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NORMAL HYPERGRAPHS AND THE WEAK PERFECT GRAPH CONJECTURE*

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A hypergraph is called normal if the chromatic index of any partial hypergraph H' of it coincides with the maximum valency in H'. It is proved that a hypergraph is normal iff the maximum number of disjoint hyperedges coincides with the minimum number of vertices representing the hyperedges in each partial hypergraph of it. This theorem implies the following conjecture of Berge: The complement of a perfect graph is perfect. A new proof is given for a related theorem of Berge and Las Vergnas. Finally, the results are applied on a problem of integer valued linear programming, slightly sharpening some results of Fulkerson.

Introduction

Let G be a finite graph and let $\chi(G)$ and $\omega(G)$ denote its chromatic number and the maximum number of vertices forming a clique in G, respectively. Obviously,

$$\chi(G) \geq \omega(G). \tag{1}$$

There are several classes of graphs such that

$$\chi(G) = \omega(G), \tag{2}$$

e.g., bipartite graphs, their line graphs and complements, interval graphs, transitively orientable graphs, etc. Obviously, relation (2) does not say too much about the structure of G; e.g., adding a sufficiently large clique to an arbitrary graph, the arising graph satisfies (1).

Berge [1, 2] has introduced the following concept: a graph is *perfect* (γ-perfect) if the equality holds in (2) for every induced subgraph of it. The mentioned special classes of graphs have this property, since every induced subgraph of them belongs to the same class. He formulated the following two conjectures in connection with this notion:

Conjecture 1. A graph is perfect if and only if neither itself nor its complement contains an odd circuit without diagonals.

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Conjecture 2. Let $\alpha(G)$ denote the stability number of G, let $\theta(G)$ denote the minimum number of cliques which partition the set of all the vertices. A graph G is perfect if and only if $\alpha(G') = \theta(G')$ for any induced subgraph G' of G.

This conjecture is an attempt to explain some similarities between the properties of the chromatic number and the stability number; his next conjecture is proved in the present paper, formulated as follows.

Perfect Graph Theorem. *The complement of a perfect graph is perfect as well.*

Obviously, the second conjecture of Berge would follow from the first one. However, due to its simpler form, it has more interesting applications and has been more investigated. Partial results are due to Berge [3], Berge and Las Vergnas [4], Sachs and Olaru [6]. Fulkerson [5] reduced the problem to the following conjecture, using the theory of anti-blocking polyhedra:

Conjecture 3. Duplicating an arbitrary vertex of a perfect graph and joining the obtained two vertices by an edge, the arising graph is perfect.

In §1 we prove a theorem which contains Conjecture 3.

Berge has observed that the perfect graph conjecture has an equivalent in hypergraph theory, interesting for its own sake too. The correspondence between graphs and hypergraphs is simple and enables us to translate proofs formulated in terms of graphs into proofs with hypergraphs, and conversely. In §2 we deduce the hypergraph version of the perfect graph theorem from the above-mentioned conjecture of Fulkerson; the proof is short and does not use the theory of anti-blocking polyhedra. It could be formulated in terms of graphs as well; however, the hypergraph version shows the idea more clearly. It should be pointed out that thus the proof consists of two steps and the more difficult second step was first carried out by Fulkerson.

In §3, we give a new proof of a related theorem of Berge. Finally, in §4 we give some formulations of the results in terms of linear programming. Most of them have been observed to be equivalent to the perfect graph theorem already proved by Fulkerson.

1.

Let G, H be two vertex-disjoint graphs and let x be a vertex of G. By substituting H for x we mean deleting x and joining every vertex of H to those vertices of G which have been adjacent with x.

Theorem 1. *Substituting perfect graphs for some vertices of a perfect graph the obtained graph is also perfect.*

Proof. We may assume that only one perfect graph H is substituted for a vertex x of a perfect graph G. Let G' be the resulting graph. It is enough to show that

$$\chi(G') = \omega(G'), \tag{3}$$

since for the induced subgraphs of G', which arise by the same construction from perfect graphs, this follows similarly.

