Triangle areas determined by arrangements of planar lines

Gábor Damásdi [*] Leonardo Martínez-Sandoval [†]
Dániel T. Nagy [‡] Zoltán Lóránt Nagy [§]

Abstract

A widely investigated subject in combinatorial geometry, originated from Erdős, is the following. Given a point set P of cardinality n in the plane, how can we describe the distribution of the determined distances? This has been generalized in many directions. In this paper we propose the following variants. Consider planar arrangements of n lines. Determine the maximum number of triangles of unit area, maximum area or minimum area, determined by these lines. Determine the maximal size of a subset of these n lines so that all triples determine distinct area triangles.

We prove that the order of magnitude for the maximum occurrence of unit areas lies between $\Omega(n^2)$ and $O(n^{9/4})$. This result is strongly connected to both additive combinatorial results and Szemerédi–Trotter type incidence theorems. Next we show a tight bound for the maximum number of minimum area triangles. Finally we present lower and upper bounds for the maximum area and distinct area problems by combining algebraic, geometric and combinatorial techniques.

[*]MTA-ELTE Lendület Combinatorial Geometry Research Group, Institute of Mathematics, Eötvös Loránd University (ELTE), Budapest, Hungary. Supported by the ÚNKP-18-3 New National Excellence Program of the Ministry of Human Capacities damasdigabor@caesar.elte.hu

[†]Sorbonne Université, Institut de Mathématiques de Jussieu – Paris Rive Gauche (UMR 7586) Paris, France. Supported by the grant ANR-17-CE40-0018 of the French National Research Agency ANR (project CAPPS) leomtz@im.unam.mx

[‡]Alfréd Rényi Institute of Mathematics, P.O.B. 127, Budapest H-1364, Hungary. Research supported by National Research, Development and Innovation Office - NKFIH grant K 116769. nagydani@renyi.hu

[§]MTA-ELTE Geometric and Algebraic Combinatorics Research Group and Eötvös Loránd University, Department of Geometry, Budapest, Hungary, H–1117 Budapest, Pázmány P. sétány 1/C. Supported by the Hungarian Research Grant (OTKA) No. K 120154 and by the János Bolyai Scholarship of the Hungarian Academy of Sciences. nagyzoli@cs.elte.hu

1 Introduction

A widely investigated subject, originated from Erdős, is to determine the maximum number of equal distances that n planar points can form, the minimum number of distinct distances they can form, the maximum number of appearances of the largest/smallest distance or the largest subset they can have such that all the distances within this subset are distinct.

Erdős and Purdy also studied the related problem of the maximum number of occurrences of the same area among the triangles determined by n points in the plane [7]. Since then, several variants has been established and the former results of Erdős and Purdy have been settled for some cases, see e.g. [3, 7, 18].

In this paper we consider the following variants of the original problem which can be considered as the dual setting. We are given n lines on the Euclidean plane and we are seeking for conditions on the distribution of the areas of triangles formed by the triples of lines. More precisely, we investigate the following four main problems and compare the results to the corresponding problems concerning triples of points.

Problem 1.1. *Determine the largest possible number $f(n)$ of triangles of unit area in arrangements of n planar lines.*

Problem 1.2. *Determine the largest possible number $m(n)$ of triangles having minimum area in arrangements of n planar lines.*

Problem 1.3. *Determine the largest possible number $M(n)$ of triangles with maximum area in arrangements of n planar lines.*

Problem 1.4. *Determine the largest possible number $D(n)$ such that in any arrangement of n lines (satisfying some generality conditions) there are $D(n)$ lines that form triangles of different areas.*

Concerning these problems, we achieved the following results.

Theorem 1.5. *For the maximum number of triangles of unit area, we have*

$$f(n) = O\left(n^{\frac{9}{4}+\varepsilon}\right)$$

for every fixed $\varepsilon > 0$, while $f(n) = \Omega(n^2)$.

Theorem 1.6.

$$\left\lfloor \frac{n^2-n}{6} \right\rfloor \leq m(n) \leq \left\lfloor \frac{n^2-2n}{3} \right\rfloor$$

holds for the occurrences of the minimum area, if $n \geq 6$.

Theorem 1.7. *For the maximum number of triangles of maximum area, we have*

$$\frac{7}{5}n - O(1) < M(n) < \frac{2}{3}n(n-2).$$

Theorem 1.8. *For the largest subset of lines forming triangles of distinct areas, we have*

$$n^{\frac{1}{5}} < D(n) < n,$$

provided that there are no six lines tangent to a common conic.

To put these results into perspective, let us recall a related problem, first asked by Oppenheim in 1967, which reads as follows: What is the maximum number of triangles of unit area that can be determined by n points in the plane? The first breakthrough after the investigation of Erdős and Purdy [7] was due to Pach and Sharir [16], who obtained an upper bound $O(n^{2+1/3})$ via a Szemerédi-Trotter type argument. Very recently this was improved by Raz and Sharir to $O(n^{2+2/9})$ in [18]. Here the lower bound is a simple lattice construction from [7], yielding $\Omega(n \log \log n)$. Our Theorem 1.5 also indicates that the straightforward application of some Szemerédi-Trotter type result can be improved. However, in the next Section we will point out that in some relaxation, it would provide the right order of magnitude.

As in the case of counting equal distances, the minimum and maximum area problems determined by point sets turned out to be easier, and settled by Brass, Rote and Swanepoel [3]. Concerning the occurrences of the maximum area, the upper bound happens to be exactly n. This is a rather common phenomenon in this field, we could mention the well-known theorem of Hopf and Pannwitz and similar results, see [2]. Surprisingly, Theorem 1.7 shows that this is not the case in our problem.

