

# STABLE SETS AND POLYNOMIALS

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**Abstract.** Several applications of methods from non-linear algebra to the stable set problem in graphs are surveyed. The most recent work sketched is joint with A. Schrijver and involves non-linear inequalities. These yield a procedure to generate facets of the stable set polytope. If a class of graphs has the property that all facets of the stable set polytope can be generated this way in a bounded number of steps then the stable set problem is polynomially solvable for these graphs. Perfect,  $t$ -perfect and  $h$ -perfect graphs have this property.

## 0. Introduction

The stable set problem is one of the simplest and most fundamental problems concerning graphs. As it is to be expected, the problem may serve as a pilot case in the application of various methods to NP-hard problems. Indeed, the stable set problem was among the first to which polyhedral and linear algebraic methods have been successfully applied.

In this paper we survey some applications of non-linear commutative algebra (polynomials) in the study of the stable set problem. While several of our considerations will have a similar flavor, they have not yet been unified in a single theory. Rather, I consider them as pieces of a jigsaw puzzle, where it is easy to feel that they belong to one bigger picture but their exact position is not found yet.

A *stable set* in a graph  $G = (V, E)$  is a set of nodes no two of which are adjacent. The maximum size  $\alpha(G)$  of a stable set is called the *stability number* of  $G$ . It is well known that the computation of  $\alpha(G)$  is NP-hard.

However, quite often we are not concerned with the stability number of an explicitly given graph but, rather, with the stability number of graphs derived from some other combinatorial structure. In fact, many results in extremal combinatorics and coding theory can be formulated this way. In such cases, general algebraic methods for the determination of  $\alpha(G)$  have a larger chance of success.

For example, let  $S$  be an  $n$ -element set,  $V = \binom{S}{k}$ , the set of its  $k$ -element subsets, and define a graph  $K_n^k$  on  $V$  by connecting two  $k$ -subsets iff they are disjoint. These graphs are called *Kneser graphs*. The stability number of  $K_n^k$  is the maximum number of mutually

intersecting  $k$ -subsets of an  $n$ -set. The Erdős–Ko–Rado Theorem asserts that this number is  $\binom{n-1}{k-1}$  if  $n \geq 2k$ . There is at least one general algebraic upper bound on the stability number from which this bound can be derived (Lovász 1979).

## 1. Turán’s Theorem and the Motzkin–Straus Formula.

One of the first results in extremal graph theory is Turán’s theorem, which can be formulated as a result on the stability number:

**1.1 Theorem.** *Let  $G$  be a graph with  $n$  nodes,  $m$  edges and stability number  $\alpha$ . Write  $n = \alpha q + r$ . Then*

$$m \geq (\alpha - r) \binom{q}{2} + r \binom{q+1}{2}.$$

This theorem is usually formulated for the complement. The best way to remember the bound is that the graph with a given number of nodes  $n$ , with given stability number  $\alpha$  and with a minimum number of edges is the union of  $\alpha$  node-disjoint cliques, as equal in size as possible. The complement of this graph (a complete multipartite graph) is called the *Turán graph*  $T_n^\alpha$ .

Turán’s theorem can be proved by double induction of  $n$  and  $\alpha$  (and in many other ways). For our purposes the most interesting is an algebraic proof by Motzkin and Straus (1965), which in fact gives an explicit formula for  $\alpha(G)$ .

**1.2 Theorem.** *Let  $G = (V, E)$  be a graph. Then*

$$1 - \frac{1}{\alpha(G)} = \max \left\{ 2 \sum_{ij \notin E} x_i x_j : \sum_{i \in V} x_i = 1, x_i \geq 0 \right\}.$$

Turán’s Theorem (at least for the case when  $\alpha|n$ ) can be obtained by substituting  $x_i = 1/n$ .

**Proof.** Consider a vector  $x$  maximizing the right hand side and (if it is not unique) having a maximum number of 0 entries. We claim that the support of  $x$  is a stable set. Assume, to the contrary, that nodes 1 and 2 are adjacent nodes belonging to the support of  $x$ . If  $-x_1 \leq t \leq x_2$  then the vector  $(x_1 + t, x_2 - t, x_3, \dots)$  also satisfies the constraints  $x_i \geq 0$  and  $\sum_i x_i = 1$ . The objective function is a linear function of  $t$  (since the term  $x_1 x_2$  does not occur in the objective function), and hence it can assume its maximum at 0 only if it is constant. But then for  $t = x_2$ , we have another optimizing vector with more 0 entries.

Now if the support of  $x$  is a stable set  $A$  then high school algebra shows that the maximum is attained when  $x_i = 1/|A|$  for all  $i \in A$ , and then the value of the objective function is exactly  $1 - 1/|A|$ . The maximum is attained when  $A$  is a maximum stable set.

## 2. Polynomial ideals, stable sets and chromatic number.

With every graph  $G = (V, E)$  on  $V = \{1, \dots, n\}$ , we associate the polynomial

$$p_G(x_1, \dots, x_n) = \prod_{ij \in E} (x_i - x_j)$$

(we need a reference orientation of the edges for this, but this plays little role). It is interesting to remark that this correspondence was one of the starting points of graph theory. Hilbert (1898; see König 1936) considered the question of decomposing such a polynomial in which every variable has degree  $d$ , into factors in which every variable has degree 1. This is of course equivalent to decomposing a regular graph in 1-factors, and was one of the motivations for the work of Petersen. The words *factor* and *degree* come from this correspondence.