We use induction on $k = \omega(G')$. For $k = 1$ the statement is obvious.

Assume $k > 1$. It is enough to find a stable set T of G' meeting all k-element cliques, since then coloring these vertices by the same color and the remaining vertices by $k - 1$ other colors (which can be done by the induction hypothesis), we obtain a k-coloring of G'.

Put $m = \omega(G)$, $n = \omega(H)$, and let p denote the maximum cardinality of a clique of G containing x. Then, obviously,

$$k = \max\{m, n + p - 1\}.$$

Consider an m-coloring of G and let K be the set of vertices having the same color as x. Let, further, L be a set of independent vertices of H meeting every n-element clique of H. Then $T = L \cup (K \setminus \{x\})$ is a stable set in G'. Moreover, T intersects every k-element clique of G'. Really, if C is a k-element clique of G' and it meets H then, obviously, it contains an n-element clique of H and thus a vertex of L. On the other hand, if C does not meet H, then C must be an m-element clique of G, and thus C contains a vertex of $K \setminus \{x\}$. $\square$

As has been mentioned in the introduction, in view of Fulkerson's results, the perfect graph theorem already follows from Theorem 1. However, to make the paper self-contained, we give a proof of the perfect graph conjecture (which seems to be different from that of Fulkerson).

2.

A *hypergraph* is a non-empty finite collection of non-empty finite sets called *edges*. The elements of edges are the *vertices*. Multiple edges are allowed, i.e., more (distinguished) edges may have the same set of vertices. The number of edges with the same vertices is called the *multiplicity* of them. The number of edges containing a given vertex is the *degree* of it. The maximum degree of vertices of a hypergraph H will be denoted by $\delta(H)$.

A *partial hypergraph* of H is a hypergraph consisting of certain edges of H.

The *subhypergraph induced by a set X of vertices* means the hypergraph

$$H \mid X = \{E \cap X \mid E \in H, E \cap X \neq \emptyset\}.$$

A *partial subhypergraph* is a subhypergraph of a partial hypergraph (or, equivalently, a partial hypergraph of a subhypergraph).

The *chromatic number* $\chi(H)$ of a hypergraph H is the least number of colors sufficient to color the vertices (so that every edge with more than one vertex has at least two vertices with different colors). The *chromatic index* $q(H)$ of H is the least number of colors by which the edges can be colored so that edges with the same color are disjoint.

Obviously,

$$q(H) \geq \delta(H). \tag{4}$$

Let a hypergraph be called *normal* if the equality holds in (4) for every partial hypergraph of it.

A set T of vertices of H is called a *transversal* if it meets every edge of H; $\tau(H)$ is the minimum cardinality of transversals. Denoting by $\nu(H)$ the maximum number of pairwise disjoint edges of H, we obviously have

$$\nu(H) \leq \tau(H). \tag{5}$$

Let a hypergraph be called τ-*normal* if the equality holds in (5) for every partial hypergraph of it.

A hypergraph is said to have the *Helly property* if any collection of edges whose intersection is empty contains two disjoint edges. It is easily seen that normal and τ-normal hypergraphs have the Helly property.

Given a hypergraph H, we can consider its *edge-graph* $G(H)$ defined as follows: the vertices of $G(H)$ are the edges of H and two edges of H are joined iff they intersect. On the other hand, for a given graph G we can construct a hypergraph $H(G)$ by considering the maximal cliques of G (in the set-theoretical sense) as vertices of H and, for any vertex x of G, the set of maximal cliques containing x, as an edge of $H(G)$. It is easily shown that if G has no multiple edges (which can be assumed throughout this paper) then

$$G(H(G)) \cong G. \tag{6}$$

Furthermore, $H(G)$ always has the Helly property.

