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A HOMOLOGY THEORY FOR SPANNING TREES OF A GRAPH

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1. Results. The following conjecture has been formulated and partially solved by A. FRANK [2]:

THEOREM 1. *Given a k-connected graph G, k points $v_1, \dots, v_k \in V(G)$, and k positive integers $n_1, \dots, n_k$ such that $n_1 + \dots + n_k = |V(G)|$, there exists a partition $\{V_1, \dots, V_k\}$ of $V(G)$ such that $v_i \in V_i$, $|V_i| = n_i$ and each V_i spans a connected subgraph.*

A very closely related conjecture has been made by S. MAURER. A. FRANK [2] proved the above statement for the case $k=2$ and also when $n_1, \dots, n_{k-1} \leqq 3$. His proof also provides an efficient algorithm to find this partition. K. MILLIKEN proved the case $k=3$ and a generalization to infinite graphs (private correspondence). Theorem 1 has been proved, independently of the author, by E. GYŐRY. His proof uses more elementary methods [3].

The case $k=2$ also follows from an unpublished lemma of A. J. BONDY and the author:

THEOREM 2. *Given a 2-connected graph G on n points, with a specified point a. Call two spanning trees of G neighbouring if they have a common subtree with $n-1$ points, including a. Then any two spanning trees of G can be connected by a chain of spanning trees, in which any two consecutive members are neighbouring.*

This theorem is a strengthening of a well-known theorem of Whitney. It was used to show the following

THEOREM 3. (A. J. BONDY, L. LOVÁSZ, unpublished). *If G is a 2-connected, non-bipartite graph on $2n$ vertices then $V(G)$ can be partitioned into two n-element classes such that the edges connecting these classes form a connected spanning subgraph. In other words, there is a spanning tree T such that the two colour-classes in the (unique) 2-colouration of F have both n elements.*

Theorem 2 above gives the idea that to attack the general problem formulated in Theorem 1, one might try to generalize its contents to k-connected graphs. In order to do so we introduce a few notions.

Let G be a digraph and $a \in V(G)$. Assume there are spanning arborescences of G rooted at a (since all arborescences considered will be rooted at a, we shall not say this explicitly in the sequel). Set $m = |E(G)|$. Then each spanning arborescence of G can be regarded as a vertex of the m-dimensional hypercube. It will cause no confusion if we denote a spanning arborescence and its representing point by the same letter.

We define certain convex cells on this set of vertices. Take an arborescence $A\subset G$, and let $V(G)-V(A)=\{x_1, \dots, x_r\}$. Assume that each point x_i can be reached on at least two edges from A, and take, for each x_i, a set N_{x_i} of at least two edges connecting A to x_i. Denote by $C(A; N_{x_1}, \dots, N_{x_r})$ the convex hull of all spanning arborescences which arise from A by adding one line of each N_{x_i}. The set of these arborescences can be written formally as

$$\{A\}\times N_{x_1}\times\dots\times N_{x_r}$$

and therefore, $C(A; N_{x_1}, \dots, N_{x_r})$ is the cartesian product of r simplices of dimension $|N_{x_1}|-1, \dots, |N_{x_r}|-1$, respectively. This also implies that the faces of these convex cells are of the same form.

The 1-dimensional skeleton of the arborescence complex is the graph whose vertices are the spanning arborescences, two of them being adjacent iff they are "neighbouring" in the sense of Theorem 2. The 2-dimensional cells of $\mathscr{K}$ are of two kinds: triangles spanned by three spanning arborescences containing a common $(n-1)$-point arborescence, and parallelograms, whose vertices are obtained as follows: we take an $(n-2)$-point arborescence A and two edges connecting it to each of the remaining two points, and select one of the two edges at each of these two points in all possible ways.