The problem of the largest subset of points with distinct pairwise distances was originally posed by Erdős [9] and generalised recently to distinct k-dimensional volumes in $\mathbb{R}^d$ by Conlon et al. [4]. As a point of comparison, we use the planar bounds for points in general position that follow from Section 5.1 of their paper and the references therein. The best lower bound so far is $\Omega(n^{1/5})$. The best upper bound so far is attained by choosing $\Omega(n)$ points in general position on the $n \times n$ grid. Lattice triangles on this grid define at most $O(n^2)$ areas, so the upper bound for the problem is $O(n^{2/3})$.

The paper is built up as follows. In Section 2 we discuss Problem 1.1 and prove Theorem 1.5. In order to do this, we consider first the maximum number of unit area triangles lying on a fixed line, and prove tight results up to a constant. Then we will apply a deep result of Pach and Zahl to complete the proof of our main theorem.
Section 3 is devoted to the Problems 1.2 and 1.3, and we prove Theorem 1.6 and 1.7. Section 4 concerns Problem 1.4 and contains the proof of Theorem 1.8. Finally we discuss some related problems and open questions in Section 5.

2 Number of unit area triangles, bounds on $f(n)$

2.1 Number of unit area triangles on a single line

A natural way to give an upper bound on $f(n)$ is to consider how many of the unit area triangles can be supported by a fixed line. Then $f(n)$ is at most $n/3$ times larger.

Problem 2.1. *Let ℓ be a line and let $\mathcal{L}$ be a set of n lines and consider the triangles given by ℓ and two elements of $\mathcal{L}$. Determine the largest possible number $g(n)$ of triangles of unit area among these.*

We determine the order of magnitude of $g(n)$ by turning the problem into an incidence problem for points and lines.

Theorem 2.2. *For the maximum number of triangles of unit area having a common supporting line ℓ, $g(n) = \Theta(n^{4/3})$ holds.*

We may assume that ℓ is horizontal, and that the rest of the lines in $\mathcal{L} = \{\ell_1, \ldots, \ell_n\}$ are not horizontal lines. Let x_i denote the x-coordinate of the intersection of ℓ and ℓ_i and let $y_i = \cot \alpha_i$ where α_i denotes the (directed) angle determined by ℓ and ℓ_i. Let T_{ij} denote the triangle formed by ℓ, ℓ_i and ℓ_j. Notice that the parameters (x, y) provide an exact description of any line not parallel to ℓ, while a parallel line $\ell' \parallel \ell$ would not contribute to the number of unit area triangles supported by ℓ. Let us denote by $e(x, y)$ the line described by parameters (x, y).

Observation 2.3. *The area of triangle T_{ij} is*

$$\mathit{Area}(T_{ij}) = \frac{(x_j - x_i)^2}{2|(y_i - y_j)|}.$$

Proof of Theorem 2.2. We apply the observation above. Supposing that $y_i > y_j$, T_{ij} is of unit area if and only if $2y_i - x_i^2 = -2x_i x_j + x_j^2 + 2y_j$. In other words T_{ij} is of unit area if and only if the point $(x_i, 2y_i - x_i^2)$ lies on the line $y = -2x_j x + 2y_j + x_j^2$.

By the Szemerédi–Trotter theorem, n lines and n points have $O(n^{4/3})$ incidences. Applying this to the lines $y = -2x_j x + 2y_j + x_j^2$ and the points $(x_i, 2y_i - x_i^2)$ we get $g(n) = O(n^{4/3})$.

On the other hand there exists $n/2$ lines and $n/2$ points that have $\Omega(n^{4/3})$ incidences. We can write these points in the form $(x_i, 2y_i - x_i^2)$ for some $(x_1, y_1), \ldots, (x_{n/2}, y_{n/2})$. Similarly we can write the lines in the form $y = -2x_j x + 2y_j + x_j^2$ for some $(x_{n/2+1}, y_{n/2+1}), \ldots, (x_n, y_n)$. Then the n lines given by the assignment $(x_i, y_i) \to e(x_i, y_i)$ determine $\Omega(n^{4/3})$ unit area triangles. Therefore $g(n) = \Theta(n^{4/3})$. $\square$

Let us mention that the upper bound is also implied by the powerful theorem of Pach and Sharir [17].

Theorem 2.4 ([17]). *Let P be a set of m points and let Γ be a set of n distinct irreducible algebraic curves of degree at most k, both in $\mathbb{R}^2$. If the incidence graph of $P \times \Gamma$ contains no copy of $K_{s,t}$, then the number of incidences is*

$$O(m^{\frac{s}{2s-1}} n^{\frac{2s-2}{2s-1}} + m + n).$$

Indeed, the lines were described by their parameters $(x_1, y_1), \dots, (x_n, y_n)$, and there is a bounded number of incidences between these points and the unit parabolas $2y = x^2 - 2xx_j + x_j^2 - 2y_j$ $(j = 1, \dots, n)$ according to this theorem. But the ith point lies on the jth parabola if and only if the triangle T_{ij} has unit area.

Corollary 2.5. *The bound above yields $f(n) = O(n^{7/3})$ for the maximum number of unit area triangles.*

2.2 Upper bound on the maximum number of unit area triangles

2.2.1 Reformulation in additive combinatorics

Proposition 2.6. *Any arrangement of n lines corresponds to a set*

$$H = \{(x_i, y_i) \mid x_i < x_j \quad \text{and} \quad y_i \geq y_j \quad \text{if} \quad i < j\} \subseteq \mathbb{R}^2$$

of size $|H| = n$, for which the number of unit area triangles equals to the number of solutions from $H \times H \times H$ to the (polynomial) equation

$$\frac{(x_j - x_i)^2}{y_i - y_j} + \frac{(x_k - x_j)^2}{y_j - y_k} + \frac{(x_i - x_k)^2}{y_k - y_i} = 2, \tag{1}$$

Proof. We may assume by rotation that none of the n lines are horizontal, and consider a horizontal line t located under all the intersections of the n lines. Taking t as the x axis of a coordinate system, we let x_i to be coordinate of the i-th intersection of t with another line (which we denote by ℓ_i). Let α_i denote the (directed) angle appearing between t and ℓ_i, see Figure 1. Let $y_i = \cot \alpha_i$. Since there are no intersections under or on t, we have $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n$ and therefore $y_1 \geq y_2 \geq \dots \geq y_n$.