It is easily seen that

$$\chi(G(H)) = q(H), \quad \omega(\overline{G(H)}) = \nu(H) \tag{7}$$

($\overline{G(H)}$ is the complement of $G(H)$). Moreover, if H has the Helly property then

$$\chi(\overline{G(H)}) = \tau(H), \quad \omega(G(H)) = \delta(H). \tag{8}$$

Hence by (6),

$$\chi(G) = q(H(G)), \quad \omega(G) = \delta(H(G)),$$
$$\chi(\bar{G}) = \tau(H(G)), \quad \omega(\bar{G}) = \nu(H(G)), \tag{9}$$

for any graph G. Equalities (7), (8) and (9) imply the following theorem:

Theorem 2. *Let H be a hypergraph with the Helly property. H is normal iff $G(H)$ is perfect; G is perfect iff $H(G)$ is normal. H is τ-normal iff $\overline{G(H)}$ is perfect; $\bar{G}$ is perfect iff $H(G)$ is τ-normal.*

As a corollary to Theorems 1 and 2 we have the following theorem:

Theorem 1′. *Multiplying some edges of a normal hypergraph, the obtained hypergraph is normal.*

Theorem 2 implies that the perfect graph theorem is equivalent to the following:

Theorem 3. *A hypergraph is τ-normal iff it is normal.*

Proof. Parts "if" and "only if" of this theorem are equivalent (by Theorem 2). Thus it is enough to show that if H is normal then

$$\tau(H) = \nu(H),$$

since for the partial hypergraphs this follows similarly. We use induction on $n = \tau(H)$. For $n = 0$ the statement can be considered to be true.

It is enough to find a vertex x with the property that the partial hypergraph H' consisting of the edges not containing x has $\nu(H') < \nu(H)$, since then H' has an $(n-1)$-element transversal T and then $T \cup \{x\}$ is an n-element transversal of H', showing that

$$\tau(H) \leq n = \nu(H).$$

Assume indirectly that for any vertex x there is a system F_x of n disjoint edges not covering x. Let

$$H_0 = \bigcup_x F_x,$$

where the edges occurring in more F_x's are taken with multiplicity. H_0 arises from H by removing and multiplying edges, hence by Theorem 1′ it is also normal, i.e.,

$$q(H_0) = \delta(H_0).$$

But obviously H_0 has $n \cdot m$ edges, where m is the number of vertices of H. Since there are at most n disjoint edges in H_0, we have

$$q(H_0) \geqslant m.$$

On the other hand, a given vertex x is covered by at most one edge of F_y $(y \neq x)$ and by no edge of F_x. Hence

$$\delta(H_0) \leqslant m - 1,$$

a contradiction. □

3.

A subhypergraph of a normal hypergraph is not always normal as shown, e.g., by the hypergraph

$$\{\{a, b, d\}, \{b, c, d\}, \{a, c, d\}\};$$

here $\{a, b, c\}$ spans a non-normal subhypergraph. Hypergraphs with the property that every subhypergraph of them is normal are described in the following theorem. A hypergraph is *balanced* if no odd circuit occurs among its partial hypergraphs (an odd circuit is a hypergraph isomorphic with the hypergraph $\{\{1, 2\}, \{2, 3\}, \ldots, \{2n, 2n+1\}, \{1, 2n+1\}\}$).

Theorem 4. *The following statements are equivalent:*

(i) *H is balanced;*
(ii) *every subhypergraph of H has chromatic number* 2;
(iii) *every subhypergraph of H is normal.*

Obviously, Theorem 3 gives more equivalent formulations of (iiii). The theorem is actually due to Berge [3]. In what follows, we are going to give a new proof for the non-trivial parts of it.

Proof of Theorem 4. (iii) ⇒ (i) being trivial, it is enough to show (i) ⇒ (ii) and (ii) ⇒ (iii).

(I) Assume that H is balanced, though it has subhypergraphs which are not 2-chromatic. Let H_0 be such a subhypergraph with minimum number of vertices. Consider the graph G consisting of the two-element edges of H_0: every vertex of H_0 is considered to be a vertex of G.

