{A}_S \), \( \mathcal{A}_S \) has a unique orthogonal basis \( \mathcal{M}_S \) consisting of idempotents, called the \textit{basic idempotents} of \( \mathcal{A}_S \). Every idempotent in \( \mathcal{A}_S \) is the sum of a subset of \( \mathcal{M}_S \), and in particular
\[
1_S = \sum_{p \in \mathcal{M}_S} p. \tag{5}
\]
For every nonzero idempotent \( p \) we have \( f(p) > 0 \), as \( f(p) = f(p^2) = \langle p, p \rangle > 0 \).

3 Maps between different \( \mathcal{A}_S \)

Let \( S \) and \( T \) be finite subsets of \( \mathbb{Z} \), and let \( \alpha : S \to T \). We define a linear function \( \hat{\alpha} : \mathcal{G}_S \to \mathcal{G}_T \) by
\[
\hat{\alpha}(G, \phi) := (G, \alpha \phi)
\]
for any \( S \)-colored graph \( (G, \phi) \). We define another linear map \( \check{\alpha} : \mathcal{G}_T \to \mathcal{G}_S \) as follows. Let \( (G, \phi) \) be a \( T \)-colored graph. For any node \( v \) of \( G \), split \( v \) into \( |\alpha^{-1}(\phi(v))| \) copies, adjacent to any copy of any neighbour of \( v \) in \( G \). Give these copies of \( v \) distinct colors from \( \alpha^{-1}(\phi(v)) \), to get the colored graph \( \check{\alpha}(G, \phi) \).

It is easy to see that the map \( \hat{\alpha} \) is an algebra homomorphism, while the map \( \check{\alpha} \) is not an algebra homomorphism in general. On the other hand, \( \check{\alpha} \) is an isomorphism of the underlying uncolored graphs, but in general \( \hat{\alpha} \) is not.

For any \( T \)-colored graph \( G \) and any \( S \)-colored graph \( H \), we have
\[
\hat{\alpha}(\hat{\alpha}(G) \times H) = G \times \hat{\alpha}(H),
\]
which implies that the underlying uncolored graphs of \( \hat{\alpha}(G) \times H \) and \( G \times \hat{\alpha}(H) \) are the same. Then \( g \in I \) implies \( \hat{\alpha}(g) \in I \) for any \( g \in \mathcal{G}_S \), since \( \langle \hat{\alpha}(g), \hat{\alpha}(g) \rangle = \langle g, \hat{\alpha} \hat{\alpha}(g) \rangle = 0 \). Hence \( \hat{\alpha} \) quotients to a linear function \( \mathcal{A}_S \to \mathcal{A}_T \). Similarly, \( g \in I \) implies \( \hat{\alpha}(g) \in I \) for any \( g \in \mathcal{G}_T \), hence \( \hat{\alpha} \) quotients to an algebra homomorphism \( \mathcal{A}_T \to \mathcal{A}_S \). We abuse notation and denote these induced maps also by \( \hat{\alpha} \) and \( \check{\alpha} \).

Then
\[
\hat{\alpha}(\hat{\alpha}(x)y) = x \hat{\alpha}(y) \tag{6}
\]
and hence
\[
\langle \hat{\alpha}(x), y \rangle = \langle x, \hat{\alpha}(y) \rangle \tag{7}
\]
for all $x \in \mathcal{A}_T$ and $y \in \mathcal{A}_S$.

It is easy to see that if $\alpha : S \to T$ is surjective, then $\hat{\alpha} : \mathcal{G}_S \to \mathcal{G}_T$ is surjective and so is the map $\mathcal{A}_S \to \mathcal{A}_T$ it induces. On the other hand, if again $\alpha : S \to T$ is surjective, then $\hat{\alpha} : \mathcal{G}_T \to \mathcal{G}_S$ is injective, and so is the map $\mathcal{A}_T \to \mathcal{A}_S$ it induces.