The area of the triangle T_{ij} determined by ℓ, ℓ_i and ℓ_j is $\frac{(x_j - x_i)^2}{2(y_i - y_j)}$. (If $y_i = y_j$ then ℓ_i and ℓ_j are parallel, and they form no triangle.) The area of the triangle determined by the lines ℓ_i, ℓ_j and ℓ_k can be calculated as

$$Area(T_{ij}) + Area(T_{jk}) - Area(T_{ik}) = \frac{(x_j - x_i)^2}{2(y_i - y_j)} + \frac{(x_k - x_j)^2}{2(y_j - y_k)} - \frac{(x_k - x_i)^2}{2(y_i - y_k)} =$$

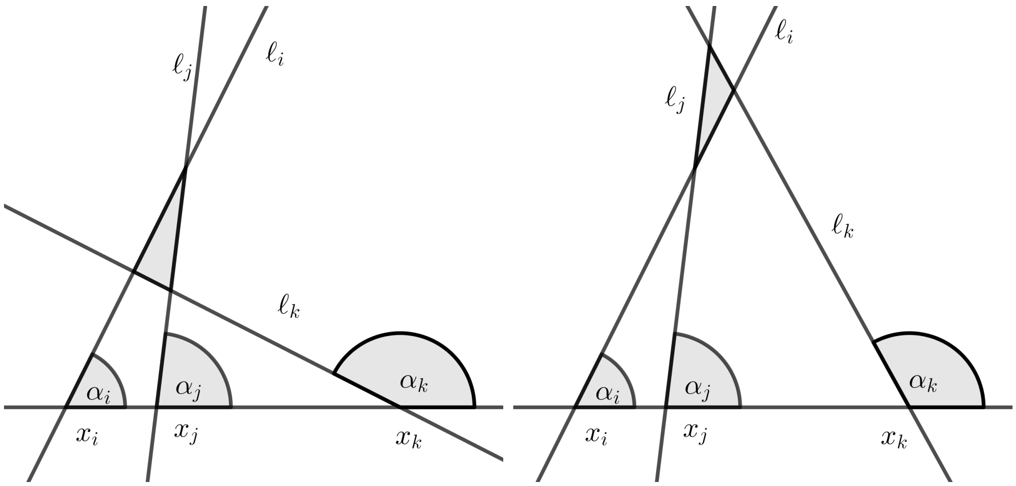


Figure 1: The calculation of the triangle area, formed by the lines ℓ_i, ℓ_j and ℓ_k

$$\frac{(x_j - x_i)^2}{2(y_i - y_j)} + \frac{(x_k - x_j)^2}{2(y_j - y_k)} + \frac{(x_i - x_k)^2}{2(y_k - y_i)}.$$

Therefore the problem of finding an arrangement of lines determining $f(n)$ triangles of unit area is equivalent to finding some reals $x_1 < x_2 < \cdots < x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$ such that (1) is satisfied for the maximal number of index triples. □

2.2.2 Improved upper bound for $f(n)$

We improve here the bound achieved by Corollary 2.5. To do this, we recall a recent result of Sharir and Zahl [20], which is a strengthening of the above mentioned Pach-Sharir Theorem 2.4.

Theorem 2.7 (Incidences between points and algebraic curves, [20])**.** *Let $\mathcal{C}$ be a set of n algebraic plane curves that belong to an s-dimensional family of curves, no two of which share a common irreducible component. Let P be a set of m points in the plane. Then for any $\varepsilon > 0$, the number $I(P, \mathcal{C})$ of incidences between the points of P and the curves of $\mathcal{C}$ satisfies*

$$I(P, \mathcal{C}) = O\left(m^{\frac{2s}{5s-4}} n^{\frac{5s-6}{5s-4}+\varepsilon}\right) + O_D\left(m^{2/3}n^{2/3} + m + n\right)$$

The implicit constant in the first term depends on ε, s, the maximum degree of the curves, and also the "complexity" of the family of curves from which the set $\mathcal{C}$ is selected.

Now we are ready to prove our main result.

Theorem 2.8. *For the maximum number of triangles of unit area, we have $f(n) = O(n^{\frac{9}{4}+\varepsilon})$ for every fixed $\varepsilon > 0$.*

Proof. Consider the additive combinatorial equivalent form of the problem in Equation 1, take the solution set with maximum number of solutions and denote it by H. For every ordered pair $(x_i, y_i), (x_j, y_j)$ where $x_i < x_j$, the solutions of Equation (1) are points (x_k, y_k) of a bounded degree rational curve defined by (1), with the condition that $x_j < x_k$ must hold. Hence we obtain at most $\binom{n}{2}$ plane curves belonging to a 4-dimensional family (since the family depends on the real values $\{x_i, y_i, x_j, y_j\}$). Notice also that no two of these curves share a common irreducible component (cf. Lemma 4.1). Applying the result of Sharir and Zahl 2.7 we get the desired bound. □

The lower bound for $f(n)$ follows from the results in the next section, by scaling the triangles of minimum area to have area 1.

3 Number of maximum and minimum area triangles, bounds on $m(n)$ and $M(n)$

3.1 Minimum area triangles

In this subsection we prove Theorem 1.6 by determining the maximal possible number of triangles of minimal area constituted by n lines, up to a factor 2. This will follow from the results on the lower and upper bound below.

Proposition 3.1. $m(n) \leq \lfloor n(n-2)/3 \rfloor - \mathbb{I}_{\{n:\ n \equiv 0,2 \pmod 6\}}$,

where $\mathbb{I}$ denotes the indicator function.

