Acta Mathematica Academiae Scientiarum Hungaricae
Tomus 22 (1—2), (1971), pp. 227—231

ON FINITE DIRICHLET SERIES

By

L. LOVÁSZ (Budapest)

To Professor P. TURÁN *on his 60th birthday*

A *finite Dirichlet series* (abreviated by FDS in the sequel) is a function of the form

$$f(s) = \sum_{i=1}^{n} a_i i^s \qquad (a_i \text{ integer})$$

(we shall not use analitical properties of these; it is enough to consider them for integral s). Such a function is an analogue of a polynomial and several properties of polynomials are valid for such functions too. We are going to list such properties here — from a number-theoretical point of view.

Obviously, sum, difference and product of FDSs are FDSs too. Thus, the FDSs form a ring. As MCKENZIE mentioned[1], unique prime factorization holds in this ring. To show this, let p_i denote the i^{th} prime number. Every FDS can be written uniquely in the following form:

$$f(s) = \sum_{1 \leqq i_1 < \cdots < i_\nu} a_{i_1, \dots, i_\nu} p_{i_1}^s p_{i_2}^s \cdots p_{i_\nu}^s.$$

From this we can deduce that the ring of FDSs is isomorphic with the ring of polynomials with infinitely many variables (over the ring of integers). It is well known that prime factorization is unique in this latter ring.

An FDS is called *primitive*, if its coefficients are coprime. The following analogue of GAUSS' lemma can be proved just in the same way as for polynomials:

The product of primitive FDSs is primitive.

The main purpose of this paper is to investigate some connections between the properties of FDSs and the number-theoretical properties of their values for different values of s. If $f(s)$ is an FDS and s_0 is a non-negative integer then $f(s_0)$ is an integer.

THEOREM 1. *Assume that* $f(s) = \sum_{i=1}^{n} a_i i^s$, $g(s) = \sum_{i=1}^{m} b_i i^s$ $(a_n, b_m \neq 0)$ *are FDSs such that*

$$f(s) \mid g(s)$$

for every $s \geqq 0$. *Then* $n \mid m$ *and*

$$h(s) = \left(\frac{m}{n} - 1\right)! \frac{g(s)}{f(s)}$$

is an FDS.

[1] Oral communication

PROOF: Put

$$u(s) = -\sum_{i=1}^{n-1} \frac{a_i}{a_n}\left(\frac{i}{n}\right)^s$$

and let p be a sufficiently large but fixed integer. Then

$$\frac{g(s)}{f(s)} = \frac{g(s)}{a_n \cdot n^s}\cdot\frac{1}{1-u(s)} = \left\{\sum_{j=0}^{p}(u(s))^j\frac{g(s)}{a_n\cdot n^s}\right\} + \frac{(u(s))^{p+1}g(s)}{a_n\cdot n^s(1-u(s))} = \left\{\sum_{i=1}^{N}\beta_i\gamma_i^s\right\} + R(s)$$

where N is some positive integer, β_i and γ_i are rationals $(1 \leqq i \leqq N)$, $\gamma_i > 0$ and for $R(s)$ we have

$$|R(s)| \leqq \left(\sum_{i=1}^{n-1}\left|\frac{a_i}{a_n}\right|\right)^{p+1}\left(\sum_{i=1}^{m}|b_i|\right)\frac{1}{|a_n|}\cdot\frac{1}{|1-u(s)|}\cdot\left[\left(\frac{n-1}{n}\right)^{p+1}\frac{m}{n}\right]^s.$$

This tends to 0 if $s \to 0$ and p is large enough.

Now let $\sum_{j=0}^{N} c_j x^j$ be a polinomial with integral coefficients and roots $\gamma_1, \dots, \gamma_N$. Then

$$\sum_{j=0}^{N} c_j \sum_{i=1}^{N}\beta_i\gamma_i^{s+j} = \sum_{i=1}^{N}\beta_i\gamma_i^s\sum_{j=0}^{N}c_j\gamma_i^j = 0.$$

and hence

$$\sum_{j=0}^{N} c_j\frac{g(s+j)}{f(s+j)} = \sum_{j=0}^{N} c_j R(s+j).$$

The right-hand side tends to 0, while the left-hand side is always an integer. This implies that both vanish for $s \geqq s_0$.

