SEPARATUM

SPECTRA OF GRAPHS WITH TRANSITIVE GROUPS

by

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0. Let G be a graph on $V(G) = \{x_1, \ldots, x_n\}$ without loops and multiple edges. The *spectrum* of G is the spectrum of its *adjacency matrix* $A_G = (a_{ij})_{i,j=1}^{n}$, where

$$a_{ij} = \begin{cases} 1 & \text{if } x_i \text{ and } x_j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

The spectrum is an important algebraic invariant of graphs which reflects many combinatorial properties of them. However, it is not easy to determine the spectrum in general. The present note gives a formula which expresses the eigenvalues of G (i.e., the eigenvalues of A_G) in terms of the characters of a transitive subgroup of the automorphism group Γ of G. The result is especially simple when Γ is commutative.

1. Let Γ be a transitive group of automorphisms of G, $\chi_1, \chi_2, \ldots, \chi_t$ be the irreducible characters of Γ, $n_1, \ldots, n_t$ be the dimensions of the corresponding irreducible representations[1]. Let, for each $\gamma \in \Gamma$, $p_{\gamma,k}$ be the number of walks of length k connecting a point to its image. We determine the eigenvalues of the adjacency matrix A of G.

Assume first Γ is regular, then $|\Gamma| = n$. Let $\gamma \to P_\gamma$ be the regular representation of Γ (P_γ is an $n \times n$ matrix over the complex field). Then the fact that γ is an automorphism of G is reflected by

$$AP_\gamma = P_\gamma A.$$

Let $\mathbf{v}$ be an eigenvector of A, with eigenvalue λ. Then

$$A(P_\gamma \mathbf{v}) = P_\gamma(A\mathbf{v}) = \lambda(P_\gamma \mathbf{v})$$

i.e., $P_\gamma \mathbf{v}$ is an eigenvector of A with the same eigenvalue.

Let M_λ be the eigensubspace of A belonging to λ; then, by the above, M_λ is invariant under $\Gamma' = \{P_\gamma : \gamma \in \Gamma\}$.

[1] For notions and basic results in group representation theory, see [2].

Let $\chi_1, \ldots, \chi_t$ be the irreducible characters of Γ. Refine the decomposition $\oplus M_\lambda$ into a decomposition into irreducible Γ-invariant subspaces $N_{ij}, i=1, \ldots, t; j=1, \ldots, n_i$, where the character belonging to the representation N_{ij} on is χ_i. Note that each element of N_{ij} is an eigenvector of A with the same eigenvalue λ_{ij}.

Let $\mathbf{b}_{ij1}, \ldots, \mathbf{b}_{ijn_i}$ be any orthogonal normalized basis of N_{ij}; then

$$\chi_i(\gamma) = \mathrm{Sp}\,(P_\gamma \mid N_{ij}) = \sum_{\nu=1}^{n_i} \mathbf{b}_{ij\nu} P_\gamma \mathbf{b}_{ij\nu}$$

and so,

$$\sum_{i=1}^{t} \sum_{j=1}^{n_i} \lambda_{ij}^k \chi_i(\gamma) = \sum_{i=1}^{t} \sum_{j=1}^{n_i} \sum_{\nu=1}^{n_i} \mathbf{b}_{ij\nu} P_\gamma(\lambda_{ij}^k \mathbf{b}_{ij\nu}) =$$

$$= \sum_{i=1}^{t} \sum_{j=1}^{n_i} \sum_{\nu=1}^{n_i} \mathbf{b}_{ij\nu} P_\gamma A^k \mathbf{b}_{ij\nu} = \mathrm{Sp}\,(P_\gamma A^k) = p_{\gamma k}.$$

Also, since λ_{ij} and $p_{\gamma k}$ are real,

$$\sum_{i=1}^{t} \sum_{j=1}^{n_i} \lambda_{ij}^k \overline{\chi_i(\gamma)} = p_{\gamma k}.$$

So

$$\sum_{\gamma \in \Gamma} p_{\gamma k} \chi_i(\gamma) = \sum_{\gamma \in \Gamma} \sum_{\mu=1}^{t} \sum_{j=1}^{n_\mu} \lambda_{ij}^k \overline{\chi_\mu(\gamma)} \chi_i(\gamma) =$$

$$= \sum_{\mu=1}^{t} \sum_{j=1}^{n_\mu} \lambda_{ij}^k \sum_{\gamma \in \Gamma} \overline{\chi_\mu(\gamma)} \chi_i(\gamma) = n \sum_{j=1}^{n_i} \lambda_{ij}^k.$$

Set

(0) $$f_{ik} = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} p_{\gamma k} \chi_i(\gamma).$$

Then the numbers λ_{ij} can be determined from the numbers f_{ik} several ways, either expressing their elementary symmetric polynomials by the f_{ik} or as the roots of the polynomial

(1) $$\det \begin{pmatrix} 1 & f_{i1} & \cdots & f_{i,n_i-1} & f_{in_i} \\ 0 & 1 & \cdots & f_{i,n_i-2} & f_{i,n_i-1} \\ & & \ddots & & \\ 0 & 0 & & 1 & f_{i1} \\ \frac{1}{n_i!} & \frac{x}{(n_i-1)!} & \cdots & \frac{x^{n_i-1}}{1!} & x^{n_i} \end{pmatrix}.$$

So we can calculate the roots of (1). Since (1) is of much smaller degree than n, this is usually easier. E.g., if Γ is commutative $n_i = 1$ and we have the eigenvalues (see below).