Also, if $C=C(A, N_{x_1}, \dots, N_{x_r})$ and $C'=C(A', N'_{y_1}, \dots, N'_{y_p})$ are two of these convex cells then so is their intersection, if non-empty. Since each point of C must have 1 for each edge of A and similarly, each point of C' must have 1 for each edge of A', it follows that each point $\omega\in C\cap C'$ must have 1 for each coordinate of $A\cup A'$. Hence $A\cup A'$ must be an arborescence. Let $V(G)-V(A\cup A')=\{z_1, \dots, z_t\}$. For each z_i, at least one edge of G with head in z_i must occur in ω with positive weight. This edge belongs then to $N''_{z_i}=N_{z_i}\cap N'_{z_i}$. Without loss of generality we may assume therefore that

$$|N''_{z_1}|, \dots, |N''_{z_q}| > 1, \quad |N''_{z_{q+1}}| = \dots = |N''_{z_t}| = 1.$$

Denote by A'' the arborescence $A\cup A'\cup N''_{z_{q+1}}\cup\dots\cup N''_{z_t}$, and set

$$C'' = C(A''; N''_{z_1}, \dots, N''_{z_q}).$$

Then it is easy to see that

$$C\cap C' = C''.$$

This proves that the collection of all convex cells of the form $C(A; N_{x_1}, \dots, N_{x_r})$ form a cellular complex $\mathscr{K}$ (see e.g. ALEXANDROFF—HOPF [1]). We call $\mathscr{K}$ the *arborescence complex* of G (relative to a).

We shall set

$$\tau(C) = A \quad \text{if} \quad C = C(A; N_{x_1}, \dots, N_{x_r}).$$

We shall use the following notations: $L^r(\mathscr{K})$ denotes the group of r-dimensional chains of the complex $\mathscr{K}$, $H^r(\mathscr{K})$ is the r-dimensional reduced homology group. $\sim$ denotes homology of chains.

If C is a convex polytop then $[C]$ denotes its combinatorial hull, i.e. the cellular complex formed by its faces. $|\mathscr{K}|$ is the body of the complex $\mathscr{K}$.

Also recall one definition from graph theory: a set X of points is said to *separate* point b from point a if $a, b\notin X$ and every directed path from a to b meets X.

Now we are able to formulate the main result of this paper:

THEOREM 4. *Let G be a digraph, $a\in V(G)$ and assume that no point can be separated from a by less than k points ($k\geqq 2$). Then the arborescence complex $\mathscr{K}$ of G relative to a satisfies $H^0(\mathscr{K})=\dots=H^{k-2}(\mathscr{K})=0$.*

We shall give a separate, elementary proof of the fact that $H^0(\mathscr{K})=0$ (this is essentially Theorem 2). Also, instead of $H^1(\mathscr{K})=0$ we prove the following, somewhat stronger result:

THEOREM 5. *The fundamental group of $\mathscr{K}$ is 0.*

It is hoped that the proofs in these two low-dimensional cases will provide motivation for the rather technical proof of Theorem 4. For similar reasons we shall sketch the proof of Theorem 1 in case $k=3$ separately.

From Theorem 4 we shall deduce a digraph analogue of Theorem 1:

THEOREM 6. *Let G be a digraph, $v_1, \dots, v_k\in V(G)$ and assume that for each point $x\neq v_1, \dots, v_k$, there are k openly disjoint paths connecting $v_1, \dots, v_k$ to x. Let, furthermore, k positive integers $n_1, \dots, n_k$ be given whose sum is $|V(G)|$. Then G contains k vertex-disjoint arborescences $A_1, \dots, A_k$, such that A_i is rooted at v_i and $|V(A_i)|=n_i$.*

In fact, Theorem 5 implies Theorem 1: one just has to replace each unoriented edge by two, oppositely directed oriented edges. So we shall only deal with the proof of Theorem 5. Note that a similar trick would enable us to deduce Theorem 2 from Theorem 4, case $k=2$.

2. Proof of the connectivity of $\mathscr{K}$. Let B, B' be two spanning arborescences of the graph G; we are going to prove that there is a chain of spanning arborescences, which starts with B and ends with B' and in which any two consecutive members have an $(n-1)$-point common arborescence. Let A denote the largest common arborescence of B and B'; we use induction of $|V(G)-V(A)|$. If A has n or $n-1$ points the assertion is trivial, so suppose A has at most $n-2$ points.

Let $e=(u, v)$ and $e'=(u', v')$ be edges of B and B', respectively, such that $u, u'\in V(A)$ but $v, v'\notin V(A)$. We distinguish two cases.

Case 1. $v\neq v'$. In this case $A+e+e'$ is an arborescence, which can be completed to a spanning arborescence B''. Now by the induction hypothesis B and B'', as well as B'' and B', can be connected by chains of arborescences, which together yield a chain connecting B to B' as desired.

