CONGRESSUS NUMERANTIUM VIII

Proceedings of the Fourth Southeastern Conference On Combinatorics, Graph Theory, And Computing

Florida Atlantic University
Boca Raton
March 5-8, 1973

Coverings and colorings of hypergraphs*

L. Lovász

A *hypergraph* H is a non-empty finite system of non-empty finite sets, called *edges*. The union of edges, denoted by $V(H)$, is the set of *vertices*. A hypergraph is *r-uniform* if each edge has cardinality r. A 2-uniform hypergraph is a graph.

We are interested in three main invariants of hypergraphs: The maximum number $\nu(H)$ of disjoint edges, the minimum number $\tau(H)$ of points to cover all edges, and the minimum number $\chi(H)$ of colors to color the points in such a way, that no edge is contained in any color class.

For graphs, all these numbers have been subject to thorough investigation and at least for the matching number $\nu(H)$, a satisfactory theory has been established.

For hypergraphs in general, there is no hope to obtain a useful formula for $\nu(H)$ or $\tau(H)$; this would contain necessary and sufficient conditions for the existence of Hamilton circuit, k-colorability and

*Research supported in part by NSF GP-29129.

PROC. 4TH S-E CONF. COMBINATORICS, GRAPH THEORY, AND COMPUTING, pp. 3-12.

other hard problems, which almost surely have no solution of this form (at least for beyond our present state of knowledge). See Karp [1].

However, it is very easy to determine whether or not a graph is 2-colorable. For hypergraphs, the situation changes drastically. We show that deciding whether $\chi(H) = 2$ is, in general, as hard as to determine the chromatic number.

Theorem 1. If there is an efficient algorithm (i.e. an algorithm of length $\le \max\{|H|^c, |V(H)|^c\}$ for some fixed c) to determine whether or not H is 2-colorable then there is an efficient algorithm to compute chromatic number.

Conversley, if there is an efficient algorithm to decide if a graph G is 3-colorable then there is an efficient algorithm to decide if H is 2-colorable.

Corollary. If there is an efficient algorithm to decide if a graph is 3-colorable, there is one to compute chromatic number.

Proof. I. Let G be a graph, $V(G) = \{x_1, \dots, x_n\}$. Let G_i be an isomorphic copy of G, $(i = 1, \dots, k)$,

$V(G_i) = \{x_{i1}, \dots, x_{in}\}$ ($x_{i\nu}$ is the point corresponding to x_ν). Take, moreover, a new point y, and let $f_\nu = \{x_{1\nu}, \dots, x_{k\nu}, y\}$. Define a hypergraph H by

$$H = E(G_1) \cup \dots \cup E(G_k) \cup \{f_1, \dots, f_n\}$$

Then H is 2-colorable if and only if G is k-colorable. Moreover, H can be computed from G efficiently, even with a bound on length uniform in k (for $k \le n$, which we may suppose).

II. Suppose we have an efficient algorithm to determine whether a graph G is 3-colorable. Let us be given a hypergraph H. For each $e \in E(H)$, take an odd circuit C_e of length $\ge |e|$; let these circuits be disjoint of each other and $V(H)$. Define G by

$$V(G) = \bigcup_{e \in E(H)} V(C_e) \cup V(H) \cup \{y\},$$

and join

(a) two points of $V(C_e)$ if they are adjacent in C_e;

(b) the points of C_e to the points of e in such a way that each point in C_e be adjacent with exactly one point of e and each point of e be adjacent with at least one point of C_e.

Then G is 3-colorable if and only if H is 2-colorable. This proves the theorem.

Theorem 1 justifies that we look for sufficient (and some necessary) conditions for 2-colorability of hypergraphs instead of trying to find necessary and sufficient conditions. There are different kinds of sufficient conditions, like

Theorem 2 [Las Vergnas-Fournier]. Let H be a hypergraph such that, whenever $E_1, \dots, E_{2k+1} \in H$ and $E_i \cap E_{i+1} \neq 0$ $(i = 1, \dots, 2k)$, $E_{2k+1} \cap E_1 \neq 0$ then there is a point belonging to three of $E_1, \dots, E_{2k+1}$. Then H is 2-chromatic.

This result says that a hypergraph without odd cycles (i.e. the exact sense stated) is 2-chromatic; it generalizes several earlier results.

Theorem 3 [Lovász 2]. If a hypergraph has the property that the union of any k edges has cardinality $\geq k + 1$ then it is 2-chromatic.

Woodall constructed 3-chromatic r-uniform hypergraphs with the property that the union of any k edges has $\geq k$ elements. Another example is the hypergraph consisting of the lines of a 7-point plane.

Theorem 4. Let H be a 3-uniform hypergraph, $|V(H)| = n$, and suppose there is a number α such that each pair of points is contained in $\geq \alpha$ but $<(2 - \frac{4}{n})\alpha$ edges of H. Then H is not 2-colorable.