Proof. Observe that if a triangle is of minimal area, then none of the lines can intersect its sides. Hence the maximal number of triangles of minimal area is at most the number of triangular faces $K(n)$ in a planar graph that can be produced by n straight line segments. The latter problem became famous as the so-called Tokyo puzzle or the problem of Kobon triangles, due to Kobon Fujimura. Saburo Tamura made progress on the Kobon Triangle problem by proving that $K(n) \leq \lfloor n(n-2)/3 \rfloor$ which was refined later by Bader and Clément [1] to obtain the upper bound $\lfloor n(n-2)/3 \rfloor - 1$ if $n \equiv 0, 2 \pmod 6$. Note that this bound is asymptotically sharp as Füredi and Palásti constructed a general arrangement to prove $K(n) \geq \lfloor n(n-3)/3 \rfloor$ [11], see also the construction of Forge and Ramírez-Alfonsín [10]. Note that a closely related problem, asking the same question on the real projective plane instead of the Euclidean plane is asked by Grünbaum [12] and solved by Roudneff [19]. □

Proposition 3.2. *Suppose that $n \geq 3$. Then*

$$m(n) \geq \begin{cases} 6l^2 & \text{if } n = 6l, \\ 6l^2 + 2jl + j - 2 & \text{if } n = 6l + j,\ 1 \leq j \leq 5. \end{cases}$$

Proof. Take the grid depicted in Figure 2. Choose n lines such that they are as close to the center of a hexagonal face as possible. If there are 2, 3 or 4 lines in the outermost layer, pick these to be in consecutive clockwise position. The number of small triangular faces can be determined by a simple calculation that we leave to the reader. □

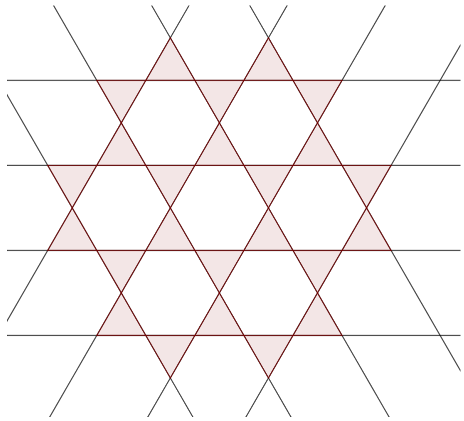


Figure 2: A hexagonal grid formed by 12 lines

Conjecture 3.3. *The lower bound of Proposition 3.4 is sharp if n is large enough.*

Note that these lower bounds are not met if n is small. Zamfirescu [21] recently proved that even the number of facial *congruent* triangles exceeds this bound if $n \leq 12$, see Table 1. On the other hand, the construction described in Proposition 3.4 provides a general lower bound as well for the number of facial *congruent* triangles in terms of the number of lines, which exceeds the bound of Zamfirescu if n is large.

# of lines, n	3	4	5	6	7	8	9	10	11	12
# of congruent facial triangles, lower bound	1	2	5	6	≥ 9	≥ 12	≥ 15	≥ 20	≥ 23	≥ 26
# of congruent triangles, lower bound via Prop. 3.2	1	2	3	6	7	10	13	16	19	24
# of congruent triangles, lower bound via Prop. 3.4	0	1	2	4	6	8	12	14	18	22

Table 1: Comparison of the constructions for the number of congruent or minimal area triangles in small cases

We remark that beside the presented construction, we can obtain the same order of magnitude in an essentially different way as well.

Proposition 3.4. *Suppose that $n \geq 3$. Then*

$$m(n) \geq \begin{cases} 6l^2 + 2jl - 2 & \text{if } n = 6l + j,\ j \in \{0, \pm 1, \pm 2\}, \\ 6l^2 + 6l & \text{if } n = 6l + 3. \end{cases}$$

Proof. Take a triangular grid. If $n \equiv 3 \pmod 6$, choose those n lines of the grid which are the closest to a fixed point on the grid. If $n \not\equiv 3 \pmod 6$, choose those n lines of the grid which are the closest to a fixed point which is the center of a triangle in the grid. A simple inductive argument shows that the number of constructed facial triangles equals the desired quantity, see Figure 3. □

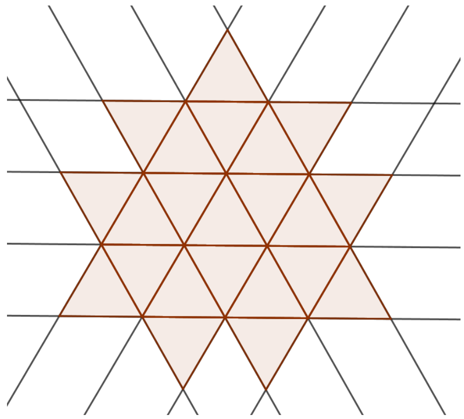


Figure 3: A triangular grid formed by 12 lines

The constructions differ in several aspects. Firstly, the former one does not contain concurrent triples of lines. Secondly, while in the upper bound of Grünbaum [12], or Bader and Clément [1] a key observation was that every line segment between consecutive intersections on a line belongs to at most one minimum area triangle, this property appears only in the former construction. If Conjecture 3.3 holds, it would imply that the extremal structure is not unique, which typically indicates the toughness of problem.