Now G is connected. Really, if $V(G) = X \cup Y$, $X \cap Y = \emptyset$, $X, Y \neq \emptyset$, and no edge of G joins a vertex of X to a vertex of Y, then considering a 2-coloration of $H_0 | X$ and one of $H_0 | Y$ (by the minimality of H_0 such 2-colorations exist) these form together a 2-coloration of H_0, since every edge E of H_0 with $|E|>1$ has at least two points in one of X, Y, and then even in this part of it there are two vertices with different colors.

Since H is balanced, G is obviously bipartite. Let G be colored by two colors. Since H_0 cannot be colored by two colors, there is an edge E, with $|E|>1$, of H_0 having only vertices of the same color. Let $x, y \in E$, $x \neq y$. Since G is connected, there is a path P of G connecting x and y. We may assume that no further vertex of E belongs to P. Then the subhypergraph spanned by the vertices of P contains an odd circuit, a contradiction.

(II) Now let H be a hypergraph with property (ii); we show it has property (iii). Obviously it is enough to show

$$\tau(H)=\nu(H).$$

Let $\tau(H)=t$ and consider a minimal partial subhypergraph H_0 of H with the property $\tau(H_0)=t$. If we show that H_0 consists of independent edges, we are ready. Suppose indirectly $E_1 : E_2 \in H_0$, $x \in E_1 \cap E_2$. By the minimality of H_0, there is a $(t-1)$-element transversal T_i of $H_0 \backslash \{E_i\}$, $i=1,2$. Put $Q=T_1 \cap T_2$, $R_i = T_i \backslash Q$, $S=R_1 \cup R_2 \cup \{x\}$. Obviously, $x \notin T_i$, hence $|S|=2|R_1|+1$. Since $H_0|S$ is 2-chromatic by (ii), there are two disjoint subsets of S both meeting every edge E of $H_0|S$ with $|E|>1$. One of them, say M, has at most $[\frac{1}{2}|S|]=|R_1|$ elements.

Now $M \cup Q$ is a transversal of H_0. Indeed, if an edge E is not represented by Q then it meets both R_1 and R_2 if $E \neq E_i$ and meets R_{1-i} and $\{x\}$ if $E=E_i$; thus, $|E \cap S| \geqslant 2$, whence E is represented by M.

But $|M \cup Q| \leqslant |R_1|+|Q|=t-1$, a contradiction. □

We conclude this section with the remark that bipartite graphs are, obviously, balanced (and thus normal). On the other hand, Theorem 4 shows that balanced hypergraphs have chromatic number 2. Recently, Las Vergnas and Fournier sharpened this statement and showed that normal hypergraphs have chromatic number 2.

4.

Let

$$A=\begin{pmatrix} a_{11} \cdots a_{1k} \\ \vdots \quad\quad \vdots \\ a_{r1} \cdots a_{rk} \end{pmatrix}$$

be a $(0,1)$-matrix, no row or column of which is the 0 vector, and consider the optimization programs

$$\frac{\begin{array}{r} yA \geq w \\ y \geq 0 \end{array}}{\min y \cdot 1} \tag{10}$$

$$\frac{\begin{array}{r} Ax \leq 1 \\ x \geq 0 \end{array}}{\max w \cdot x} \tag{11}$$

where 1 denotes the vector

$$\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

It is well-known that if x, y run through non-negative real vectors, (10) and (11) have a common optimum. But now we are interested in integer vector solutions.

Let B be a $(0,1)$-matrix such that

(i) any column u of B satisfies $Au \leq 1$,

(ii) every maximal $(0,1)$-vector with this property is a column of B.

Consider two further programs:

$$\frac{\begin{array}{r} yB \geq w \\ y \geq 0 \end{array}}{\min y \cdot 1} \tag{12}$$

$$\frac{\begin{array}{r} Bx \leq 1 \\ x \geq 0 \end{array}}{\max w \cdot x} \tag{13}$$

Theorem 5. *Assume that the optimum of* (10) (= *the optimum of* (11)) *is an integer for any* $(0,1)$-*vector* w. *Then, for any non-negative integer vector* w, *each of* (10)–(13) *has an integer optimum and an integer solution vector.*

Remark. The greatest part of this theorem is formulated in Fulkerson [5] as a consequence of the perfect graph conjecture and the theory of anti-blocking polyhedra.