Suppose now Γ is transitive but not regular. Define a graph G' by

$$V(G') = \Gamma, \ (\gamma_1, \gamma_2) \in E(G') \Leftrightarrow (x_1\gamma_1, x_1\gamma_2) \in E(G)$$

(this is the lexicographic product of G and the null graph on $s = \frac{n}{|\Gamma|}$ points). Let A' be the adjacency matrix of G' then A' is the Kronecker product of A and the matrix

$$J = \begin{pmatrix} 1 & \dots & 1 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & \dots & 1 \end{pmatrix}$$

of size $t = \frac{\Gamma}{s}$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of G then the eigenvalues of A' are products of λ's and eigenvalues of J i.e., $(s-1)n$ 0's and $s\lambda_1, \ldots, s\lambda_n$. Since Γ acts regularly on G' we can determine the spectrum of G'; from this we get the spectrum of G by throwing out $(s-1)n$ 0's and dividing the rest by s. So we get the following *rule to calculate the eigenvalues of* G:

THEOREM. *Let Γ be a transitive group of automorphisms of the (finite) graph G and let, for each $\gamma \in \Gamma$, p_γ denote the number of points adjacent to their γ-image. Let $\chi_1, \ldots, \chi_t$ be the irreducible characters of Γ of dimensions $n_1, \ldots, n_t$ respectively. Write down each root of* (1) *(where f_{ik} is determined by* (0)*) n_i times for $i = 1, \ldots, t$. Remove $|\Gamma| - |G|$ zeros from this sequence. The remaining numbers are the eigenvalues of G.*

If Γ is commutative this rule simplifies. In this case, obviously, Γ is regular, $s = 1$.

Let $x_i \sim x_j$ denote that x_i, x_j are adjacent. We have

$$p_\gamma = \begin{cases} n \text{ if } x_1\gamma \sim x_1, \\ 0 \text{ otherwise;} \end{cases}$$

since if $x_1\gamma \sim x_1$ then

$$x_i\gamma = (x_1\gamma_i)\gamma = (x_1\gamma)\gamma_i \sim x_1\gamma_i = x_i .$$

Also $n_i = 1$. So we get one eigenvalue associated with each character χ_i, namely

$$\sum_{x_1\gamma \sim x_1} \chi_i(\gamma) \, . \tag{2}$$

We remark that

$$\begin{pmatrix} \chi_t(\gamma_1) \\ \vdots \\ \chi_t(\gamma_n) \end{pmatrix}$$

is an eigenvector (γ_i is defined by $x_1\gamma_i = x_i$).

2. Consider graphs G with $V(G) = \{1, \varepsilon, \dots, \varepsilon^{n-1}\}$ where $\varepsilon = e^{2\pi i/n}$. Such a graph is called *cyclic* if the rotation (i.e., the multiplication by ε) is an automorphism of it. G admits the "rotations" as automorphisms, i.e., the mappings

$$\varphi_\nu : \varepsilon^k \to \varepsilon^{k+\nu}.$$

These mappings form a cyclic group of order n; the characters are, as known, given by

$$\chi_t(\varphi_\nu) = \varepsilon^{t\nu}.$$

Let 1 be adjacent to $\varepsilon^{a_1}, \dots, \varepsilon^{a_m}$ in G (clearly, $n - a_i$ is also among $a_1, \dots, a_m$ for $i = 1, \dots, m$). Then by (2), the eigenvalues of G are

$$\lambda_t = \sum_{i=1}^{m} \varepsilon^{t a_i} \qquad (t = 0, \dots, n-1).$$

This formula was used by DJOKOVIĆ [3] in investigating isomorphic cyclic graphs and by H. SACHS [1] in investigating self-complementary graphs.

Another case of interest is when G is "cube-like", i.e., the vertices of G can be regarded as all subsets of a set S; $A, B \subseteq S$ being adjacent iff $A \triangle B \in H$, where $H \subseteq 2^S$ is given. If γ_X is the mapping

$$\gamma_X(Y) = X \triangle Y$$

then γ_X is an automorphism. For any $A \subseteq S$, $\chi_A(\gamma_X) = (-1)^{|A \cap X|}$ defines a character of the group of these automorphisms. Thus by (2) the eigenvalues are

$$\lambda_A = \sum_{X \in H} (-1)^{|X \cap A|}.$$

Note that $\lambda_A \in \{|H|, |H| - 2, \dots, -|H|\}$ (cf. [4]).

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