Case 2. $v=v'$. Since by hypothesis there is at least one more point outside A and also by the hypothesis of the theorem this cannot be separated from the root by v, there must be an edge $f=(x, y)$ with $x\in V(A)$ and $y\in V(G)-V(A)-\{v\}$. Consider the arborescences $A+e+f$ and $A+e'+f$. These can be completed to get two spanning arborescences B_1 and B_2. Then by the induction hypothesis we find chains of arborescences connecting B to B_1 to B_2 to B', which completes the proof.

3. Proof of Theorem 5. 1° Let $(B_0, B_1, \dots, B_p=B_0)$ be a sequence of spanning arborescences, such that B_{i-1} and B_i are equal or adjacent in $\mathscr{K}$, for all $i=1, \dots, p$. We are going to prove that there exists a cellular complex homomorphic to the 2-cell, bounded by a p-gon $(X_0, X_1, \dots, X_p=X_0)$, and with 3- and 4-lateral 2-dimensional cells, and a mapping of it into $\mathscr{K}$ such that X_i is mapped onto B_i and cell onto cell. If this is the case we say shortly that the cycle $(B_0, \dots, B_p)$ is *contractible*.

Let A denote the largest arborescence which is contained in all B_i; we use induction on $|V(G)-V(A)|$. If this number is 0, 1 or 2, the assertion is trivial.

2° Note that there is no common edge of $B_0, \dots, B_p$ except those in A. Assume e is such an edge. Let us consider the path P in B_0 connecting a to e and let f be the edge of P leaving A. Then if e is contained in all B_i, the whole path P must be contained in all B_i. But then $A+f$ is a larger common sub-arborescence than A.

3° Suppose now that there is an edge $e=(x, y)\in E(B_i)$ $(1\leqq i\leqq p)$ such that $x, y\notin V(A)$ and y can be reached from A on an edge $e'=(z, y)$ $(z\in V(A))$. Replace e by e' in each B_i which contains e. Let $B_0', B_1', \dots, B_p'$ denote the resulting sequence. It is straightforward to see that B_{i-1}' and B_i' are adjacent or equal for all i. Using induction e.g. on the sum of distances from a of all points and in all B_i, we may assume the cycle $(B_0', B_1', \dots, B_p')$ is contractible.

We may assume e.g. $e\notin B_0$. Consider all pairs $i\leqq j$ of indices such that $e\notin B_{i-1}$, $e\in B_i$, $e\in B_{i+1}, \dots, e\in B_j$, $e\notin B_{j+1}$. As in 2° it follows that there is an edge f, connecting A to $V(G)-V(A)$, which is contained in $B_i, B_{i+1}, \dots, B_j$. Then $A+f$ is a common subtree of the members of the cycle $(B_i, B_{i+1}, \dots, B_j, B_j', \dots, B_i', B_i)$ and therefore this cycle is contractible by induction. This implies that the cycle $(B_0, \dots, B_p)$ is contractible (see Fig. 1).

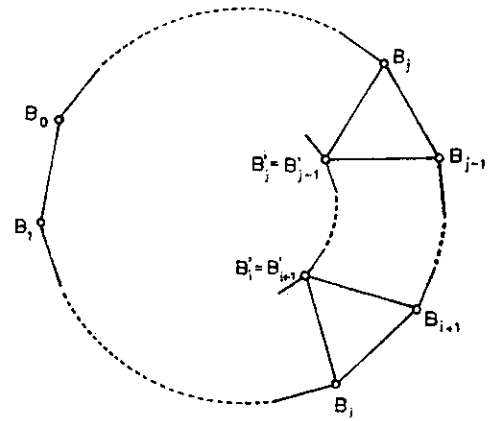

Fig. 1

4° So we may assume that if v is any point accessible from A on an edge then all B_i contain an edge connecting A to v (of course, different B_i may contain different edges of this type). Note that the connectivity assumption implies that there are at least three such points v.

Let $e=(u, v)$ be any edge with $u\in V(A)$, $v\notin V(A)$. Replace the (unique) edge of B_i entering v by e, to get a spanning arborescence B_i'. Then trivially B_{i-1}' and B_i' are adjacent vertices in $\mathscr{K}$. Moreover, the cycle $(B_0', B_1', \dots, B_p')$ is contractible, since its members contain the common arborescence $A+e$.