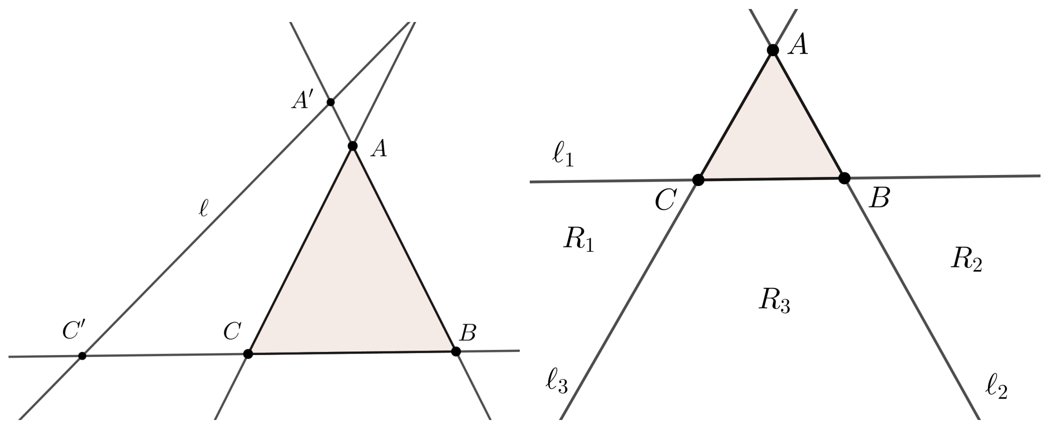
3.2 Maximum area triangles

We start with a construction to prove the lower bound of Theorem 1.7 on the number of maximum area triangles. The main idea is the following. Suppose you have a construction with some number of maximal area triangles. Then we can add a new line that doesn't create large triangles, i.e. the maximal area doesn't increase. We can slide this line until it creates an extra triangle of maximal area. This way we can create a new maximal area triangle per line. To improve this we will show that we can add five lines together to get seven new maximal area triangles. Five of the new maximal triangles will appear between these five new lines and then by sliding the five lines together we will get two extra ones.

The precise construction requires a couple of lemmas first.

Proposition 3.5. *Let ABC be one of the maximal area triangles in the arrangement and let ℓ be one of the lines of the arrangement. Then either ℓ intersects the interior of ABC or it is parallel to one of the sides of ABC.*

Proof. If ℓ is not parallel to one of the three sides then it intersects each of the three lines. Suppose ℓ avoids the interior of the triangle. By symmetry we can assume that ℓ runs as in Figure 4a. Then $A'BC'$ is a triangle of larger area which contradicts the maximality of ABC. □



(a) A line avoiding a maximal triangle (b) Regions around a triangle

Figure 4: Line positions with respect to a maximum area triangle

Proposition 3.6. *Suppose that there are no parallel lines in the arrangement and that the triangle Δ, formed by lines (ℓ_1, ℓ_2, ℓ_3), is a maximal area triangle. Then all the maximal area triangles that are supported by ℓ_1 lie on the same side of ℓ_1.*

Proof. Suppose that a triangle Δ', formed by the lines (ℓ_1, ℓ_4, ℓ_5), is also a maximal area triangle and it lies on the opposite side of ℓ_1. Let $P = \ell_4 \cap \ell_5$ and consider the possible positions of P. We will denote the three regions by R_1, R_2 and R_3 as seen in Figure 4b.

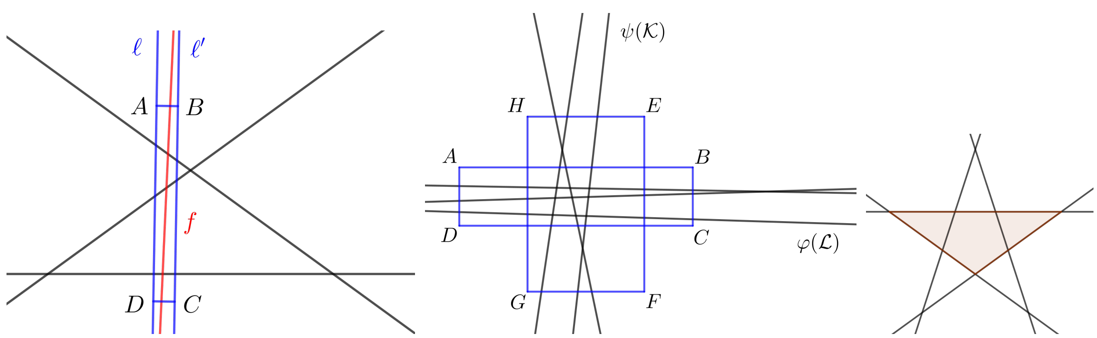
Assume that P lies in the interior of $R_1 \cup R_3$. By Proposition 3.5 we know that ℓ_4 and ℓ_5 must intersect the interior of the triangle ABC, therefore they intersect ℓ_1 on the interior of the $\overrightarrow{BC}$ ray. But then the line ℓ_2 avoids the maximal triangle Δ', contradicting Proposition 3.5. Similarly P cannot lie in the interior of $R_2 \cup R_3$. □

Proposition 3.7. *If there are no parallel lines in an arrangement then we can add a new line ℓ to the arrangement such that it supports no maximal area triangle in the new arrangement.*

Proof. Pick an arbitrary direction that is not parallel to any of the lines of the arrangement. Choose ℓ to be the line that has the chosen direction and for which the largest new triangle area created is the smallest possible. Let's say this area is q. Then ℓ must support two triangles on opposite sides that have area q. Otherwise we could translate ℓ slightly in one direction to decrease all the new areas below q. By Proposition 3.6 this implies that the q cannot be the maximal area in the whole arrangement. □

Proposition 3.8. *If there are no parallel lines in an arrangement then we can find a rectangle $ABCD$ such that if we add any line to the arrangement that intersects both AB and CD we create no new maximal area triangles.*

Proof. By Proposition 3.2 we can find a line ℓ that creates no new maximal area triangles. Let ℓ' be a line parallel to ℓ which also doesn't create a new maximal triangle and lies so close to ℓ that no two line of the arrangement intersects each other between ℓ and ℓ'. Then any line f that intersects all lines of the arrangement between ℓ and ℓ' doesn't create a new maximal area triangle. This follows from the fact that if f supports a triangle then either ℓ or ℓ' avoids that triangle, so by Proposition 3.5 the triangle cannot be maximal. Then we can choose points A, D on ℓ and B, C on ℓ' appropriately, see Figure 5a. □



(a) No new maximal triangles. (b) Combining two constructions (c) Pentagon

Figure 5: Ingredients for the recursive construction

For an arrangement $\mathcal{L}$ let $T(\mathcal{L})$ denote the number of maximal area triangles. For example $T(\mathcal{L}) = 5$ if $\mathcal{L}$ consists of five lines forming a regular pentagon. For an affine transformation φ let $\varphi(\mathcal{L})$ denote the image of $\mathcal{L}$.