More recently S.R.Li and W.W.Li (1981) related this polynomial to the independence number of the graph  $G$ . Their starting point is the following observation:

**2.1 Lemma.** *The graph  $G$  has independence number at most  $\alpha$  if and only if the identification of any  $\alpha + 1$  variables in  $p_G$  yields the 0 polynomial.*

Let  $I_n^\alpha$  denote the set of all polynomials  $f \in \mathbb{R}[x_1, \dots, x_n]$  such that  $f(x_1, \dots, x_n) = 0$  if at least  $\alpha + 1$  variables are equal. Clearly  $I_n^\alpha$  is an ideal. Now S.R.Li and W.W.Li prove the following characterization of this ideal:

**2.2 Theorem.** *The ideal  $I_n^\alpha$  is generated by the polynomials  $P_H$ , where  $H$  is a graph isomorphic to the complement of the Turán graph  $T_n^\alpha$ .*

Let  $\hat{I}_n^\alpha$  denote the ideal generated by the polynomials  $p_H$ ,  $H \equiv \bar{T}_n^\alpha$ . Trivially  $\hat{I}_n^\alpha \subseteq I_n^\alpha$ . It is easy to see that an  $n$ -tuple  $(y_1, \dots, y_n)$  is a zero of  $\hat{I}_n^\alpha$  if and only if some  $\alpha + 1$  of the  $y_i$  are equal. Hence by the Nulstellensatz of Hilbert, every  $f \in I_n^\alpha$  has some power  $f^p$  which belongs to  $\hat{I}_n^\alpha$ . The theorem says that we have  $p = 1$ .

One consequence of this result is Turán's theorem. In fact, if  $G$  is a graph with  $n$  nodes,  $m$  edges, and  $\alpha(G) = \alpha$ , then  $p_G \in I_n^\alpha$ . By theorem 2.2, this ideal is generated by homogeneous polynomials of degree  $m_0 = |E(\bar{T}_n^\alpha)|$ , and hence  $m = \deg(p_G) \geq m_0$ .

The proof of theorem 2.1 is quite involved. We shall state and prove an analogue for the chromatic number, due to Kleitman and Lovász (unpublished), whose proof is simpler. We start with the following observation:

**2.3 Lemma.** *The graph  $G$  has chromatic number at least  $\chi$  if and only if every way of identifying variables in  $p_G$  so that at most  $\chi - 1$  distinct variables remain, yields the 0 polynomial.*

Let  $J_n^\chi$  denote the set of all polynomials  $f \in \mathbb{R}[x_1, \dots, x_n]$  such that  $f(x_1, \dots, x_n) = 0$  if at most  $\chi - 1$  variables are distinct. Clearly  $J_n^\chi$  is an ideal. This ideal can be characterized as follows:

**2.4 Theorem.** *The ideal  $J_n^\chi$  is generated by the polynomials  $P_H$ , where  $H$  is a graph consisting of a clique of size  $\chi$  and isolated nodes.*

The relation of this theorem to the Nullstellensatz is similar to that of Theorem 2.2.

**Proof.** Let  $\hat{J}_n^\chi$  denote the ideal generated by the polynomials  $p_H$ , where  $H$  consists of a  $\chi$ -clique and isolated nodes. Trivially  $\hat{J}_n^\chi \subseteq J_n^\chi$ . To show the converse inclusion, let  $f \in J_n^\chi$ . For  $S \subseteq \{1, \dots, n-1\}$ , let  $f_S$  denote the polynomial obtained from  $f$  by substituting  $x_n$  for each  $x_i$ ,  $i \in S$ .

Clearly  $f_S \in J_n^\chi$ , and so by induction on the number of variables we may assume that  $f_S \in \hat{J}_n^\chi$  for every non-empty  $S$ .

Consider also the polynomial

$$g = \sum_S (-1)^{|S|} f_S.$$

If we substitute  $x_n$  for any  $x_i$  ( $1 \leq i \leq n-1$ ) in  $g$ , we get the 0 polynomial; hence  $g$  is divisible by the product  $(x_1 - x_n)(x_2 - x_n) \cdots (x_{n-1} - x_n)$ . Write

$$g = (x_1 - x_n)(x_2 - x_n) \cdots (x_{n-1} - x_n)h.$$

It is clear that  $g \in J_n^\chi$  and hence  $h$  vanishes whenever at most  $\chi - 2$  of the variables  $x_1, \dots, x_{n-1}$  are distinct. So if we expand  $h$  by the powers of  $x_n$ , the coefficient of every  $x_n^k$  will belong to  $J_{n-1}^{\chi-1}$ . By induction on  $n$ , we may assume that these coefficients belong to  $\hat{J}_{n-1}^{\chi-1}$ . This implies that  $g \in \hat{J}_n^\chi$ .

It follows that

$$f = g - \sum_{S \neq \emptyset} (-1)^{|S|} f_S \in \hat{J}_n^\chi,$$

which proves theorem 2.4.

Slight modification of the proof gives the following results:

**2.5 Corollary.** (a) A graph  $G$  has  $\alpha(G) \leq k$  iff  $p_G$  has a representation of the form

$$p_G = p_{H_1} + p_{H_2} + \cdots + p_{H_N},$$

where each  $H_i$  is a graph on  $V(G)$  containing  $k$  cliques covering all nodes.

(b) A graph  $G$  has  $\chi(G) \geq k$  iff  $p_G$  has a representation of the form

$$p_G = p_{H_1} + p_{H_2} + \cdots + p_{H_N},$$

where each  $H_i$  is a graph on  $V(G)$  containing a  $k$ -clique.