Proof of Theorem 5. (1) First we show that (11) has a solution vector with integral entries for any $(0,1)$-vector w_0. For let x_0 be a solution of it with the greatest possible number of 0's. Put

$$w_0 = (w_1, \ldots, w_k), \quad x_0 = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}$$

Obviously, $x_0^T \leq w_0$. We show that x_0 is an integer vector.

Assume indirectly $0 < x_1 < 1$, say; then $w_1 = 1$. Put

$$w'_i = \begin{cases} 1 & \text{if } x_i \neq 0 \text{ and } i > 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$w' = (w'_1, \dots, w'_k).$$

Let x' be a solution of (11) with $w = w'$, then

$$w'x' \leqslant w_0x' \leqslant w_0x_0$$

and

$$w'x' \geqslant w'x_0 > w_0x_0 - 1.$$

Hence, both $w'x'$ and w_0x_0 being integers,

$$w'x' = w_0x' = w_0x_0,$$

i.e., x' is a solution of (11) with $w = w_0$ too, and has, obviously, more 0's than x_0 has, a contradiction.

(2) Now we prove that also (10) has an integer solution vector for any $(0,1)$-vector w. Assume indirectly that there are $(0,1)$-vectors w failing to have this property and let w_0 be one with minimum number of 1's. Let y_0 be a solution of (10) with $w = w_0$. Obviously, we may assume that $y_0^T \leqslant 1$. Put

$$w_0 = (w_1, \dots, w_k), \quad y_0 = (y_1, \dots, y_k), \quad y_1 \neq 0,$$

say, and define a $(0,1)$-vector $w' = (w'_1, \dots, w'_k)$ by

$$w'_i = \begin{cases} w_i & \text{if } a_{1i} = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We show first that y_0 is not a solution of (10), with $w = w'$. For let x' be a solution of (11) with $w = w'$; we may assume $x'^T \leqslant w'$. Then

$$y_0 \cdot 1 = y_0Ax' = w'x'$$

or

$$y_0(1 - Ax') = 0,$$

but this is impossible since both y_0 and $1 - Ax'$ are non-negative and their first entries are y_1 and $\geqslant 1 - \sum_{i=1}^{k} a_{1i}w'_i = 1$, i.e., the inner product is non-zero.

Thus, considering a solution $y' = (y'_1, \dots, y'_r)$ of (10) with $w = w'$ we have

$$y' \cdot 1 < y \cdot 1$$

and these being integers,

$$y' \cdot 1 \leqslant y \cdot 1 - 1.$$

This implies $w' \neq w_0$, i.e., by the minimality property of w_0, y' can be chosen to be an integer vector. Let

$$y'' = (1, y_2', \dots, y_r'),$$

then

$$y''A \geqslant w$$

since

$$\sum_{j=1}^{r} y_j'' a_{ji} = \sum_{j=1}^{r} y_j' a_{ji} \geqslant w_j' = w_i \quad \text{if } a_{1i} = 0,$$

$$\sum_{i=1}^{r} y_j'' a_{ji} \geqslant 1 \geqslant w_j \quad \text{if } a_{1i} \neq 0.$$

Since

$$y'' \cdot 1 \leqslant 1 + y' \cdot 1 \leqslant y_0 \cdot 1,$$

y'' is an integer vector solution of (10).

(3) Put

$$B = \begin{pmatrix} b_{11} \cdots b_{1k} \\ \vdots \quad\quad \vdots \\ b_{s1} \cdots b_{sk} \end{pmatrix}$$