Proposition 3.9. *If $\mathcal{L}$ and $\mathcal{K}$ are arrangements of lines that contain no parallel lines then there exist affine transformations φ and ψ such that $T(\varphi(\mathcal{L}) \cup \psi(\mathcal{K})) \geq T(\mathcal{L}) + T(\mathcal{K}) + 2$.*

Proof. We can assume that the maximal area triangles have the same area in $\mathcal{L}$ and $\mathcal{K}$. Using Proposition 3.8 we can define rectangle $ABCD$ for $\mathcal{L}$ and rectangle $EFGH$ for $\mathcal{K}$. Then applying an area preserving affine transformation we can place the two construction such that the two rectangles cross each other (see Figure 5b).

Now every line of $\varphi(\mathcal{L})$ crosses EF and GH and every line of $\psi(\mathcal{K})$ crosses AB and CD. By Proposition 3.8 this means that in the new construction the maximal area triangles are the same as they are in $\mathcal{K}$ and $\mathcal{L}$. So we have exactly $T(\mathcal{L}) + T(\mathcal{K})$ maximal triangles.
Finally we increase this number by two in two steps. Translate first the lines of $\varphi(\mathcal{L})$ together in an arbitrary direction until a new maximal area triangle appears, formed by lines both from the translates of $\varphi(\mathcal{L})$ and $\psi(\mathcal{K})$. We may assume that only one such triangle Δ^* is formed, and it has exactly one supporting line ℓ^* in $\psi(\mathcal{K})$. Now if we translate again the lines of $\varphi(\mathcal{L})$, this time along the line ℓ^*, then obviously neither the area of triangles formed by the lines from $\varphi(\mathcal{L})$ or $\psi(\mathcal{K})$, nor the area of Δ^* will change. However, some translated lines of $\varphi(\mathcal{L})$ will eventually form yet another triangle of maximal area together with some lines from $\psi(\mathcal{K})$. $\square$

It is easy to see that the lower bound of Theorem 1.7 follows. We start with five lines forming a regular star pentagon (see Figure 5c). Then we use Proposition 3.9 repeatedly, always using the previous construction as $\mathcal{L}$ and five lines forming a regular pentagon as $\mathcal{K}$.

Theorem 3.10. *$M(n) \leq \frac{2}{3}n(n-2)$.*

Proof. We will show that in an arrangement of n lines, any fixed line ℓ supports at most $2(n-2)$ triangles of maximal area. This immediately implies the statement of the theorem.

Let ℓ be a fixed line in the arrangement. We may assume that all other lines intersect it as otherwise they would not form any triangle together. Consider ℓ as the x axis of a coordinate system, and let x_i denote the x coordinate of the intersection of ℓ and ℓ_i for all $i = 1, 2, \ldots, n-1$. We also use the notation y_i for the cotangent of the (directed) angle determined by ℓ and ℓ_i. As we have seen before, the area of the triangle T_{ij} determined by the lines ℓ, ℓ_i and ℓ_j is $\text{Area}(T_{i,j}) = \frac{(x_i - x_j)^2}{2|y_i - y_j|}$. (If $x_i = x_j$ or $y_i = y_j$ then there is no triangle to speak of.) If the sign of $x_i - x_j$ and $y_i - y_j$ is the same, then the triangle is located under ℓ, otherwise it is located over it.

Without loss of generality, we may assume that the maximal triangle area is $1/2$. Then $(x_i - x_j)^2 \leq |y_i - y_j|$ applies to all pairs (i, j), with equality if and only if $\text{Area}(T_{i,j})$ is maximal.

Let us define the graph G_ℓ^+, and resp. G_ℓ^- on the vertex set $\{v_1, v_2, \ldots, v_{n-1}\}$ and connect v_i to v_j if $(x_i - x_j)^2 = |y_i - y_j|$ and the sign of $x_i - x_j$ and $y_i - y_j$ is the same or respectively, the opposite. We will show that there is no cycle in G_ℓ^+, therefore $|E(G_\ell^+)| \leq n-2$ holds for the cardinality of the edge set. The same argument applies to G_ℓ^- as well, yielding $|E(G_\ell^-)| \leq n-2$. Therefore the total number of edges, which is equal to the number of triangles of maximal area supported by ℓ, is at most $2(n-2)$.

Assume by contradiction that there is a cycle $v_{i_1}v_{i_2}\dots v_{i_k}$ in G_ℓ^+. We will get a contradiction using two simple propositions, where we consider the indexing of the vertices modulo k.

Proposition 3.11. *The signs of $x_{i_t} - x_{i_{t+1}}$ and $x_{i_{t+1}} - x_{i_{t+2}}$ are the opposite.*

Proof. Assume that the signs are the same. Then

$$|y_{i_{t+2}} - y_{i_t}| = |y_{i_{t+2}} - y_{i_{t+1}}| + |y_{i_{t+1}} - y_{i_t}| = (x_{i_{t+1}} - x_{i_{t+2}})^2 + (x_{i_t} - x_{i_{t+1}})^2 < (x_{i_t} - x_{i_{t+2}})^2$$

would hold, a contradiction. □

Proposition 3.12. *There are no four vertices v_a, v_b, v_c and v_d in G_ℓ^+ such that $x_a < x_b < x_c < x_d$ and $v_av_c, v_bv_c, v_bv_d \in E(G_\ell^+)$.*

Proof. Assume that there are four such vertices. Then

$$(x_d - x_a)^2 \le y_d - y_a = (y_d - y_b) + (y_c - y_a) - (y_c - y_b) = (x_d - x_b)^2 + (x_c - x_a)^2 - (x_c - x_b)^2$$

After rearranging, we get

$$x_ax_c + x_bx_d \le x_ax_d + x_bx_c,$$

which can be written as $(x_a - x_b)(x_c - x_d) \le 0$, a contradiction. □

Returning to the cycle $v_{i_1}v_{i_2}\dots v_{i_k}$, Proposition 3.11 implies that k is even. We may assume that $|x_{i_1} - x_{i_2}| > |x_{i_2} - x_{i_3}|$ after shifting the indexing of the vertices if necessary. This means that x_{i_3} is between x_{i_1} and x_{i_2}.