Figure 1

Figure 1 shows a representation of  $P_{C_5}$  illustrating both parts of the theorem. This shows that there is a certain “graph calculus” involved here. We can write  $\sum_i G_i = 0$  instead of  $\sum_i p_{G_i} = 0$ ; here the  $G_i$  are directed graphs but we identify two orientations differing on an even number of edges. Let us call a relation

$$\sum_{i=1}^N G_i = 0$$

a *graph identity*. It might be interesting to study the structure of graph identities. For example, it is not difficult to show that (after cancellation of edges common to all terms) there are two graph identities with 3 terms, illustrated in Figure 2. Is the number of graph identities with a fixed number of terms finite? Can one verify a graph identity in polynomial time? (It can be verified in randomized polynomial time, by substituting random values for the variables.) One would expect that the number of terms  $N$  in the representations in Corollary 2.5 is exponentially large in the worst case (else, NP would be equal to randomized co-NP). Can one prove this about some particular series of graphs? How to find representations for graphs for which a lower bound for  $\alpha(G)$ , or an upper bound for  $\chi(G)$ , has been proved by other means (e.g. for the Kneser graphs)?

Figure 2

Recently Alon (1989) obtained another related result. Let  $f(x)$  be any polynomial in one variable with  $k$  distinct roots. Consider the polynomial ideal  $Q_n^k$  in  $\mathbb{R}[x_1, \dots, x_n]$  generated by the polynomials  $f(x_1), f(x_2), \dots, f(x_n)$ ; an  $n$ -tuple  $(u_1, \dots, u_n)$  is a zero of this ideal iff every entry  $u_i$  is a root of  $f$ . It is easy to see that  $G$  has chromatic number greater than  $k$  iff it vanishes on all zeros of  $Q_n^k$ . It is not difficult to prove again the stronger “Nullstellensatz” asserting that this is equivalent to saying that  $p_G \in Q_n^k$ .

Alon uses this to prove the following sufficient condition on colorability:

**2.6 Theorem.** *Assume that the expansion of  $p_G$  contains a monomial with non-zero coefficient in which every exponent is less than  $k$ . Then  $G$  is  $k$ -colorable.*

It should be remarked that every expansion term  $x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$  of  $p_G$  corresponds to an orientation of  $G$  in which node  $i$  has outdegree  $a_i$ . However, different orientations may give expansion terms with different sign which, therefore, may cancel. So the condition in Theorem 2.6 requires more than just the existence of an orientation with outdegrees less than  $k$  (which would not suffice anyway, as the complete graphs show).

An advantage of this method is that it extends to restricted colorations in which every node has a set of at most  $k$  colors assigned and we want a legal coloration where the color of each node is one of the colors assigned to it.

### 3. $\alpha$ -critical graphs.

A graph  $G$  is called  $\alpha$ -critical if it has no isolated nodes and for every edge  $e$ ,  $\alpha(G-e) > \alpha(G)$ . This is equivalent to saying that  $\tau(G') < \tau(G)$  for every proper subgraph  $G'$  of  $G$ , and so these graphs can also be called  $\tau$ -critical. Having a good description of  $\alpha$ -critical graphs would yield a good description of  $\alpha$ ; this means that we probably cannot have a complete list of  $\alpha$ -critical graphs (in particular, the class of  $\alpha$ -critical graphs is probably neither in NP nor in co-NP).

Cliques and odd circuits are  $\alpha$ -critical, and so are many other graphs; we just mention the icosahedron graph as an interesting example. Several monographs treat  $\alpha$ -critical graphs (Berge 1973, Lovász and Plummer 1986), so here we only sketch some of the most important results and discuss one proof in detail which relates to our main topic. For the following paragraphs, let  $G$  be an  $\alpha$ -critical graph with  $n$  nodes and  $m$  edges, and set  $\alpha = \alpha(G)$ ,  $\tau = \tau(G)$ .

The first result on  $\alpha$ -critical graphs was obtained by Erdős and Gallai (1961):

**3.1 Theorem.** *For every  $\alpha$ -critical graph*

$$\alpha \leq n/2$$

*(equivalently,  $\tau \geq n/2$ ). Equality holds iff  $G$  is a matching.*

The quantity  $\delta = \delta(G) = n - 2\alpha = 2\tau - \alpha = \tau - \alpha$  turns out to be an important measure of the complexity of the  $\alpha$ -critical graph, as we shall see below. This was first suggested by Gallai, and we call  $\delta(G)$  the *Gallai class number* of the  $\alpha$ -critical graph  $G$ .

The result of Erdős and Gallai can be viewed as a Helly type result: it is equivalent to saying that *if the edges of every subgraph of a graph  $G$  with at most  $2k$  nodes can be blocked by  $k$  nodes then all edges of  $G$  can be blocked by  $k$  nodes*. There is an analogous result on the number of edges, proved by Erdős, Hajnal and Moon (1964):

**3.2 Theorem.** *For every  $\alpha$ -critical graph,*

$$m \leq \binom{\tau + 1}{2}.$$

Hajnal (1965) proved two extensions of the Erdős-Gallai Theorem. For every set  $A \subseteq V$ , let  $\Gamma(A)$  denote the set of nodes in  $G$  adjacent to some node in  $A$ .

**3.3 Theorem.** *If  $A$  is stable then  $|\Gamma(A)| \geq |A|$ .*

The Erdős-Gallai Theorem corresponds to the case when  $A$  is a maximum stable set. Theorem 3.3 implies that  $G$  has a perfect 2-matching, i.e., a collection of node-disjoint circuits and edges covering all nodes.