Let H be a hypergraph on vertices $1, \dots, s$; for any $1 \leqslant i \leqslant k$ it has an edge

$$E_i = \{j ; b_{ij} = 1\}.$$

Now H is normal. For consider a partial hypergraph H' of it; let

$$w_i = \begin{cases} 1 & \text{if } E_i \in H', \\ 0 & \text{otherwise,} \end{cases}$$

$$w_0 = (w_1, \dots, w_k).$$

Let x_0, y_0 be integer solution vectors of (11) and (10), respectively. Since

$$Ax_0 \leqslant 1,$$

there is a column u of B with $x_0 \leqslant u$ by property (ii) of it. Then the vertex corresponding to u has degree $w_0 u \geqslant w_0 x_0$ in H', i.e.,

$$\delta(H') \geq w_0 x_0.$$

On the other hand, associate a color with every 1 entry of y_0. For a given edge E_i, consider a $1 \leq j \leq r$ with $y_j a_{ji} > 0$ and give the color associated with y_j to E_i. If E_i and E_t have the same color, then there is a j with $a_{ij} = a_{tj} = 1$, i.e., no column of B can have 1's on both the ith and tth place by (i). Hence E_i, E_t are disjoint, i.e., the coloring defined above is a good one, showing that

$$\rho(H') \leq y_0 \cdot 1 = w_0 x_0,$$

whence $\rho(H') = \delta(H')$.

(4) Let now $w_0 = (w_1, \ldots, w_k)$ be a (0, 1)-vector. Consider the partial hypergraph H' consisting of those E_i's for which $w_i = 1$. By Theorem 3,

$$\tau(H') = \nu(H') = \nu,$$

i.e., there are ν columns $u_{j_1}, \ldots, u_{j_\nu}$ of B such that every row corresponding to an edge of H' has a 1 in at least one of them. Let

$$y_j = \begin{cases} 1 & \text{if } j = j_1, \ldots, j_\nu, \\ 0 & \text{otherwise,} \end{cases}$$

$$y_0 = (y_1, \ldots, y_s).$$

Then

$$y_0 B \geq w_0, \quad y_0 \geq 0, \quad y_0 \cdot 1 = \nu.$$

On the other hand, there are ν rows $b_{i_1}, \ldots, b_{i_\nu}$ of B such that they correspond to edges of H' and every column has at most one 1 in them. Putting

$$x_i = \begin{cases} 1 & \text{if } i = i_1, \ldots, i_\nu, \\ 0 & \text{otherwise,} \end{cases}$$

$$x_0 = (x_1, \ldots, x_k)$$

we have

$$Bx_0 \leq 1, \quad x_0 \geq 0, \quad w_0 x_0 = \nu,$$

showing that x_0, y_0 are solution vectors of (12) and (13).

(5) Finally, let $w_0 = (w_1, \ldots, w_k)$ be an arbitrary non-negative integer vector. We show that (10)–(13) have integer solution vectors. It is enough to deal with (10) and (11). Let us multiply the edge E_i of H by w_i, $i = 1, \ldots, k$; let H' denote the arising hypergraph. Then

$$\delta(H') = \rho(H'),$$

since by Theorem 1′ H' is normal. Let j be a vertex with maximum valency in H' and u_j the corresponding column of B. Then

$$Au_j \leqslant 1, \quad u_j \geqslant 0,$$

and

$$w_0 u_j = \delta(H'). \tag{14}$$

On the other hand, let the edges of H' be colored by $\rho = \rho(H')$ colors. This means that there are ρ (0, 1)-vectors $a_1, \dots, a_\rho$ such that $a_1 + \cdots + a_\rho = w_0$ and $Ax \leqslant 1$, $x \geqslant 0$ implies $a_l x \leqslant 1$ for any $1 \leqslant l \leqslant \rho$. Hence there is a (0, 1)-vector y_l by part (2) of the proof such that

$$y_l A \geqslant a_l, \quad y_l \geqslant 0, \quad y_l \cdot 1 = 1.$$

Putting

$$y = y_1 + \cdots + y_l,$$

this vector satisfies

$$yA \geqslant w_0, \quad y \geqslant 0, \quad y \cdot 1 = \rho,$$

i.e., by (14) the theorem is proved. □

Appendix. A characterization of perfect graphs

Let $\bar{G}$ be the complement of G. We prove the following theorem:

Theorem. *A graph G is perfect if and only if*

$$\omega(G')\omega(\bar{G}') \geqslant |G'|$$

for every induced subgraph G' of G.