Let A_i be the largest common arborescence in B_i and B_i'. Let $C_{i,0}=B_i, C_{i,1}, \dots$ $\dots, C_{i,r_i}=B_i'$ be a chain of spanning arborescences, such that $C_{i,j-1}$ and $C_{i,j}$ are

adjacent for all $1\leqq j\leqq r_i$, and $A_i\subseteqq C_{i,j}$. Then the cycle $(C_{i,0}, C_{i,1}, \dots, C_{i,r_i}$, $C_{i-1,r_{i-1}}, \dots, C_{i-1,0}, C_{i,0})$ is contractible, since its members contain the common arborescence $A_i\cap A_{i-1}$, which is clearly still larger than A. But this proves that $(B_0, \dots, B_p)$ is contractible (Fig. 2).

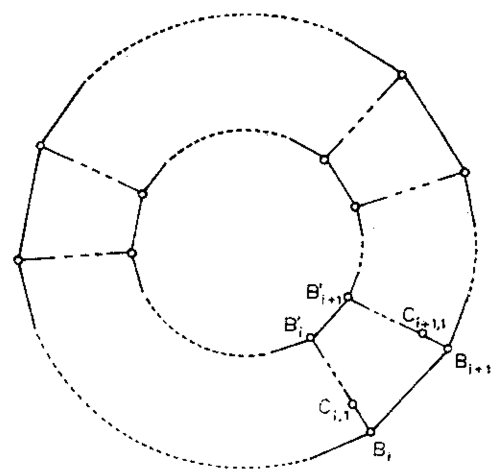

Fig. 2

4. Proof of Theorem 4. Let A be a (not necessarily spanning) arborescence rooted at a. Denote by $\mathscr{K}_A$ the subcomplex of $\mathscr{K}$ spanned by those vertices of $\mathscr{K}$ (spanning arborescences) which contain A. We prove by induction on $d=|V(G)-V(A)|$ and m that the m-dimensional homology group of $\mathscr{K}_A$ is 0 for $m=0, \dots, k-2$. For $d=0$ and $d=1$ the complexes in consideration consist of a single point and a single cell, respectively, so the assertion is true. So suppose $d\geqq 2$.

Let M be the set of vertices accessible from A on an edge. For each $x\in M$, let e_x be one of these edges.

Let B be any spanning arborescence containing A. For each $x\in M$, if the edge of B entering x does not start from A, replace it by e_x. Denote by $\omega(B)$ the resulting arborescence. It is easy to see that

$$B\mapsto \omega(B)$$

is an affine mapping of the n-dimensional space (it is simply the addition of all coordinates corresponding to edges entering x from a point outside A to the coordinate e_x, and then annulating these coordinates, for all $x\in M$). Also ω maps cells of $\mathscr{K}_A$ onto cells of $\mathscr{K}_A$. Thus we can define a homomorphism $\omega^r: L^r(\mathscr{K}_A)\to L^r(\mathscr{K}_A)$ for all r by

$$\omega^r(C^r) = \begin{cases} \omega(C^r) & \text{if } \omega \text{ does not degenerate on } C^r, \\ 0 & \text{otherwise.} \end{cases}$$

We need a

LEMMA. *Let*

$$\alpha\colon |\mathscr{K}_A| \to |\mathscr{K}_A|$$

be a continuous map which is affine on the cells. Define

$$\alpha^r\colon L^r(\mathcal{K}_A)\to L^r(\mathcal{K}_A),$$

as usual, by

$$\alpha^r(C^r)=\begin{cases}C^r, & \text{if } \alpha \text{ is not degenerate on } C^r,\\ 0, & \text{otherwise.}\end{cases}$$

Then there exist homomorphisms

$$\beta^r\colon L^r(\mathcal{K}_A)\to L^{r+1}(\mathcal{K}_A)$$

for all $r\leqq m-1$ *such that*

(1) *β is a deformation, i.e.*

$$\beta^{r-1}\delta^r x^r+\delta^{r+1}\beta^r x^r = x^r-\alpha^r x^r \quad (r\leqq m-1);$$

(2) *if C^r is a cell such that both $\tau(C^r)$ and $\tau(\alpha(C^r))$ contain an arborescence $A'\supseteqq A$ then so does each cell of $\beta^r C^r$;*

(3) *if both C^r and $\alpha(C^r)$ are faces of a cell C^q then each cell in $\beta^r C^r$ is a face of C^q. In particular, $\beta^r C^r=0$ if $\alpha(C^r)\subseteqq C^r$ $(r\leqq m-1)$.*

Proof of the Lemma. For $r<0$ we set $\beta^r=0$. Suppose β^{r-1} is defined, we define β^r.

