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INDEPENDENT SETS IN CRITICAL CHROMATIC GRAPHS

by

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Let $\alpha_k(n)$ denote the maximum number of independent points in a critical k-chromatic graph on n vertices. W. G. BROWN and J. W. MOON [1] showed $\alpha_k(n) \geqq n - c_1\sqrt{n}$ for any $k \geqq 4$. M. SIMONOVITS [2] proved $\alpha_4(n) \leqq n - c_2 n^{2/5}$. In the present note we are going to show $n - 2kn^{\frac{1}{k-2}} \leqq \alpha_k(n) \leqq n - \frac{k}{6} n^{\frac{1}{k-2}}$ for $k \geqq 4$. The lower bound follows from a construction which is an improvement of that of BROWN's and MOON's. The proof of the upper bound is a modification of SIMONOVITS's proof, and shows an interesting connection to SPERNER's lemma in combinatorial topology.

Let $\tau(G)$ denote the minimum number of points covering every edge in the graph G. Then $|V(G)| - \tau(G)$ is the maximum number of independent points of G. It will be more convenient to formulate our arguments for $\tau(G)$.

A vertex (edge) of a graph G will be called *critical* if its removal decreases the chromatic number of G. A graph is *k-critical* if it has chromatic number k and every edge and vertex is critical.

THEOREM. *A k-critical graph G on n points has*

$$\tau(G) \geqq \frac{1}{6} kn^{\frac{1}{k-2}}.$$

On the other hand, there are infinitely many k-critical graphs G such that, putting $|V(G)| = n$,

$$\tau(G) = 2kn^{\frac{1}{k-2}}.$$

PROOF. Let T be a set of points of G covering all edges, $S = V(G) - T$, $|T| = t$. Following an idea of SIMONOVITS [2], we may assume S has vertices of degree $k-1$ only; for if $x \in S$ has degree $d \geqq k$, we may substitute $\binom{d}{k-1} = r$ points $x_1, \dots, x_r$ for x, connecting them to $(k-1)$-tuples of the neighborhood of x. The obtained graph G' has chromatic number k, and all vertices but the x_i's are critical. Removing some of the x_i's without decreasing the chromatic number, we get a graph G'' in which all vertices are critical. However, in this graph the edges are also critical; this is trivial for the edges not incident with the x_i's but also follows for the x_i's since these points have degree $k-1$. Thus G'' is k-critical. It is obvious that the number of remaining x_i's is $\geqq 2$. Going on, we can split every vertex of S and construct a k-critical graph G_1 with $|V(G_1)| > |V(G)|$ and $\tau(G_1) \leqq \tau(G)$. If the theorem is valid for G_1 it also holds for G.

Let, now, H be the system of all $(k-1)$-tuples of T connected to the same point of S. Then H has the following property:

$(*)$ *For any $A\in H$, the vertices of T can be $(k-1)$-colored in such a way that A is the only $(k-1)$-tuple which gets all the $k-1$ colors.*

Really, remove the point corresponding to A and color the remaining graph by $k-1$ colors. Now it follows

$(**)$ *Whenever $A_1, \dots, A_s \in H$, there is a $(k-2)$-tuple of points of T which is contained in an odd number of A_i's.*

For assume this is not the case and consider a coloring of the vertices of T by $(k-1)$ colors such that A_1 meets every color but $A_2, \dots, A_s$ don't. Let us count incidences between A_i's and $(k-2)$-tuples of colors $1, 2, \dots, k-2$ in two different ways. Since any $(k-2)$-tuple is contained in an even number of A_i's, the resulting number is even. On the other hand, A_1 contains exactly one $(k-2)$-tuple of colors $1, 2, \dots, k-2$, the other A_i's contain 0 or 2. Hence the sum is odd, a contradiction.

Let $p = \binom{t}{k-2}$, $B_1, \dots, B_p$ all $(k-2)$-tuples of T elements and define, for any $A\in H$, a vector $v_A=(v_1, \dots, v_p)$ where

$$v_i = \begin{cases} 1 & \text{if} \quad B_i \subseteq A, \\ 0 & \text{if} \quad B_i \subsetneqq A. \end{cases}$$

Then $(**)$ can be re-formulated as

$(***)$ *The vectors v_A $(A\in H)$ are linearly independent over $GF(2)$.*

Thence it follows

$$n-t = |S| = |H| \leqq \binom{t}{k-2},$$

hence

$$n \leqq \binom{t}{k-2} + t \leqq 2\binom{t}{k-2} \leqq t^{k-2}\cdot\frac{2}{(k-2)!} \leqq t^{k-2}\cdot\frac{3^{k-2}}{(k-2)^{k-2}},$$

whence

$$t \geqq \frac{k-2}{3}\cdot n^{\frac{1}{k-2}} \geqq \frac{k}{6}\cdot n^{\frac{1}{k-2}}.$$

Remarks: I. The vectors v_A don't span the whole p-dimensional space over $GF(2)$ but only a subspace; but this won't give any improvement even in the constant.

II. SIMONOVITS's proof used $(*)$ but deduced only a smaller class of excluded partial systems — in fact, some special triangulations of the $(k-2)$-sphere. (The exclusion of all triangulated spheres would not give better result then SIMONOVITS's one; this follows from the results of BROWN, ERDŐS, and V. T. SÓS [3].) The idea of the present proof was that $(*)$ follows for triangulated manifolds from SPERNER's lemma, moreover, the proof of SPERNER's lemma is applicable under the more general condition $(**)$. It would be interesting to characterize all set-systems with property $(*)$, and determine the maximum number of $(k-1)$-tuples in them. For $k=3$, graphs with property $(*)$ are forests.