Proposition 3.11 tells us that x_{i_4} must be in the same direction from x_{i_3} as x_{i_2}. However Proposition 3.12 implies that it can't be past x_{i_2}. Note that $x_{i_2} = x_{i_4}$ is also impossible since this would imply $y_{i_2} = y_{i_4}$ and $\ell_{i_2} = \ell_{i_4}$. Therefore x_{i_4} must be between x_{i_2} and x_{i_3}.

Following this argument, we find that $x_{i_{t+2}}$ must be between x_{i_t} and $x_{i_{t+1}}$ for all $t = 1, 2, \dots, k-2$. Then the vertices $v_1, v_{k-1}, v_k, v_{k-2}$ violate Proposition 3.12, a contradiction. □

Remark 3.13. *Theorem 3.10 can be even strengthened, as $M(n) \le \frac{1}{3}n(n-1)$ also holds. Indeed, one can verify that Proposition 3.6 is true in a more general form, namely if there are parallel lines in the line arrangement, then there may exist maximal area triangles on both sides of a fixed line ℓ, but on one of the sides there is no more than one maximal area triangle. This result yields $|E(G_\ell^-)| + |E(G_\ell^+)| \le n-1$ in the proof above, implying our stated improvement. The details are left to the interested reader.*

4 Lines defining distinct area triangles

In this section we assume that the lines in the original arrangement are in general position. More specifically, we will require that no six of them are tangent to a common quadratic curve on the plane.

To prove Theorem 1.8, we begin with the following result.

Lemma 4.1. *Let r_1 and r_2 be two rays from a point O and $\lambda \in \mathbb{R}^+$ fixed. Then the lines that create with r_1 and r_2 a triangle of area λ are all tangent to a fixed hyperbola.*

Proof. Affine transformations preserve lines, conics and ratios of areas. Therefore, we may assume that r_1 and r_2 are perpendicular and correspond to the positive parts of the x and y axis respectively.

Now, for a positive real number c consider the hyperbola $xy = c$ and (x_1, y_1) any point on it. Let P_1 and P_2 the intersections of the line that is tangent to the hyperbola at (x_1, y_1) and the x and y axis. A simple calculation shows that the area of the triangle OP_1P_2 is $4c$.

Any line that intersects the positive parts of the x and y axis must be tangent to exactly one of these hyperbolas, and as seen above the area of the triangle it defines depends completely and injectively on c. Therefore, triangles with the same area must all be tangent to a fixed one of these hyperbolas. □

If we take two intersecting lines ℓ_1 and ℓ_2 and apply the result above to the four quadrants they define, we obtain the following corollary.

Corollary 4.2. *Let ℓ_1 and ℓ_2 be two intersecting lines. Then for any fixed value λ, there can be at most 20 lines in general position such that for any of them ℓ, the triangle defined by ℓ, ℓ_1 and ℓ_2 has area λ.*

The second ingredient that we use is a rainbow Ramsey result. We apply the following particular version of a result proven by Conlon et al. [4] and independently by Martínez-Sandoval, Raggi and Roldán-Pensado [15] when they were studying geometric results with a similar combinatorial flavour.

Theorem 4.3. *Let H be an m-uniform hypergraph on the vertex set V and k a positive integer. Suppose that the hyperedges of H are coloured in such a way that no 2 vertices lie in k edges of the same color.*

Then there exists a set of $\Omega_k(n^{1/(2m-1)})$ vertices for which all the hyperedges receive distinct colors.

We are ready to prove the main result of this section.

Proof of Theorem 1.8. Consider the 3-uniform hypergraph whose vertex set is the given set of lines, and provide a colouring by assigning to each triple the area of the triangle it defines. By Corollary 4.2, no pair of points belongs 21 or more triples of the same color. Therefore, by Theorem 4.3 we obtain a set of $\Omega(n^{1/5})$ lines such that the triangles that they define have all distinct areas. □

5 Discussion and open problems

One could also raise here an analogue question to the well known problem due to Erdős, Purdy and Strauss, which is formulated as

Problem 5.1 (Erdős, Purdy, Straus, [8])**.** *Let S be a set of n points in $\mathbb{R}^d$ not all in one hyperplane. What is the minimal number of distinct volumes of non-degenerate simplices with vertices in S?*

Concerning the case $d = 2$, we refer to e.g. [5] and its reference list. Note that to obtain reasonable results on the cardinality of *distinct areas*, one has to prescribe certain restrictions to avoid huge classes of parallel lines hence obtaining only few triangles. However, having assumed e.g. that no pair of parallel lines appear, the distribution of the areas may be modified significantly. Indeed, we conjecture that not only the cardinality of the minimum area triangles, but also the cardinality of the unit area triangles drops to $O(n^2)$ in that case, and in fact we could not even find evidence that the order of magnitude is $\Omega(n^2)$.

The proof of the upper bound on the number of maximum area triangles was relying on an argument about maximum area triangles sharing a common line ℓ that provides a linear upper bound. Although it is easy to see that a linear lower bound is realisable by a set of $n-2$ tangent and two asymptotes of a hyperbole branch see Figure 6, we conjecture that this won't provide the right (quadratic) order of magnitude for $M(n)$. Note that this phenomenon appeared concerning the unit area triangles as well when we compared $g(n)$ and $f(n)$, see Section 2.

In fact, we believe that the following holds.