We consider first a cell C^r. Set

$$d^r = C^r-\alpha^r C^r-\beta^{r-1}\delta^r C^r, \quad b^{r-1}=\delta^r C^r.$$

Then

$$\delta^r d^r = b^{r-1}-\alpha^{r-1}b^{r-1}-\delta^r\beta^{r-1}b^{r-1}.$$

Since β^{r-1} fulfills (1) by the induction hypothesis, it follows that this equals to

$$\beta^{r-2}\delta^{r-1}b^{r-1}=\beta^{r-2}\delta^{r-1}\delta^r C^r=0.$$

Thus d^r is a cycle. We are going to define $\beta^r C^r$ as an $(r+1)$-chain with

$$\delta^{r+1}\beta^r C^r = d^r;$$

then (1) will be satisfied for C^r. Such a chain $\beta^r C^r$ exists since $H^r(\mathcal{K}_A)=0$ by assumption.

To take care of (2) and (3) we need only to add a few remarks. Suppose both C^r and $\alpha(C^r)$ are contained in a cell C^q. Then, trivially, there is a unique minimal C^q with this property. By the induction hypothesis, also

$$\beta^{r-1}\delta^r C^r\in L^r([C^q]),$$

and hence

$$C^r-\alpha^r C^r-\beta^{r-1}\delta^r C^r\in L^r([C^q]).$$

Since $[C^q]$ is homologically trivial, the chain $\beta^r C^r$ can be chosen from $L^{r+1}([C^q])$. If we do so, condition (3) above will be satisfied.

Secondly, assume that both $\tau(C^r)$ and $\tau(\alpha(C^r))$ contain some arborescence $A'\supseteqq A$. There is a unique maximal arborescence A' with this property.

If C^r and $\alpha(C^r)$ have both been contained in some cell C^q then the minimal C^q with this property must belong to $\mathcal{K}_{A'}$. Then, however, we have already guaranteed

that

$$\beta^r C^r\in L^{r+1}([C^q])\subseteqq L^{r+1}(\mathcal{K}_{A'}).$$

So in this case (2) is automatically satisfied.

If this is not the case then, since we already know that $H^r(\mathcal{K}_{A'})=0$, the chain $\beta^r C^r$ can be chosen from $L^{r+1}(\mathcal{K}_{A'})$. This completes the proof of the Lemma.

Now we consider any chain $b^m\in L^m(\mathcal{K}_A)$ with $\delta^m b^m=0$. Set

$$b^m=\sum_{i=1}^N a_i C_i^m$$

where the C_i^m are cells. Let φ^r be the map β^r belonging to $\alpha=\omega$ in the Lemma. Also let

$$d_i^m = C_i^m-\omega^m C_i^m-\varphi^{m-1}\delta^m C_i^m.$$

Then $\delta^m d_i^m=0$ follows as in the proof of the Lemma. We show d_i^m is, in fact, a boundary.

We distinguish two cases:

Case 1. $\tau(C_i^m)=A$. Then ω is the identity map on C_i^m and hence, $d_i^m=0$ by (3) in the Lemma.

Case 2. $\tau(C_i^m)\supset A$. Let z be any point of $\tau(C_i^m)$ accessible from A on an edge $(w,z)\in\tau(C_i^m)$. Then the edge (w,z) is not altered by ω and hence, $A'=A+(w,z)$ is contained in both $\tau(C_i^m)$ and $\tau(\omega(C_i^m))$. But then each cell in $\varphi^{m-1}\delta^m C_i^m$ is contained in $\mathcal{K}_{A'}$, by property (3) in Lemma 2. Then, however, all cells of d_i^m are contained in $\mathcal{K}_{A'}$. By the induction hypothesis on A, we know $H^m(\mathcal{K}_{A'})=0$ and hence, d_i^m is a boundary in $\mathcal{K}_{A'}$; therefore it is a boundary in $\mathcal{K}_{A'}$.