Proof. Part "only if" is trivial. To prove part "if" we use induction on $|G|$. Thus we may assume that any proper induced subgraph of G, as well as its complement, is perfect.

Let *multiplication* of a vertex x by h ($h \geqslant 0$) mean substituting for it h independent vertices, joined to the same set of vertices as x. This notion is closely related to the notion of *pluperfection*, introduced by D. R. Fulkerson.

(I) As a first step of the proof we show that if G_0 arises from G by multiplication of its vertices then G_0 satisfies

$$\omega(G_0)\omega(\bar{G}_0) \geqslant |G_0|.$$

Assume this is not the case and consider a G_0 failing to have this property and with minimum number of vertices. Obviously, there is a vertex y of G which is multiplied by $h \geq 2$; let $y_1, \ldots, y_h$ be the corresponding vertices of G_0. Then

$$\omega(G_0 - y_1)\omega(\bar{G}_0 - y_1) \geq |G_0| - 1$$

by the minimality of G_0; hence

$$\omega(G_0) = \omega(G_0 - y_1) = p, \quad \omega(\bar{G}_0) = \omega(\bar{G}_0 - y_1) = r$$

and

$$|G_0| = pr + 1.$$

Put $G_1 = G_0 - \{y_1, \ldots, y_h\}$. Then G_1 arises from $G - y$ by multiplication of its vertices, hence by [1, Theorem 1], $\bar{G}_1$ is perfect. Thus, $\bar{G}_1$ can be covered by $\omega(\bar{G}_1) \leq \omega(\bar{G}_0) = r$ disjoint cliques of G_1; let $C_1, \ldots, C_r$ be these cliques, $|C_1| \geq |C_2| \geq \cdots \geq |C_r|$.

Obviously, $k \leq r$. Since $|G_1| = |G_0| - h = pr + 1 - h$,

$$|C_1| = \cdots = |C_{r-h+1}| = p.$$

Let G_2 be the subgraph of G_0 induced by $C_1 \cup \cdots \cup C_{r-h+1} \cup \{y_1\}$, then

$$|G_2| = (r - h + 1)p + 1 < |G_0|;$$

thus, by the minimality of G_0,

$$\omega(G_2)\omega(\bar{G}_2) \geq |G_2|.$$

Since $\omega(G_2) \leq \omega(G_0) = p$, this implies

$$\omega(\bar{G}_2) \geq r - h + 2.$$

Let F be a stable set of $r - h + 2$ vertices of G_2; then $|F \cap C_i| \leq 1$ $(1 \leq i \leq r - h + 1)$, hence $y_1 \in F$. This implies that $F \cup \{y_2, \ldots, y_h\}$ is stable in G_0. On the other hand

$$|F \cup \{y_2, \ldots, y_h\}| = r + 1 > \omega(\bar{G}_0),$$

a contradiction.

(II) We show that $\chi(G) = \omega(G)$. It is enough to find a stable set F such that $\omega(G - F) < \omega(G)$ since then, by the induction hypothesis, $G - F$ can be colored by $\omega(G) - 1$ colors and, adding F as a further one, we obtain a $\mu(G)$-coloring of G.

Assume indirectly that $G - F$ contains a $\omega(G)$-clique C_F for any stable set F in G. Let, for $x \in G$, $h(x)$ denote the number of C_F's containing x. Let G_0 arise from G by multiplying each x by $h(x)$.

Then, by Part I above,

$$\omega(G_0)\omega(\bar{G}_0) \geqslant |G_0|.$$

On the other hand, obviously

$$|G_0| = \sum_x h(x) = \sum_F |C_F| = pf,$$

where f denotes the number of all stable sets in G_0, and

$$\omega(G_0) \leqslant \omega(G) = p,$$

$$\omega(\bar{G}_0) = \max_F \sum_{x \in F} h(x) = \max_F \sum_{F'} |F \cap C_{F'}| \leqslant \max_F \sum_{F' \neq F} 1 = f - 1,$$

a contradiction. □

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