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Note

A Note on the Line Reconstruction Problem

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It is shown that if a graph has more lines than its complement does, then it can be reconstructed from its line-deleted subgraphs.

As in Harary's book [4], *graph* means finite, undirected graph without loops or multiple lines. $V(G)$ and $E(G)$ denote the sets of points and lines of G, respectively. Ulam [6] conjectured that, if two graphs G_1 and G_2 are such that $V(G_1) = \{v_1, ..., v_n\}$, $V(G_2) = \{w_1, ..., w_n\}$, $n \geqslant 3$, and $G_1 - v_i \cong G_2 - w_i$, for each i, then $G_1 \cong G_2$. In other words, every graph with at least three points can be uniquely reconstructed from its maximal induced subgraphs. It seems that this conjecture is particularly difficult, and it is solved for special cases only; see, e.g., [5].

An analogous conjecture, formulated by Harary [3], replaces "maximal induced subgraphs" by "maximal subgraphs". This conjecture is actually weaker than Ulam's conjecture (see [1]). In this note we prove it for graphs with "many" lines.

THEOREM. *Let G_1, G_2 be two graphs, $E(G_1) = \{e_1, ..., e_m\}$, $E(G_2) = \{f_1, ..., f_m\}$, and $|V(G_1)| = |V(G_2)| = n$. Assume that $G_1 - e_i \cong G_2 - f_i$ for each $1 \leqslant i \leqslant m$, and $m > \frac{1}{2}\binom{n}{2}$. Then $G_1 \cong G_2$.*

Proof. Let $G \to H$ denote the set of all monomorphisms of G into H. Then, by the sieve formula,

$$|G \to H| = \sum_{X \subseteq G} (-1)^{|E(X)|} |X \to \bar{H}|, \tag{1}$$

where $\bar{H}$ is the complement of H and X runs over all graphs with $V(X) = V(G)$, $E(X) \subseteq E(G)$. In effect, the right-hand side of (1) just counts all maps from the points of G to the points of $\bar{H}$, then takes away those maps sending (at least) one line of G to a line of $\bar{H}$, then adds those sending (at least) two lines to lines of $\bar{H}$, etc. Thus it counts exactly those maps which send no lines of G to lines of $\bar{H}$, so every line of G goes to a line of H.

Applying (1) to G_1 and G_2 we have

$$| G_1 \to G_2 | = \sum_{X \subseteq G_1} (-1)^{|E(X)|} | X \to \bar{G}_2 |, \tag{2}$$

and for G_2 and G_2 we have

$$| G_2 \to G_2 | = \sum_{X \subseteq G_2} (-1)^{|E(X)|} | X \to \bar{G}_2 |. \tag{3}$$

Since the hypothesis on maximal subgraphs assures that G_1 and G_2 have the same proper subgraphs (see [2, p. 92]), the terms in (2) and (3), with $X \neq G_1$ and $X \neq G_2$, are equal. Also, since $m > \frac{1}{2}\binom{n}{2}$, $| G_1 \to \bar{G}_2 | = | G_2 \to \bar{G}_2 | = 0$. Hence $| G_1 \to G_2 | = | G_2 \to G_2 | > 0$, which proves the theorem.

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