So $d_i^m\sim 0$, and

$$0\sim\sum_{i=1}^N a_i d_i^m = b^m-\omega^m b^m-\varphi^{m-1}\delta^m b^m = b^m-\omega^m b^m,$$

whence $\omega^m b^m\sim b^m$. So it suffices to prove that $\omega^m b^m\sim 0$.

Note that each spanning tree of the form $\omega(B)$ necessarily contains at least one edge e_y entering y from A, for each $y\in M$. (Recall that M is the set of points accessible from A on an edge.) So if this edge e_y is unique, $\omega^m b^m$ is an m-chain in $\mathcal{K}_{A+e_y}$ and $\omega^m b^m\sim 0$ follows by the induction hypothesis.

So suppose that there are at least two edges connecting A to y for each $y\in M$.

If $M=V(G)-V(A)$ then all spanning trees of the form $\omega(B)$ belong to the same cell

$$C(A;S_y\colon y\in M)$$

where S_y is the set of edges connecting A to y. Thus $\omega^m b^m\sim 0$ follows from the fact that each convex cell is homologically trivial. Thus we may suppose that $M\neq V(G)-V(A)$. Since M separates a from any point of $V(G)-V(A)-M$, this implies $|M|\geqq k$.

We consider a map σ as follows. We select a point $x\in M$, and consider an edge e_x entering x from A. Given any spanning arborescence B, replace its (unique) edge entering x by the edge e_x. Denote by $\sigma(B)$ the resulting arborescence. Again, this mapping is affine when spanning arborescences are considered as points of the

m-dimensional space and maps cells onto cells. Let σ^r be defined similarly as ω^r and denote by ψ^r the deformation map β^r constructed in the Lemma when $\alpha=\sigma$. Let

$$\omega^m b^m = \sum_{j=1}^{M} e_j C_j^m,$$

where the C_j^m are cells. Then

$$f_j^m = C_j^m - \sigma^m C_j^m - \psi^{m-1}\delta^m C_j^m$$

satisfies $\delta^m f_j^m = 0$. We show that $f_j^m \sim 0$. Trivially

$$|V(\tau(C_j^m))| \geqq |V(G)|-m \geqq |V(G)|-k+2 \leqq |A|+2,$$

whence there are at least two edges f, f' leaving A which are common in all vertices of C_j^m. We may assume $f \neq e_x$. But then both C_j^m and $\sigma(C_j^m)$ are cells of $\mathscr{K}_{A+f}$. By property (2) of ψ, this implies that f_j^m is a chain in $\mathscr{K}_{A+f}$, and so $f_j^m \sim 0$ follows by the induction hypothesis.

Thus

$$0 \sim \sum_{j=1}^{m} f_j^m = \omega^m b^m - \sigma^m \omega^m b^m - \psi^{m-1}\delta^m \omega^m b^m = \omega^m b^m - \sigma^m \omega^m b^m.$$

Now $\sigma^m \omega^m b^m$ is a chain in $\mathscr{K}_{A+e_x}$. Thus

$$b^m \sim \omega^m b^m \sim \sigma^m \omega^m b^m \sim 0.$$

This completes the proof of Theorem 4.

5. The proof of Theorem 6 in the cases $k \leqq 3$. *Case* $k=2$. Take a new point a and connect it to v_1 and v_2. Let B_i be a spanning arborescence of the resulting graph G', rooted at a and not containing the edge (a, v_i). Consider a chain of spanning arborescences, linking B_1 and B_2 as constructed in Section 2. Consider the number of points of these arborescences on the branch over (a, v_1). This number changes by at most 1 at a time, and for B_1 it is 0 while for B_2 it is n. So there is a spanning arborescence for which it is n_1. Deleting a we obtain the pair of spanning arborescences, rooted at v_1 and v_2 as desired.

Case $k=3$ (sketch). Take a new point a and connect it to v_1, v_2, v_3. For each spanning arborescence B denote by $x_i(B)$ the number of points on the branch over (a, v_i), and let $\mathbf{x}(B)=(x_1(B), x_2(B))$.