Conjecture 5.2. *The order of magnitude of $M(n)$, largest possible number of triangles with maximum area in arrangements of n planar lines is $O(n^{1+\varepsilon})$ for every $\varepsilon > 0$.*

In general we have seen that in these types of combinatorial geometry problems, small (or minimum) distances (or areas) may occur much more frequently than large (or maximum) distances (areas).

Supposing that this assertion holds, it raises yet another interesting inverse research problem from a statistical point of view.

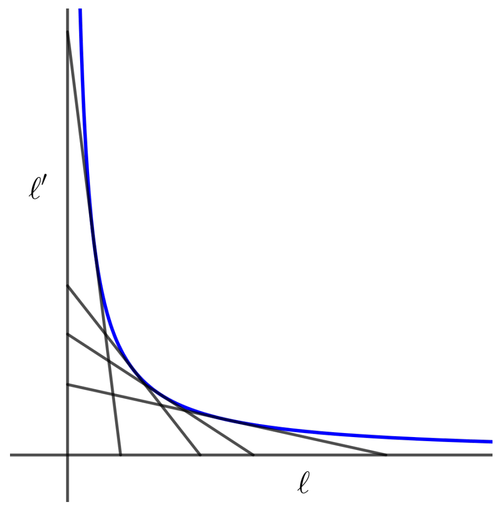


Figure 6: Maximum area triangles lying on ℓ, formed by ℓ, ℓ', and a third line from the tangent line set.

Problem 5.3. *A set of n lines are given on the plane. Assume that the cardinality of triangles determined by a triple of lines having unit area is $\phi(n)$, $\phi(n) \gg n$. Prove a lower bound (in terms of $\phi(n)$) on the number of triangles having area greater than* 1*.*

The analogue of Problem 5.3 for the original Erdős-Purdy problem on distances in a planar point set seems also widely open. Some results were obtained by Erdős, Lovász and Vesztergombi [6].

We also note that the problem may be investigated in a finite field setting as well, similarly to [13].

The bound in Theorem 1.8 can be improved by a logarithmic factor, as mentioned in [4]. The problem could also be generalized to higher dimensions as follows.

Problem 5.4. *A set of n hyperplanes in general position are given in $\mathbb{R}^d$. What is the maximum number $D_d(n)$ such that we can always find a subset of these hyperplanes of this size for which all the simplices that they define have distinct n-dimensional volume?*

We finish by mentioning that there are very few geometric problems with this combinatorial flavour in which the bounds are asymptotically tight. A related question concerning circumradii is discussed in [14].

Acknowledgement We are grateful to Nóra Frankl for the fruitful discussion on the topic.

References

[1] Bader, J., G. Clement, G. (2007). Tighter upper bound for the number of Kobon triangles, draft version.

[2] Brass, P., Moser, W. O. J., Pach, J. (2005). Research Problems in Discrete Geometry, Springer.

[3] Brass, P., Rote, G., Swanepoel, K. J. (2001). Triangles of extremal area or perimeter in a finite planar point set. Discrete & Computational Geometry, 26(1), 51–58.

[4] Conlon D., Fox J., Gasarch W., Harris D.G., Ulrich D., Zbarsky S. (2015). Distinct volume subsets. SIAM J. Discrete Math., 29(1), 472–480.

[5] Dumitrescu, A., Tóth, C. D. (2007). Distinct triangle areas in a planar point set. In International Conference on Integer Programming and Combinatorial Optimization (pp. 119–129). Springer, Berlin, Heidelberg.

[6] Erdős, P., Lovász, L., Vesztergombi, K. (1989). On the graph of large distances. Discrete & Computational Geometry, 4(6), 541–549.

[7] Erdős, P., Purdy, G. (1971). Some extremal problems in geometry, J. Combin. Theory Ser. A 10, 246–252.

[8] Erdös, P., Purdy, G., Straus, E. G. (1982). On a problem in combinatorial geometry. Discrete Mathematics, 40(1), 45–52.

[9] Erdős, P. (1957). Néhány geometriai problémáról (On some geometrical problems, in Hungarian), Math. Lapok, 8, 86–92

[10] Forge, D., Alfonsin, J. R. (1998). Straight line arrangements in the real projective plane. Discrete & Computational Geometry, 20(2), 155–161.

[11] Füredi, Z., Palásti, I. (1984). Arrangements of lines with a large number of triangles, Proc. Amer. Math. Soc. 92, 561–566

[12] Grünbaum, B. (1972). Arrangements and spreads. American Mathematical Society.

[13] Iosevich, A., Rudnev, M., Zhai, Y. (2015). Areas of triangles and Beck's theorem in planes over finite fields. Combinatorica 35(3), 295–308.

[14] Martínez-Sandoval L., Roldán-Pensado E. (2015). Points defining triangles with distinct circumradii. Acta Mathematica Hungarica. 145(1). 136–141.

[15] Martínez-Sandoval L., Raggi M., Roldán-Pensado E. (2015). A sunflower anti-Ramsey theorem and its applications. ArXiv e-prints. 1505.05170.

[16] Pach, J., Sharir, M. (1992). Repeated angles in the plane and related problems, J. Combinatorial Theory Ser. A 59 12–22.

[17] Pach, J., Sharir, M. (1998). On the number of incidences between points and curves. Combinatorics, Probability and Computing, 7, 121-127

[18] Raz, O. E., Sharir, M. (2017). The number of unit-area triangles in the plane: Theme and variation. Combinatorica, 37(6), 1221–1240.

[19] Roudneff, J. P. (1996). The maximum number of triangles in arrangements of pseudolines. journal of combinatorial theory, Series B, 66(1), 44–74.

[20] Sharir, M., Zahl, J. (2017). Cutting algebraic curves into pseudo-segments and applications. Journal of Combinatorial Theory, Series A, 150, 1–35.

[21] Zamfirescu, C. T. (2017). Congruent triangles in arrangements of lines. Ars Mathematica Contemporeana, 14(2), 359–373.