Proof. Suppose there is a 2-coloration of H with color classes S_1, S_2. Set $|S_i| = n_i$. Each pair of points of the same S_i is contained in $\geq \alpha$ triples and each triple is counted once, hence

$$|H| \geq \alpha\left[\binom{n_1}{2} + \binom{n_2}{2}\right]$$

On the other hand, each pair (x,y) with $x \in S_1$, $y \in S_2$ is contained in $<(2 - \frac{4}{n})\alpha$ triples of H and each triple is counted twice. Hence,

$$|H| < \frac{1}{2}(2 - \frac{4}{n})\alpha\, n_1 n_2$$

Thus

$$\alpha\left[\binom{n_1}{2} + \binom{n_2}{2}\right] < (1 - \frac{2}{n})\alpha\, n_1 n_2 ,$$

or

$$1 - \frac{2}{n} > \frac{\binom{n_1}{2} + \binom{n_2}{2}}{n_1 n_2} \geq \frac{\binom{\frac{n}{2}}{2} + \binom{\frac{n}{2}}{2}}{\frac{n}{2} + \frac{n}{2}} = 1 - \frac{2}{n} ,$$

a contradiction.

Let us turn now to the question of connection between $\tau(H)$, $\nu(H)$ and 2-colorability. For graphs, these concepts are linked by König's theorem: A bipartite graph has $\tau(H) = \nu(H)$. This is not true for 2-colorable hypergraphs, and it seems quite probable that there is no other minimax theorem for this class of hypergraph. However, certain stronger colorability assumptions imply this equality.

To formulate these results let us introduce the following notions: A <u>partial hypergraph</u> H' of H is a subsystem of H. A <u>subhypergraph</u> determined by $X \subseteq V(H)$ is the hypergraph

$$H_X = \{e \cap X \; ; \; e \in H \; , \; |e \cap X| \geq 2\} .$$

<u>Theorem 5</u> [Berge-Las Vergnas 3]. The following properties of a hypergraph H are equivalent:

(a) each subhypergraph is 2-colorable

(b) whenever $x_1, \ldots, x_{2k+1}$ are distinct points and $E_1, \ldots, E_{2k+1}$ are distinct edges such that $x_i \in E_i$, $x_i \in E_{i+1}$, $x_{2k+1} \in E_1$ then one of the E_i's contains at least three x_j's ;

(c) $\tau(H') = \nu(H')$ for every partial subhypergraph of H.

<u>Theorem 6</u> [Lovász]. The following two properties of a hypergraph are equivalent:

(a) each partial hypergraph has equal chromatic index and maximum degree;

(b) each partial hypergraph has $\tau(H') = \nu(H')$.

It follows from Theorem 2 that these hypergraphs are 2-colorable. Theorem 6 is equivalent to the fact that the complement of a perfect graph is perfect, conjectured by Berge. There must be other interesting classes of hypergraphs satisfying $\tau(H) = \nu(H)$.

It is a surprising observation, due to Erdős, that while Theorems 5 and 6 say that "$\nu(H)$ is big" implies H is 2-colorable, $\nu(H) = 1$ will also imply this in most cases; more exactly, if a hypergraph has $\nu(H) = 1$

and is not 2-colorable, it must have rather strict properties. Erdős and I would like to discuss these questions in a forthcoming paper so I restrict myself to showing a few examples and mentioning some results. For brevity, call a hypergraph "strange" if $\nu(H) = 1$, i.e. any two edges intersect and $\chi(H) \geq 3$.

The seven-point plane and the hypergraph consisting of all r-tuples chosen from $2r - 1$ points are, obviously, strange. The following example is less trivial: Let $V = A_1 \cup A_2 \cup \ldots \cup A_r$, where $|A_i| = i$. Take as edges all r-tuples of the following form: we choose a whole A_i and one point from each A_j with $j > i$.

<u>Theorem 7</u>. An r-uniform strange hypergraph has $\leq r^r$ edges.

<u>Theorem 7'</u>. An r-uniform hypergraph with more than r^r edges and $\nu(H) = 1$ has $\tau(H) \leq r - 1$.

As the strange hypergraph constructed above has $[(e - 1)r!]$ edges, the upper bound given in Theorem 7 is not very far from best possible.

<u>Theorem 8</u>. Any strange r-uniform hypergraph has two edges with $\geq c\frac{r}{\log r}$ points in common.

There exists an r-uniform strange hypergraph such that any two edges have an odd number of points in common. On the other hand, there is always a pair of edges with one common point and at least two other numbers occur as cardinalities of intersections of edges if r is large enough. It would be interesting to know more about which cardinalities have to occur among intersections of edges in a strange hypergraph.

References

[1] R.M. Karp, Reducibility among combinatorial problems, Technical Report, UC Berkeley, 1972.

[2] L. Lovász, Graphs and set-systems, Beiträge zur Graphentheorie (Leipzig 1968), 99-106.

[3] C. Berge - M. Las Vergnas, Sur un theoreme du type König pour hypergraphes, Int. Conf. on Comb. Math., New York, 1970, 32-40.

[4] L. Lovász, Normal hypergraphs and the perfect graph conjecture, Discrete Mathematics 2, (1972), 253-267.