Let H be the triangle $\{(x, y): x \geqq 0,\ y \geqq 0,\ x+y \leqq n\}$. Dissect each 4-lateral 2-dimensional cell of $\mathscr{K}$ into two triangles, to get a 2-dimensional simplical complex $\bar{\mathscr{K}}$. Then it is easily seen that $\mathbf{x}$ maps the vertices of a triangle of $\bar{\mathscr{K}}$ onto vertices of some empty lattice triangle. Extend $\mathbf{x}$ affinely over all simplices of $\bar{\mathscr{K}}$.

Let now B_i be a spanning arborescence (rooted at a) such that every point is above (a, v_i). Let $B_1=C_1, C_2, \dots, C_{k_1}=B_2$ be a chain of spanning arborescences in $G'-(a, v_3)$; let us define the chains $B_2=D_1, D_2, \dots, D_{k_2}=B_3$, $B_3=E_1, E_2, \dots, E_{k_3}=B_1$ analogously. The images of these chains by $\mathbf{x}$ give continuous curves connecting B_1 to B_2 to B_3 to B_1, which remain on the corresponding side of H. Therefore if we span a surface on the cycle in $\bar{\mathscr{K}}$ (which is possible by Theorem 5), the image of this surface covers the interior of H, in particular it contains the point (n_1, n_2). But the construction of the mapping $\mathbf{x}$ is such that if (n_1, n_2) is the image of something, it is necessarily the image of a vertex. This vertex is a spanning arborescence, from which we obtain the desired arborescences rooted at v_1, v_2 and v_3 by deleting a.

6. Proof of Theorem 6. Take a new point a and join it to $v_1, \dots, v_k$ by edges. It is clear that no point of the resulting digraph G can be separated from a by less than k points. Let $\mathscr{K}$ be the arborescence complex of G relative to a, and $\mathscr{K}'$ its $(k-1)$-dimensional skeleton. Subdividing each cell of $\mathscr{K}'$ into simplices whose vertices are also vertices of $\mathscr{K}$, we get a simplicial complex $\bar{\mathscr{K}}$.

For each spanning arborescence B, let $m_i(B)$ denote the number of vertices accessible from a on the edge (a, v_i). We want to show that there is a spanning arborescence B such that

$$m_i(B) = n_i \quad (i = 1, \dots, n). \tag{1}$$

Once such a spanning arborescence is found we are done since then the k components of $B-a$ are k disjoint arborescences rooted at $v_1, \dots, v_k$ and having the desired cardinalities.

Suppose indirectly no spanning arborescence satisfying (1) exists. Then for each spanning arborescence B there is an index $i(B)$, $1 \leqq i(B) \leqq k$ such that

$$m_{i(B)}(B) > n_{i(B)}. \tag{2}$$

Let $x_1, \dots, x_k$ be k affinely independent points in the $(k-1)$-dimensional space. Denote by S their convex hull and set $\mathscr{S}=[S]$. Define

$$\xi(B) = x_{i(B)}$$

and extend this to a simplicial map of $\bar{\mathscr{K}}$ into $\mathscr{S}$. This induces then homomorphisms $\xi^r: L^r(\bar{\mathscr{K}}) \to L^r(\mathscr{S})$ for each r.

We claim $\xi^{k-1}=0$. It suffices to show that no simplex of $\bar{\mathscr{K}}$ can be mapped onto S. Suppose indirectly $(B_1, \dots, B_k)$ were a simplex in $\bar{\mathscr{K}}$ with $\xi(B_i)=x_i$ $(i=1, \dots, k)$. Then $(B_1, \dots, B_k)$ is contained in a $(k-1)$-dimensional cell $C^{k-1}=C(A, N_{y_1}, \dots, N_{y_r})$ of $\mathscr{K}$. Let A_i be the branch of A above v_i $(i=1, \dots, k)$ and denote by a_i the number of edges in $N_{y_1} \cup \dots \cup N_{y_r}$, ending in A_i. Thus

$$\sum_{i=1}^{k} a_i = \sum_{i=1}^{r} |N_{y_i}| = k+r-1.$$

But $\xi(B_i)=x_i$ implies

$$n_i < m_i(B_i) \leqq |A_i|+a_i$$

and hence

$$\sum_{i=1}^{k} a_i \geqq \sum_{i=1}^{k} (n_i-|A_i|+1) = (n-1)-(n-r-1)+k = k+r,$$

a contradiction. So $\xi^{k-1}=0$.