The other theorem of Hajnal asserts the following:

**3.4 Theorem.** *The degree of any node in an  $\alpha$ -critical graph  $G$  is bounded by  $\delta(G) + 1$ .*

Besides the Erdős-Gallai theorem, this also implies the theorem of Erdős, Hajnal and Moon by summing over all elements of a minimum blocking set. Surányi (1975) obtained the following common generalization of all results mentioned so far:

**3.5 Theorem.** *If  $A$  is a stable set and  $a \in A$  then the degree of  $a$  is at most  $|\Gamma(A)| - |A| + 1$ .*

The proof of this theorem is elegant but elementary (see Lovász and Plummer 1986, Theorem 12.1.13). The following generalization of the theorem of Erdős, Hajnal and Moon, however, is not known to follow by elementary methods (Lovász 1977).

**3.6 Theorem.** *A blocking set  $B$  in an  $\alpha$ -critical graph  $G$  spans at most  $\binom{|B| + \tau - n + 1}{2}$  edges.*

The Erdős-Hajnal-Moon Theorem is obtained when  $B = V(G)$ . Another interesting case is when  $|B| = \tau$ : *A minimum blocking set in an  $\alpha$ -critical graph spans at most  $\binom{\delta + 1}{2}$  edges.*

Since the proof of Theorem 3.6 fits in our discussions we give it here. The key is the following lemma:

**3.7 Lemma.** *Let  $H$  be any graph with  $n$  nodes and  $m$  edges, and let, to each  $i \in V(H)$ , a vector  $v_i \in \mathbb{R}^k$  be assigned. Assume that these  $v_i$  span  $\mathbb{R}^k$ , and*

- (i) *every stable set corresponds to linearly independent vectors;*
- (ii) *deleting any edge this will not hold any more.*

*Then  $m \leq \binom{n-k+1}{2}$ .*

Each  $\alpha$ -critical graph provides an example when the conditions of the lemma hold: choose vectors  $v_i \in \mathbb{R}^\alpha$  in general position. More interesting examples will come up in the proof of Theorem 3.6.

**Proof of Lemma 3.7.** Let us assign a variable  $x_i$  to each node  $i$ , and consider the following equations:

$$\begin{aligned} x_i x_j &= 0 & (ij \in E(H)) \\ \sum_{i \in V(H)} x_i v_i &= 0 \end{aligned}$$

Note that the first set of equations is equivalent to saying that the support of  $x$  is a stable set in  $H$ ; the second set implies that the support of  $x$  corresponds to linearly dependent vectors. So by assumption (i) in the Lemma, this system has no non-trivial solution; moreover, (ii) implies that dropping any equation from the first set, the remaining system will have a non-trivial solution.

The solutions of  $\sum_i x_i v_i = 0$  form a linear space with dimension  $n - k$ ; over this space, the first set of equations give  $m$  quadratic equations. By the above remarks, these quadratic equations must be linearly independent. Since the linear space of all quadratic equations over an  $(n - k)$ -dimensional space has dimension  $\binom{n-k+1}{2}$ , the lemma follows.

**Proof of Theorem 3.6.** For simplicity, we describe the proof in the case when  $B$  is a minimum blocking set in  $G$ . Let  $A = V(H) \setminus B$  and let  $H$  be the subgraph of  $G$  induced by  $B$ . Let the numbers  $\xi_{ij}$  ( $i \in B, j \in A$ ) be algebraically independent transcendentals, and let, for each  $i \in B$ ,  $v_i$  be vector in  $\mathbb{R}^A$  whose  $j$ th entry is  $\xi_{ij}$  if  $ij \in E(G)$  and 0 otherwise. The reader familiar with matroid theory will recognize that this is a representation of the transversal matroid induced on  $B$  by the edges connecting  $A$  to  $B$ . In particular, a set  $S \subseteq B$  corresponds to linearly independent vectors iff every  $S' \subseteq S$  has at least  $|S'|$  neighbors in  $A$ .

We claim that  $H$  and the set of vectors  $\{v_i\}$  satisfy the conditions of the Lemma. The fact that  $\{v_i\}$  span the space  $\mathbb{R}^A$  follows e.g. from Theorem 3.3. To verify (i), let  $S$  be a stable set in  $G$  and assume that the vectors  $\{v_i : i \in S\}$  are not linearly independent. Then there exists a set  $S' \subseteq S$  having fewer than  $|S'|$  neighbors in  $A$ . But then  $A \setminus \Gamma(S') \cup S'$  is a stable set in  $G$  larger than  $A$ , a contradiction. The proof that (ii) holds is similar.

Thus Lemma 3.7 implies that  $H$  has at most  $\binom{|B|-\alpha+1}{2} = \binom{\delta+1}{2}$  edges.

Corollary 3.6 plays an important role in the proof of the following theorem,

**3.9 Theorem.** *For every  $\delta \geq 0$ , the number of  $\alpha$ -critical graphs with Gallai class number  $\delta$  and with all degrees at least 3 is finite.*

For  $\delta = 1$ , this follows trivially from Theorem 3.4. It was proved by Andrásfai (1967) that every connected  $\alpha$ -critical graph with  $\delta = 2$  arises from  $K_4$  by subdividing every edge by an even number of nodes; this implies theorem 3.9 in this case. Case  $\delta = 3$  was settled by Surányi, and the general case by Lovász (1978).

It should be mentioned that those  $\alpha$ -critical graphs with Gallai class number  $\delta$  and having nodes with degree 1 or 2 can be generated from this finite “basis” in a rather simple way: one may add node-disjoint copies of  $K_2$ , or split a node into two and connect these two to a new node.