III. Assume we are given a system H of $(k-1)$-tuples on m points such that

(i) no $(k-1)$-coloring of the points yields exactly one $(k-1)$-tuple meeting all colors;

(ii) removing an $E_0\in H$, the remaining system has property $(*)$.

Then we can construct a k-critical graph G as follows: connect any two points of E_0; take a point x_E for any $E\in H$, $E\neq E_0$ and connect it to every point of E. The obtained graph G will have $\tau(G)\leqq m$, $|V(G)|=m+|H|-1$. It would be interesting to know what is the maximum cardinality of H; may be it gives the best possible construction, but I cannot prove this for $k\geqq 4$. Therefore, another way is chosen below to construct a k-critical graph with many independent points.

Construction. Let G_i be an $(i+1)$-critical graph $(i=2,\dots,k-2)$ $V(G_i)=\{b_{i,1},\dots,b_{i,m_i}\}$. Let P be a $(k-1)$-critical graph, $V(P)=\{p_1,\dots,p_{m_1}\}$. Let $B=\{b_{1,1},\dots,b_{1,m_1}\}$ be other points, $b_{1,i}$ joined to p_i. This anomaly for $i=1$ is because there is only one 2-critical graph; the set $\{b_{1,1},\dots,b_{1,m_1}\}$, however, behaves like a 2-critical graph in the following sense: using $k-1$ colors, the set $\{b_{1,1}\dots b_{1,m_1}\}$ must get at least two different colors; on the other hand, giving $b_{1,1}$ the color 1 and the other points color 2, this is a "good" coloring, i.e. it can be extended to a coloring of P.

Let A be the set of all vectors $(v_1,\dots,v_{k-2})$ such that $1\leqq v_i\leqq m_i$. Let $(v_1,\dots,v_{k-2})$ be joined to b_{ij} if and only if $v_i=j$.

Finally, take a point d and connect it to every point of A.

We show the obtained graph G is not $(k-1)$-colorable. Assume indirectly it has a coloration by colors $1,\dots,k-1$. Let d have color 1, say. One of the points in B, b_{11} say, has color different from 1, color 2, say. A point $b_{2,1}$ of G_2 has a third color, 3 say. And so on, we can select a point $b_{i,1}$ from every G_i which has different color from d and from the colors of $b_{1,1},\dots,b_{i-1,1}$. These points $d, b_{1,1},\dots,b_{k-2,1}$ have all the $k-1$ colors; but $(1,\dots,1)\in A$ is joined to each of them, which yields a contradiction.

As a second step we show that the edge $(d,(1,\dots,1))$ is critical. Remove it and consider the following $(k-1)$-coloring f of the remaining graph G'.

Let $f\colon V(G)\to\{1,\dots,k-1\}$ be defined on $V(G_i)$ as an $(i+1)$-coloring such that

$$f(b_{i,1})=i+1,$$

$$f(b_{i,j})\leqq i \quad \text{for} \quad 2\leqq j\leqq m_i;$$

let it be defined on $V(P)$ as a $(k-1)$-coloring such that

$$f(p_1)=1,$$

$$f(p_j)\geqq 2 \quad \text{for} \quad 2\leqq j\leqq m_1;$$

let

$$f(b_{1,1})=2,$$

$$f(b_{1,j})=1 \quad \text{for} \quad 2\leqq j\leqq m_1;$$

$$f(d)=1,$$

$$f((1,\dots,1))=1$$

and, finally,

$$f((v_1,\dots,v_{k-2}))=i+1 \quad \text{where } i \text{ is the last index such that } v_i\neq 1.$$

Then f is a $k-1$-coloring of G'.

Similarly we get that all edges incident with d are critical. Thence, the points of A are critical. Since these points have degree $k-1$, all edges incident with them are critical.

Now we prove that the edges of G_i $(i = 2, \dots, k-2)$ are critical. Remove one of them, then what remains from G_i can be colored by colors $1, \dots, i$. Color the graph spanned by $B \cup V(P)$ by $k-1$ colors so that B gets colors 1 and 2 only, and d by color 1. Now any point v of A is connected to $k-i-2$ points of the graphs $G_{i+1}, \dots, G_{k-2}$, and only these points can have colors $\geqq i+1$; therefore, there is a color which does not occur among the neighbors of v. This means that the coloring given above can be extended to A.

We argue similarly if an edge incident with P is removed. This proves G is k-critical.

The number of points of G is

$$n = m_1 \dots m_{k-2} + 2m_1 + m_2 + \dots + m_{k-2} + 1,$$

while

$$\tau(G) \leqq t = 2m_1 + m_2 + \dots + m_{k-2} + 1.$$

Taking $m > k$, $m_i = m$ or $m+1$ (this is possible, if $m_i \equiv i+1 \pmod 2$ $m_i \geqq i+1$ then a circuit of length $m_i - i + 2$ and $i-2$ points, joined to each other and to each point of the circuit, gives an $(i+1)$-critical graph on m_i points), we get

$$|V(G)| = n \geqq m^{k-2}$$

and

$$\tau(G) \leqq k(m+1) \leqq 2km \leqq 2k \cdot n^{\frac{1}{k-2}}.$$

This proves the theorem.

*

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REFERENCES

[1] BROWN, W. G. and MOON, J. W.: Sur les ensembles de sommets independants dans les graphes chromatiques minimaux, *Can. J. Math.* **21** (1969) 274—278.
[2] SIMONOVITS, M.: On colour-critical graphs, *Studia Sci. Math. Hungar.* **7** (1972), 67—81.
[3] BROWN, W. G., ERDÖS, P. and SÓS, V. T.: On the existence of triangulated spheres in 3-graphs, and related problems, to-appear.

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