Since $\hat{\alpha}$ is an algebra homomorphism, $\hat{\alpha}(p)$ is an idempotent in $\mathcal{A}_S$ for any idempotent $p \in \mathcal{A}_T$, and $\hat{\alpha}(1_T) = 1_S$. So (5) implies that

$$\sum_{p \in M_T} \hat{\alpha}(p) = \hat{\alpha}(1_T) = 1_S = \sum_{q \in M_S} q. \quad (8)$$

Define for any $p \in M_T$ and $\alpha : S \to T$:

$$M_{\alpha,p} := \{ q \in M_S \mid \hat{\alpha}(p)q = q \}.$$ 

By (8),

$$\hat{\alpha}(p) = \sum_{q \in M_{\alpha,p}} q. \quad (9)$$

This implies that if $\alpha$ is surjective, then $M_{\alpha,p} \neq \emptyset$.

**Proposition 2** Let $p \in M_T$, $\alpha : S \to T$, and $q \in M_{\alpha,p}$. Then

$$\hat{\alpha}(q) = \frac{f(q)}{f(p)}p.$$

**Proof.** If $p' \in M_T \setminus \{p\}$, then

$$\langle \hat{\alpha}(q), p' \rangle = \langle q, \hat{\alpha}(p') \rangle = 0 = \left( \frac{f(q)}{f(p)}p, p' \right),$$

since $\langle p, p' \rangle = 0$. Moreover,

$$\langle \hat{\alpha}(q), p \rangle = \langle q, \hat{\alpha}(p) \rangle = f(\hat{\alpha}(p)q) = f(q) = \left( \frac{f(q)}{f(p)}p, p \right),$$

since $\langle p, p \rangle = f(p)$. \hfill \square
4 Maximal basic idempotents

For each $x \in A$, let $C(x)$ be the minimal set $S$ of colors for which $x \in A_S$. This is well defined because $A_S \cap A_T = A_{S\cap T}$.

**Proposition 3** $|C(p)| \leq n$ for each basic idempotent $p$.

**Proof.** Suppose $|C(p)| > n$. Let $S := C(p)$. Then any $\mathcal{A}_T$ has at least $\binom{|T|}{|S|}$ basic idempotents, since for each subset $S'$ of $T$ of size $|S|$ we can choose a bijection $\alpha : S \to S'$. Then $\hat{\alpha}(p)$ belongs to $\mathcal{A}_T$, and they are all distinct.

So $\dim(\mathcal{A}_T) \geq \binom{|T|}{|S|}$. As $|S| > n$, the latter value is larger than $|T|^n$ if $|T|$ is large enough. This contradicts our rank assumption. □

This proposition implies that we can choose an idempotent $p$ with $|C(p)|$ maximal, which we fix from now on. Define $S := C(p)$.

**Proposition 4** Let $\alpha : T \to S$ be surjective. Then

$$\hat{\alpha}(p) = \sum_{\beta : S \to T, \alpha \beta = \text{id}_S} \hat{\beta}(p).$$

Note that the maps $\beta$ in the summation are necessarily injections.

**Proof.** Consider any $q \in \mathcal{M}_{\alpha,p}$. By Proposition 2, $\hat{\alpha}(q)$ is a nonzero multiple of $p$. This implies $C(p) = C(\hat{\alpha}(q)) \subseteq \alpha(C(q))$. So $|C(q)| \geq |C(p)|$, hence by the maximality of $|C(p)|$, $|C(q)| = |C(p)|$. So $\alpha|_{C(q)}$ is a bijection between $C(q)$ and $C(p)$. Setting $\beta = (\alpha|_{C(q)})^{-1}$, we get $q = \hat{\beta}(p)$. By symmetry, $\phi(p)$ occurs in the sum for every injective $\phi : S \to T$ such that $\alpha \phi = \text{id}_S$. □

**Proposition 5** For any finite set $T$,

$$\sum_{\alpha : S \to T} \hat{\alpha}(p) = f(p)1_T. \quad (10)$$

**Proof.** Let $\sigma$ and $\tau$ be the projections of $S \times T$ on $S$ and on $T$, respectively. Then for any $S$-colored graph $G$ and any $T$-colored graph $H$ one has that $\hat{\sigma}(G) \times \hat{\tau}(H)$ is, as uncolored graph, equal to the product of the underlying uncolored graphs of $G$ and $H$. Hence, with (11),

$$f(\hat{\sigma}(G) \times \hat{\tau}(H)) = f(G)f(H). \quad (11)$$
Now for each $\alpha : S \to T$, there is a unique $\beta : S \to S \times T$ with $\sigma \beta = \text{id}_S$ and $\tau \beta = \alpha$. Hence, with Proposition 4,

$$\sum_{\alpha : S \to T} \hat{\alpha}(p) = \sum_{\beta : S \to S \times T \atop \sigma \beta = \text{id}_S} \hat{\tau} \hat{\beta}(p) = \hat{\tau} \sum_{\beta : S \to S \times T \atop \sigma \beta = \text{id}_S} \hat{\beta}(p) = \hat{\tau} \hat{\sigma}(p).$$