Let us remark that there are many unsolved problems concerning  $\alpha$ -critical graphs. The known bounds on the maximum number of nodes of an  $\alpha$ -critical graph  $G$  with  $\delta(G) = \delta$  and all degrees at least 3 are very poor: the proof of Theorem 3.9 gives an upper bound of  $2^{\delta^2}$ , while a construction of Surányi (1978) gives a lower bound of  $\delta^2$ . Is it true that if  $G'$  is an  $\alpha$ -critical subgraph of an  $\alpha$ -critical graph  $G$  then  $\delta(G') < \delta(G)$ . The following conjecture of Chvátal is also unsettled: *any  $k$  edges of an  $\alpha$ -critical graph  $G$  adjacent to the same node are contained in an  $\alpha$ -critical subgraph  $G'$  with  $\delta(G') = k - 1$ .* (For  $k = 2$  this is a result of Berge.) Recently Sewell (1990) proved that *every  $\alpha$ -critical graph  $G$  with  $\delta(G) > 2$  contains and even subdivision of  $K_4$*  (i.e., a connected  $\alpha$ -critical subgraph  $H$  with  $\delta(H) = 2$ ).



## 4. The stable set polytope.

At this point we have to recall some results on the stable set polytope. A detailed account can be found e.g. in Grötschel, Lovász and Schrijver (1988).

Let  $G = (V, E)$  be a graph. For every subset  $S \subseteq V$ , let  $\chi^S \in \mathbb{R}^V$  denote its incidence vector, i.e., the vector defined by

$$\chi_i^S = \begin{cases} 1, & \text{if } i \in S, \\ 0, & \text{otherwise,} \end{cases}$$

The *stable set polytope*  $\text{STAB}(G)$  of  $G$  is the convex hull of incidence vectors of all stable sets.

Since the stable set polytope is defined in terms of its vertices, it is natural to ask for a description of its facets. There is no hope to get a complete description, but many interesting classes of facets, and more generally, of inequalities valid for  $\text{STAB}(G)$ , are known. We mention some of them. To exclude some trivial complications, we assume that  $G$  has no isolated nodes.

There are two classes of trivial inequalities: the *non-negativity constraints*

$$(4.1) \quad x_i \geq 0 \quad (i \in V),$$

and the *edge constraints*

$$(4.2) \quad x_i + x_j \leq 1 \quad (ij \in E).$$

These inequalities define a polytope  $\text{FRAC}(G)$  which is in general larger than  $\text{STAB}(G)$ . In fact one has  $\text{STAB}(G) = \text{FRAC}(G)$  iff  $G$  is bipartite. An important property of  $\text{FRAC}(G)$  is that its vertices are half-integral.

A natural extension of (4.2) is the class of *clique constraints*:

$$(4.3) \quad \sum_{i \in B} x_i \leq 1, \quad \text{where } B \text{ is a clique.}$$

Inequalities (4.1) and (4.3) suffice to describe  $\text{STAB}(G)$  iff  $G$  is perfect.

A next important class of inequalities valid for  $\text{STAB}(G)$  is the class of *odd hole constraints*:

$$(4.4) \quad \sum_{i \in B} x_i \leq \frac{|C| - 1}{2}, \quad \text{where } B \text{ induces a cordless odd cycle.}$$

A graph is called *t-perfect* if (4.1), (4.2) and (4.4) suffice to describe  $\text{STAB}(G)$ , and *h-perfect* if (4.1), (4.3) and (4.4) suffice to describe  $\text{STAB}(G)$ .

A further, perhaps less well studied class of inequalities is the class of *odd antihole constraints*:

$$(4.5) \quad \sum_{i \in B} x_i \leq 2, \quad \text{where } B \text{ induces the complement of a cordless odd cycle.}$$

All these inequalities are facets at least if  $B = V$ . If there are nodes not occurring in the inequality then they may sometimes be added to the constraint with non-zero coefficient; this is called *lifting* (we don't discuss this procedure here).

A wider class of facets was found by Chvátal (1975).

**4.6 Theorem.** *Let  $G = (V, E)$  be an  $\alpha$ -critical graph. Then the inequality  $\sum_{i \in V} x_i \leq \alpha(G)$  defines a facet of  $\text{STAB}(G)$ .*

This theorem indicates how complex the facets of  $\text{STAB}(G)$  can be. Another important consequence of it is that we may in a sense consider facets of  $\text{STAB}(G)$  as generalizations of  $\alpha$ -critical graphs and extend the theory of  $\alpha$ -critical graphs to facets. This idea is rather unexplored; most of the results are due to Sewell (1990). We only state one result (found also by Lovász and Schrijver 1989) which will be needed later on.

Let  $\sum_i a_i x_i \leq b$  be an inequality defining a facet of  $\text{STAB}(G)$ . We define its *Gallai class number* as  $\sum_i a_i - 2b$ . The following lemma implies that this value is non-negative, and in fact positive except for the edge constraints:

**4.7 Lemma.** *Let  $\sum_i a_i x_i \leq b$  be a facet of  $\text{STAB}(G)$ . Then*

$$\max\left\{\sum_i a_i x_i : x \in \text{FRAC}(G)\right\} = \frac{1}{2} \sum_i a_i.$$

In other words, the left hand side of any facet of  $\text{STAB}(G)$  is maximized over  $\text{FRAC}(G)$  by the vector  $(1/2, \dots, 1/2)^T$ . It can also be shown that if  $a_i > 0$  for all  $i$ , and the facet is different from the edge constraints, then this is the unique vector maximizing the left hand side.