We use now the following well-known theorem: if $\eta(s)$ is a function defined on the set of integers $s \geqq s_0$ and it satisfies the linear difference-equation

$$\sum_{j=0}^{M} d_j\eta(s+j) = 0$$

where the roots $\varrho_1, \dots, \varrho_M$ of the polynomial $\sum_{j=0}^{M} d_j x^j$ are different then

$$\eta(s) = \sum_{i=1}^{M}\alpha_i\varrho_i^s$$

with some coefficients α_i. The application of this theorem yields

$$g(s) = f(s)\sum_{i=1}^{N}\alpha_i\gamma_i^s \tag{1}$$

with some coefficients α_i, $s \geqq s_0$. But it is easy to see that if (1) holds for every $s \geqq s_0$, then it holds formally, i.e. it holds for every $s \geqq 0$.

We have still to show that γ_i and $\left(\frac{m}{n}-1\right)!\ \alpha_i$ are integers. Let $\varphi_i(x) = \sum_{j=0}^{N-1} d_j x^j$ be a polynomial with integral coefficients and with roots γ_ν $(\nu \neq i)$. Then

$$\sum_{j=0}^{N-1} d_j \frac{g(s+j)}{f(s+j)} = \alpha_i \gamma_i^s \varphi_i(\gamma_i) \tag{2}$$

is an integer for every $s \geqq 0$. Hence γ_i is an integer, $n|m$ and

$$\frac{g(s)}{f(s)} = \sum_{i=1}^{m/n} \alpha_i' i^s$$

Put

$$\psi_i(x) = \prod_{\substack{\nu=1 \\ \nu \neq i}}^{m/n} (x-\nu) = \sum_{j=0}^{m/n} e_j x^j.$$

then

$$\sum_{j=0}^{m/n} e_j \frac{g(s+j)}{f(s+j)} = (i-1)!\left(\frac{m}{n}-i\right)!\, \alpha_i' \cdot i^s.$$

Hence $(i-1)!\left(\frac{m}{n}-i\right)!\ \alpha_i'$ is an integer, consequently $\left(\frac{m}{n}-1\right)!\ \alpha_i' = \binom{\frac{m}{n}-1}{i}(i-1)!$ $\left(\frac{m}{n}-i\right)!\ \alpha_i'$ is an integer too, which proves the theorem.

Combining the analogue of GAUSS' lemma and theorem 1, we obtain

COROLLARY: *If $f(s)$, $g(s)$ are FDSs and $f(s)$ is primitive, furthermore $f(s)|g(s)$ for every $s \geqq 0$, then $\frac{g(s)}{f(s)}$ is an FDS.*

If $f(s)$ is not primitive, then the factor $\left(\frac{m}{n}-1\right)!$ cannot be omitted or replaced by a smaller one in general. Put

$$f(s) = (N-1)!\, h(s), \qquad g(s) = \left\{\sum_{i=1}^{N} (-1)^i \binom{N-1}{i-1} \cdot i^s\right\} h(s)$$

where $h(s)$ is primitive, then one can check easily that $f(s)|g(s)$ for every $s \geqq 0$, on the other hand $g(s)$ is primitive, and thus $K\frac{f(s)}{g(s)}$ has integral coefficients only if $(N-1)!|K$.

We prove another theorem which is an analogue of the theorem of BRISSE [1] and JENTZSCH [2] for polynomials.

THEOREM 2. *Let $f(s) = \sum_{i=1}^{n} a_i \cdot i^s$ be an FDS and assume that $f(s)$ is the k^{th} power of an integer for every $s \geqq 0$. Then $f(s)$ is the k^{th} power of an FDS.*

PROOF: Put

$$m = \sqrt[k]{n}, \quad b = \sqrt[k]{a_n}, \quad \varepsilon_\mu = e^{\frac{2\pi i\mu}{k}}, \quad u(s) = \sum_{i=1}^{n-1} \frac{a_i}{a_n} \cdot \left(\frac{i}{n}\right)^s.$$

Now $u(s) \to 0$ if $s \to \infty$, hence $|u(s)| < \frac{1}{2}$ for $s \geqq s_0$ and the Newton formula valid:

$$\sqrt[k]{f(s)} = b \cdot m^s \sqrt[k]{1+u(s)} = \sum_{j=0}^{\infty} \binom{\frac{1}{k}}{j} u(s)^j \cdot b \cdot m^s.$$