Let C^r be a face of S, say

$$C^r = \text{convex hull of } (x_{i_0}, \dots, x_{i_r}).$$

Denote by $\mathscr{K}_{C^r}$ the subcomplex of $\mathscr{K}$ spanned by those vertices B in which each edge leaving a is one of $(a, v_{i_0}), \ldots, (a, v_{i_r})$, and let $\bar{\mathscr{K}}_{C^r}$ be obtained from $\mathscr{K}_{C^r}$ by the same subdivision as $\bar{\mathscr{K}}$ is obtained from $\mathscr{K}$. Then $\mathscr{K}_{C^r}$ is the arborescence complex of the $(r+1)$-connected graph $G+(a, v_{i_0})+\ldots+(a, v_{i_r})$, and therefore it satisfies

$$H^0(\mathscr{K}_{C^r}) = \ldots = H^{r-1}(\mathscr{K}_{C^r}) = 0$$

by Theorem 4. Thus also

$$H^0(\bar{\mathscr{K}}_{C^r}) = \ldots = H^{r-1}(\bar{\mathscr{K}}_{C^r}) = 0.$$

Let us define a homomorphism

$$\eta^r\colon L^r(\mathscr{S}) \to L^r(\bar{\mathscr{K}}) \quad (r \leqq k-1)$$

such that

$$\delta^r \eta^r = \eta^{r-1}\delta^r$$

and for each face C^r of S, $\eta^r C^r \in L^r(\bar{\mathscr{K}}_{C^r})$. We do this by induction on r. For $r<0$ let $\eta^r=0$. Suppose η^{r-1} is defined. Then for any face C^r of S,

$$b^{r-1} = \eta^{r-1}\delta^r C^r$$

is defined. Observe that

$$\delta^{r-1} b^{r-1} = \delta^{r-1}\eta^{r-1}\delta^r C^r = \eta^{r-2}\delta^{r-1}\delta^r C^r = 0.$$

Also $b^{r-1} \in L^{r-1}(\bar{\mathscr{K}}_{C^r})$. Since $H^{r-1}(\bar{\mathscr{K}}_{C^r})=0$, there is a chain in $L^r(\bar{\mathscr{K}}_{C^r})$ with boundary b^{r-1}; let this chain be $\eta^r C^r$. It is easy to check that this defines a homomorphism with the desired property.

Now take the composition map $\xi^r\eta^r$. We prove by induction on r that this is the identity map. This clearly holds for $r \leqq 0$. Since by the definition of η^r,

$$\eta^r C^r \in L^r(\bar{\mathscr{K}}_{C^r})$$

for each r-face C^r=convex hull of $(x_{i_0}, \ldots, x_{i_r})$, it follows that each vertex of any cell in $\eta^r C^r$ is a spanning arborescence containing no edge leaving a different from $(a, v_{i_0}), \ldots, (a, v_{i_r})$. Hence by the definition of ξ^r,

$$\xi^r \eta^r C^r = pC^r$$

with some integer p. Now

$$p\delta^r C^r = \delta^r \xi^r \eta^r C^r = \xi^{r-1}\delta^r \eta^r C^r = \xi^{r-1}\eta^{r-1}\delta^r C^r = \delta^r C^r$$

by the induction hypothesis. Hence $p=1$, i.e. $\xi^r\eta^r=\text{id}$ is proved.

In particular $\xi^{k-1}\eta^{k-1}=\text{id}$, which contradicts $\xi^{k-1}=0$. This completes the proof of Theorem 5.

7. We conclude with the much more elementary proof of Theorem 2. Let G be a non-bipartite 2-connected graph on n vertices, $(x_0, \ldots, x_{2p})$ an odd circuit in G and T a spanning arborescence of $G-x_0$ containing the path $(x_1, \ldots, x_{2p})$. Set $T=T+(x_0, x_1)$, $T''=T+(x_0, x_{2p})$.

By Theorem 1, there exists a sequence

$$T_0 = T, T_1, \ldots, T_N = T''$$

of spanning trees, such that T_i and T_{i+1} have a common subtree with $n-1$ poin containing x_0. Let us 2-colour each T_i with red and blue such that x_0 is red. L $f(i)$ be the number of red points. Then

$$|f(i)-f(i+1)| \leqq 1$$

and

$$f(N) = 2n+1-f(0).$$

Hence $f(i)$ must take the value n somewhere for $0 \leqq i \leqq N$, which proves the assertio

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