This lemma also generalizes Hajnal's theorem (3.3). It also follows that the Gallai class number is a kind of "integrality gap":

$$\sum_i a_i - 2b = 2 \max\left\{\sum_i a_i x_i : x \in \text{FRAC}(G)\right\} - 2 \max\left\{\sum_i a_i x_i : x \in \text{STAB}(G)\right\}.$$

Certain extensions of theorems 3.5 and 3.6 were given by Sewell. It would be interesting to find extensions of theorem 3.9 to facets.

We introduce one further class of inequalities valid for  $\text{STAB}(G)$ . An *orthonormal representation* of a graph  $G$  is an assignment of a vector  $v_i \in \mathbb{R}^k$  to each node  $i$  (for some  $k \geq 1$ ) so that  $|v_i| = 1$  and  $v_i^T v_j = 0$  for every pair  $i, j$  of non-adjacent nodes. Consider any vector  $c$  with  $|c| = 1$ , and the inequality

$$\sum_i (c^T v_i)^2 x_i \leq 1.$$

If  $x$  is the incidence vector of a stable set  $A$  then the left hand side is  $\sum_{i \in A} (c^T v_i)^2$ , and the inequality holds by Parseval's formula since the vectors  $v_i, i \in A$  are mutually orthogonal. So this inequality holds true for every vector  $x \in \text{STAB}(G)$ . These inequalities are called *orthogonality constraints*. The set of vectors satisfying all orthogonality constraints is denoted by  $\text{TH}(G)$ .

The definition of  $\text{TH}(G)$  is rather complicated. In particular, there are infinitely many orthogonality constraints for a fixed graph and, accordingly,  $\text{TH}(G)$  is in general not a polytope. But  $\text{TH}(G)$  has some surprisingly nice properties (see Grötschel, Lovász and

Schrijver 1987, 1988). We mention a few: If  $\bar{G}$  denotes the complement of the graph  $G$ , then  $\text{TH}(\bar{G})$  is the antiblocker of  $\text{TH}(G)$ .  $\text{TH}(G)$  is polyhedral iff  $G$  is perfect, and in this case it is equal to the stable set polytope of  $G$ . Perhaps most important is the following fact: every linear objective function can be optimized over  $\text{TH}(G)$  in polynomial time.

## 5. Projection representations.

If we project a polytope to a subspace then the number of vertices can, of course not increase; but we have little control over the the number of facets, which may increase substantially. This seemingly negative fact can be turned around: a polytope having an inconveniently large number of facets may be represented as the projection of a polytope with a much smaller number of facets. This technique in polyhedral optimization is relatively new and promising (Balas and Pulleyblank 1987, Liu 1988, Barahona and Mahjoub 1987); at the same time, many of the basic questions are unsettled.

As an example, consider the stable set polytope of a comparability graph  $G$ , obtained by taking a partially ordered set  $(V, \leq)$ , and connecting two elements of  $V$  iff they are comparable in this partial order. This graph is perfect and so its stable set polytope is described by the non-negativity and clique constraints:

$$\begin{aligned} x_i &\geq 0 \quad \text{for every } i \in V, \\ \sum_{i \in B} x_i &\leq 1 \quad \text{for every chain } B \subseteq V. \end{aligned}$$

This latter family of constraints is typically exponentially large. However  $\text{STAB}(G)$  can be represented as the projection of a polytope with  $O(|V|^2)$  facets:

**5.1 Lemma.** *Let  $(V, \leq)$  be a poset. Assign two variables  $x_i, y_i$  to each  $i \in V$  and consider the polytope  $P$  defined by the inequalities*

$$\begin{aligned} 0 &\leq x_i \leq y_i \leq 1 \quad \text{for all } i \in V, \\ y_i + x_j &\leq y_j \quad \text{for all pairs } i < j. \end{aligned}$$

*Then projecting  $P$  on the  $x$ -coordinates, we obtain  $\text{STAB}(G)$ .*

It is a very interesting question whether  $\text{STAB}(G)$  in general can be represented as the projection of a polytope with a polynomial number of facets. The answer is in the affirmative for comparability graphs and their complements, chordal graphs and their complements, and several other classes of perfect graphs, but is not known for all perfect graphs. The answer is positive for  $t$ -perfect graphs (cf. next section). One suspects that the answer will be negative for a general graph. However, the only negative result to date is that of Yannakakis (1988), who proves that if we also require that the automorphism group of  $G$  “lifts up” to isometries of the polytope then the stable set polytope of the line-graph of  $K_n$  (i.e., the matching polytope of  $K_n$ ) cannot be represented as the projection of a polytope with a polynomial number of facets.

Yannakakis also formulated the following combinatorial necessary condition. Let  $\{U_1, W_1\}, \{U_2, W_2\}, \dots, \{U_N, W_N\}$  be partitions of  $V(G)$  into two classes. We say that this family *separates cliques and stable sets* if for every stable set  $A$  and clique  $B$  such that  $A \cap B = \emptyset$  there exists an  $i$  with  $A \subseteq U_i$  and  $B \subseteq W_i$ .

**5.2 Lemma.** *If  $\text{STAB}(G)$  can be represented as the projection of a polytope with  $N$  facets then there exist  $N$  partitions of  $V(G)$  separating cliques from stable sets.*

It is not known whether in every graph (or even in every perfect graph) cliques can be separated from stable sets by a polynomial number of partitions. It follows from the theory of communication complexity that  $O(n^{\log n})$  partitions suffice in every graph. This was improved by A. Hajnal (unpublished) to  $O(n^{(1/2)\log n})$ .

## 6. Quadratic inequalities

The polyhedral theory of stable sets can be viewed as a theory of linear inequalities valid for the incidence vectors of stable sets. In view of the results discussed in sections 1 and 3, it is natural to ask for quadratic inequalities (and, of course higher degree inequalities) valid for the incidence vectors of stable sets.