So for any $x \in A_T$, with (7) and (11),

$$\langle \hat{\tau} \hat{\sigma}(p), x \rangle = \langle \hat{\sigma}(p), \hat{\tau}(x) \rangle = f(\hat{\sigma}(p) \hat{\tau}(x)) = f(p) f(x) = (f(p) 1_T, x).$$

This implies that $\hat{\tau} \hat{\sigma}(p) = f(p) 1_T$. □

**Remark 6** While it follows from the theorem, it may be worthwhile to point out that the maximal basic idempotent $p$ is unique up to renaming the colors, and all other basic idempotents arise from it by merging and renaming colors. Indeed, we know by Proposition 2 that every term in (10) is a positive multiple of a basic idempotent in $A_T$, and so it follows that every basic idempotent in $A_T$ is a positive multiple of $\hat{\alpha}(p)$ for an appropriate map $\alpha$.

In particular, if $p'$ is another basic idempotent with $|C(p')| = |C(p)|$, then it follows that $p' = \hat{\alpha}(p)$ for some bijective map $\alpha : C(p) \to C(p')$.

### 5 Möbius transforms

For any colored graph $H$, define the quantum graph $\mu(H)$ (the Möbius transform) by

$$\mu(H) := \sum_{Y \subseteq E(H)} (-1)^{|Y|} (H - Y).$$

We call a colored graph $G$ flat, if $V(G) = T$, and the color of node $t$ is $t$. For any $T$-colored graph $G$ and any finite set $S$, define

$$\lambda_S(G) := \sum_{\alpha : S \to T} \hat{\alpha}(G).$$

**Proposition 7** Let $F$ and $G$ be flat colored graphs, and $S := V(F)$, $T := V(G)$. Then

$$\lambda_S(G) \times \mu(F) = \text{hom}(F, G) \mu(F). \quad (12)$$
Proof. Consider any map $\alpha : S \to T$. If $\alpha$ is a homomorphism $F \to G$, then $\hat{\alpha}(G) \times \mu(F)$ is equal to $\mu(F)$. If $\alpha$ is not a homomorphism $F \to G$, then $\hat{\alpha}(G) \times \mu(F) = 0$, since $F$ contains edges that are not represented in $\hat{\alpha}(G) \times F$. □

6 Completing the proof

Since $\bar{K}_S = \sum_F \mu(F)$, where $F$ ranges over all flat $S$-colored graphs, and since $\bar{K}_Sp = p$, there exists a flat $S$-colored graph $F$ with $\mu(F)p \neq 0$. As $p$ is a basic idempotent, $\mu(F)p = \gamma p$ for some real $\gamma \neq 0$. We prove that $f = \hom(F, \cdot)$.

Choose a flat $T$-colored graph $G$. As $p$ is in the ideal generated by $\mu(F)$, (12) gives $\lambda_S(G)p = \hom(F,G)p$. Then by (10) and (7):

$$f(p)f(G) = f(p)(G,1_T) = \sum_{\alpha : S \to T} \langle G, \hat{\alpha}(p) \rangle = \sum_{\alpha : S \to T} \langle \hat{\alpha}(G), p \rangle = \langle \lambda_S(G), p \rangle = f(\lambda_S(G)p) = \hom(F,G)f(p).$$

Since $f(p) \neq 0$, this gives $f = \hom(F, \cdot)$.

7 Concluding remarks

For a fixed finite set $S$ of colors, colored graphs can be thought of as arrows $G \to \bar{K}_S$ in the category of graph homomorphisms. The product of two colored graphs is pullback of the corresponding pair of maps. The setup in [1, 3] can be described by reversing the arrows. This raises the possibility that there is a common generalization in terms of categories.

The methods from [1] have been applied in extremal graph theory and elsewhere. Are there similar applications of the methods used in this paper?

If we relax the rank condition from $k^n$ to just polynomial in $k$, we obtain a characterization of the convex hull of the functions $\hom(F, \cdot)$.

References