Let p be a sufficiently large fixed integer and put

$$R_1(s) = \sum_{j=p+1}^{\infty} \binom{\frac{1}{k}}{j} u(s)^j b \cdot m^s.$$

Then

$$|R_1(s)| \leqq \sum_{j=p+1}^{\infty} |u(s)|^j b \cdot m^s \leqq 2 \cdot b \cdot \left(\sum \left|\frac{a_i}{a_n}\right|\right)^{p+1} \left[\left(\frac{n-1}{m}\right)^{p+1} m\right]^s$$

since

$$\left|\binom{\frac{1}{k}}{j}\right| \leqq 1.$$

Thus, if p is sufficiently large, $R_1(s) \to 0$ for $s \to \infty$. On the other hand,

$$\sum_{j=0}^{r} \binom{\frac{1}{k}}{j} u(s)^j b \cdot m^s = \sum_{i=1}^{N} b \cdot \alpha_i (m\gamma_i)^s + \sum_{i=1}^{N'} b \cdot \beta_i (m\delta_i)^s = Q(s) + R_2(s),$$

where $m\gamma_i \geqq 1$, but $m\delta_i < 1$, and α_i, β_i, γ_i, δ_i are rationals. Put

$$R(s) = R_1(s) + R_2(s)$$

and

$$\varphi(x) = A \prod_{i=1}^{N} (x^k - \gamma_i^k n) = \sum_{j=0}^{kN} c_j x^j$$

where A is chosen so that the c_j-s are integers. Then

$$\sum_{j=0}^{kN} c_j \sqrt[k]{f(s+j)} = \sum_{j=0}^{kN} c_j R(s+j) \to 0 \qquad (s \to \infty).$$

Hence

$$\sum_{j=0}^{kN} c_j R(s+j) = 0 \qquad (s \geqq s_1).$$

As in the proof of Theorem 1, this implies

$$R(s+j) = \sum_{\nu=1}^{N} \sum_{\mu=1}^{k} \varrho_{\nu\mu}(m_\nu \varepsilon_\mu)^s.$$

Put for a given pair μ, ν

$$\psi(x) = \frac{\varphi(x)}{x - m\gamma_\nu \varepsilon_\mu} = \sum_{i=0}^{kN-1} d_i x^i.$$

Then

$$\sum_{i=0}^{kN-1} d_i R(s+i) = \varrho_{\nu\mu}\psi(m\gamma_\nu \varepsilon_\mu)[m\gamma_\nu \varepsilon_\mu]^s$$

which can tend to 0 only if $\varrho_{\gamma\mu} = 0$. This shows that for $s \geqq s_1$ $R(s) = 0$, i.e.

$$\sqrt[k]{f(s)} = \sum_{i=1}^{N} b_i \alpha_i (m\gamma_i)^s \qquad (s \geqq s_1)$$

Putting here $s' = ks_1$, $s'' = ks_1 + 1$:

$$\sqrt[k]{f(s')} = bn^{s_1} \sum_{i=1}^{N} \alpha_i \gamma_i^{s'}, \qquad \sqrt[k]{f(s'')} = bmn^{s_1} \sum_{i=1}^{N} \alpha_i \gamma_i^{s''},$$

from which it follows that b and m are rationals, consequently integers. From this we can deduce similarly to the proof of Theorem 1 that, choosing the indeces appropriately,

$$\sqrt[k]{f(s)} = \sum_{i=1}^{N} b\alpha_i i^s.$$

The analogue of GAUSS' lemma gives now that $b\alpha_i$ is an integer. This proves Theorem 2.

(Received 27 October 1969)

EÖTVÖS LORÁND TUDOMÁNYEGYETEM
GEOMETRIA TANSZÉK
BUDAPEST, VIII., MÚZEUM KRT. 6—8.

References

[1] CH. BRISSE, Probleme, *Intermedière des mat.*, **1** (1894), p. 10. Solution: J. Franel, **2** (1895), p. 94.
[2] R. JENTZSCH, Aufgabe, *Archiv d. Math. u. Phys.*, Series 3, **19** (1912), p. 361. Solution: W. Grosch, **21** (1913), p. 361.