At the first sight it seems that we are getting to much too easily. Let  $G = (V, E)$  be a graph and consider the following system of equations:

$$(6.1) \quad x_i^2 = x_i \text{ for every node } i \in V,$$

$$(6.2) \quad x_i x_j = 0 \text{ for every edge } ij \in E.$$

Trivially, the solutions of (6.1) are precisely the 0-1 vectors, and so the solutions of (6.1)–(6.2) are precisely the incidence vectors of stable sets. Unfortunately, there is little known about the solutions of systems of quadratic equations. In fact, what this shows is that even the solvability of such a simple system of quadratic equations (together with a linear equation  $\sum_i x_i = \alpha$ ) is NP-hard.

However, we can use this system to derive some other constraints. (6.1) implies that for every node  $i$ ,

$$(6.3) \quad x_i = x_i^2 \geq 0, \quad 1 - x_i = (1 - x_i)^2 \geq 0,$$

and using this (6.2) implies that for every edge  $ij$ ,

$$(6.4) \quad 1 - x_i - x_j = 1 - x_i - x_j + x_i x_j = (1 - x_i)(1 - x_j) \geq 0.$$

So we can derive the edge constraints from (6.1)–(6.2) formally. We can go on and use these to derive the odd hole constraints. Consider e.g. a pentagon  $(1, 2, 3, 4, 5)$ . Then we have

$$1 - x_1 - x_2 - x_3 + x_1 x_3 = 1 - x_1 - x_2 - x_3 + x_1 x_2 + x_1 x_3 = (1 - x_1)(1 - x_2 - x_3) \geq 0$$

and similarly

$$1 - x_1 - x_4 - x_5 + x_1 x_4 \geq 0.$$

Furthermore,

$$x_1 - x_1 x_3 - x_1 x_4 = x_1(1 - x_3 x_4) \geq 0.$$

Summing these inequalities, we get the odd hole constraint

$$(6.5) \quad 2 - x_1 - x_2 - x_3 - x_4 - x_5 \geq 0.$$

We can also derive the clique constraints. Assume that nodes 1,2,3,4,5 induce a complete 5-graph. We start with the trivial inequality

$$(1 - x_1 - x_2 - x_3 - x_4 - x_5)^2 \geq 0.$$

Expanding, we get

$$1 + \sum_{i=1}^5 x_i^2 - 2 \sum_{i=1}^5 x_i + 2 \sum_{i \neq j} x_i x_j \geq 0.$$

Here the first sum is just  $\sum_i x_i$  by (6.1) and the third sum is 0 by (6.2), so we get

$$(6.6) \quad 1 - x_1 - x_2 - x_3 - x_4 - x_5 \geq 0.$$

We can in fact derive any orthogonality constraint. Let  $\{v_i : i \in V\}$  be any orthonormal representation of  $G$  and let  $c$  be a unit vector in the same space. Then we have the trivial inequality

$$\left( c - \sum_i (c^T v_i) x_i v_i \right)^2 \geq 0.$$

Expanding, we get

$$c^2 + \sum_i (c^T v_i)^2 x_i^2 v_i^2 - 2 \sum_i (c^T v_i)^2 x_i + 2 \sum_{i \neq j} (c^T v_i)(c^T v_j) x_i x_j v_i^T v_j \geq 0.$$

Here  $x_i^2 v_i^2 = x_i$  by (6.1) and by  $|v_i| = 1$ . Moreover,  $x_i x_j v_i^T v_j = 0$ ; this follows from (6.2) if  $i$  and  $j$  are adjacent, and from the definition of orthonormal representations if  $i$  and  $j$  are non-adjacent. Using also that  $|c| = 1$  we get

$$(6.7) \quad 1 - \sum_i (c^T v_i)^2 x_i \geq 0,$$

which is just an orthogonality constraint.

We can formalize these procedures: if we have a family of linear inequalities valid for  $\text{STAB}(G)$ , then by multiplying pairs of them we obtain quadratic inequalities. Other sources of quadratic inequalities are (6.1)–(6.2) and the fact that the square of a linear form is non-negative. By taking non-negative linear combinations of such quadratic inequalities, we may be able to get rid of all quadratic terms and obtain a linear inequality. The examples above show that quite complicated linear inequalities can be derived this way.

This way of deriving inequalities was introduced and studied by Lovász and Schrijver (1989, 1990), and we do not repeat the details here. But we have to address the question: what is the point in deriving inequalities algebraically, which can be proved true anyway by trivial combinatorial considerations?

Let  $F$  be a family of linear inequalities valid for  $\text{STAB}(G)$ ; we shall assume that  $F$  contains the family  $F_0$  of inequalities (6.3). As a minimal set of operations, allow the following: multiply each member of  $F$  by  $x_i$  and by  $1 - x_i$  to get a family of quadratic inequalities; take non-negative linear combinations of these inequalities; use (6.1) and (6.2) to get rid of certain quadratic terms. Let  $T(F)$  be the family of linear inequalities obtained this way. Let  $T_+(F)$  be the set of linear inequalities obtained if also squares of

linear functions can be taken at the start. Clearly  $F \subseteq T(F) \subseteq T_+(F)$ . We denote by  $T^k(F)$  the family of inequalities obtained by repeating the  $T$  operator  $k$  times.

The following facts are the key to the algorithmic applications of these methods.

**6.8 Theorem.** *Every linear inequality valid for  $\text{STAB}(G)$  occurs in  $T^k(F_0)$  for some  $k \leq n$ .*

Let  $K$  be the solution set of  $F$ ; we denote by  $N(K)$  [ $N_+(K)$ ] the solution set of  $T(F)$  [ $T_+(K)$ ] (it is easy to see that this does indeed depend on  $K$  only). Let  $K^0$  denote the convex hull of 0-1 solutions of  $F$ . Then it follows from these considerations that

$$K^0 \subseteq N_+(K) \subseteq N(K) \subseteq K.$$

and that iterating the  $N$  operator  $n$  times, we get down to  $K^0$ .

We define the  $N$ -index of a linear inequality valid for  $\text{STAB}(G)$  as the least  $k$  for which it belongs to  $T^k$  (6.3, 6.4). The  $N_+$ -index is defined analogously. It follows that every inequality has a  $N$ -index at most  $n$ ; but in fact, the  $N$ -index of an inequality is usually much smaller.

**6.9 Theorem.** *The  $N$ -index of an inequality is at most its Gallai class number. The  $N$ -index of an inequality is 1 iff it is a non-negative combination of odd hole constraints. The  $N_+$ -index of an odd antihole constraint is 2. The  $N$ -index of a clique constraint involving  $k$  nodes is  $k - 2$ .*

**6.10 Corollary.** *A graph is  $t$ -perfect if and only if  $T(F)$  defines its stable set polytope, where  $F$  consists of the edge constraints.*

The  $T_+$  procedure generates inequalities even faster.

**6.11 Theorem.** *Clique, orthogonality, odd hole, and odd antihole constraints have  $N_+$ -index 1.*

It follows that if  $G$  is perfect then  $\text{STAB}(G) = N_+(\text{FRAC}(G))$ ; if  $G$  is  $t$ -perfect then  $\text{STAB}(G) = N(\text{FRAC}(G))$ .

It is also worth noticing that  $N(K)$  is the projection of a convex set in the  $\binom{n}{2} + n$  dimensional space, obtained as the solution set of the following inequalities: take all quadratic inequalities generated from  $F$  by multiplying by  $x_i$  or  $1 - x_i$ , and replace the products  $x_i x_j$  by a new variable  $y_{ij}$ . So if  $|F| = m$  then  $N(K)$  is the projection of a polytope defined by at most  $2nm$  inequalities. It follows in particular:

**6.12 Corollary.** *If  $G$  is  $t$ -perfect then  $\text{STAB}(G)$  can be obtained as the projection of a polytope with  $O(mn)$  facets.*

It is also interesting to remark that the convex (but generally non-polyhedral) set  $\text{TH}(G)$  is just the solution set of all linear inequalities derived from (6.1), (6.2) and all inequalities  $\ell^2 \geq 0$ , where  $\ell$  is a linear form.

An important feature of the  $N$  and  $N_+$ -operators is that they preserve algorithmic "niceness". Stating this somewhat imprecisely, if every linear objective function can be optimized over  $K$  in polynomial time, then the same is true for  $N(K)$  and  $N_+(K)$  (with a different polynomial in the time bound). In particular, the following holds:

**6.13 Theorem.** *Let  $c > 0$  be any constant. Then for the class of graphs  $G$  such that  $\text{STAB}(G)$  can be described by inequalities with  $N_+$ -index at most  $c$ , the value  $\alpha(G)$  is polynomial time computable.*

The theorem applies in particular to perfect,  $t$ -perfect and  $h$ -perfect graphs. It also applies to graphs whose stable set polytope can be defined by constraints with bounded Gallai class number. We remark that the class of graphs whose stable set polytope can be defined by rank constraints with bounded Gallai class number was studied in detail by Sewell (1990).

Let  $G = (V, E)$  be a graph and let  $I(G)$  denote the polynomial ideal generated by the polynomials  $x_i^2 - x_i$  ( $i \in V$ ) and by the polynomials  $x_i x_j$  ( $ij \in E$ ). Consider the quotient ring  $R = \mathbb{R}[x_1, \dots, x_n]/I(G)$  (this is related to the so-called *Reisner–Stanley ring* of the graph; cf. Stanley 1983). We can introduce an order on polynomials by writing  $f \geq 0 \pmod{I(G)}$  iff  $f(x) \geq 0$  for every root of the ideal  $I(G)$ ; clearly, this induces an order on the quotient ring. It is not difficult to prove the following:

**6.14 Lemma.** *For any polynomial  $f$ , we have  $f \geq 0 \pmod{I(G)}$  iff there exist polynomials  $g_1, \dots, g_N$  such that  $f \equiv g_1^2 + \dots + g_N^2 \pmod{I(G)}$ .*

In fact, this holds for every ideal with a finite number of zeros. But the following theorem shows an interesting connection between this property and the perfectness of a graph:

**6.15 Theorem.** *A graph  $G$  is perfect if and only if the following holds:*

(\*) *For any linear polynomial  $f$ , we have  $f \geq 0 \pmod{I(G)}$  iff there exist linear polynomials  $g_1, \dots, g_N$  such that  $f \equiv g_1^2 + \dots + g_N^2 \pmod{I(G)}$ .*

The proof of this theorem follows from the characterization of perfectness in terms of the body  $\text{TH}(G)$ . One could formulate analogous characterizations of  $t$ -perfect graphs using Corollary 6.10. It would be interesting to know which polynomial ideals in general have property (\*).

The method sketched here is not restricted to the stable set problem; in fact, it can be applied to any 0-1 optimization problem. Moreover, it can be extended from quadratic to higher-order inequalities. For these extensions, see Lovász and Schrijver (1989, 1990) and Sherali and Adams (1989).

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