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A SIMPLE PROOF OF THE ZEILBERGER–BRESSOUD q-DYSON THEOREM

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Abstract. As an application of the Combinatorial Nullstellensatz, we give a short polynomial proof of the q-analogue of Dyson's conjecture formulated by Andrews and first proved by Zeilberger and Bressoud.

1. Introduction

Let $x_1, \dots, x_n$ denote independent variables, each associated with a nonnegative integer a_i. Motivated by a problem in statistical physics Dyson [6] in 1962 formulated the hypothesis that the constant term of the Laurent polynomial

$$\prod_{1 \le i \ne j \le n} \left(1 - \frac{x_i}{x_j}\right)^{a_i}$$

is equal to the multinomial coefficient $(a_1 + a_2 + \dots + a_n)!/(a_1!a_2! \dots a_n!)$. Independently Gunson [unpublished] and Wilson [25] confirmed the statement in the same year, then Good gave an elegant proof [9] using Lagrange interpolation.

Let q denote yet another independent variable. In 1975 Andrews [2] suggested the following q-analogue of Dyson's conjecture: The constant term of the Laurent polynomial

$$f_q(\boldsymbol{x}) := f_q(x_1, x_2, \dots, x_n) = \prod_{1 \le i < j \le n} \left(\frac{x_i}{x_j}\right)_{a_i} \left(\frac{qx_j}{x_i}\right)_{a_j} \in \mathbb{Q}(q)[\boldsymbol{x}, \boldsymbol{x}^{-1}]$$

must be

$$\frac{(q)_{a_1 + a_2 + \dots + a_n}}{(q)_{a_1} (q)_{a_2} \dots (q)_{a_n}},$$

where $(t)_k = (1-t)(1-tq)\dots(1-tq^{k-1})$ with $(t)_0$ defined to be 1. Specializing at $q = 1$, Andrews' conjecture gives back that of Dyson.

Despite several attempts [11, 22, 23] the problem remained unsolved until 1985, when Zeilberger and Bressoud [27] found a combinatorial proof. Shorter proofs for the equal parameter case $a_1 = a_2 = \dots = a_n$ are due to Habsieger [10], Kadell [12] and Stembridge [24]; they cover the special case A_{n-1} of a problem of Macdonald [20] concerning root systems, which was solved in full generality by Cherednik [5]. A shorter proof of the Zeilberger–Bressoud theorem, manipulating formal Laurent series, was given by Gessel and Xin [8].

Following up a recent idea of Karasev and Petrov we present a very short combinatorial proof using polynomial techniques. We find that their proof of the Dyson conjecture in [15] naturally extends for Andrews' q-Dyson conjecture. We note that built on the same basic principles but with more sophisticated details it is possible to prove a whole family of constant term identities for Laurent polynomials, including the Bressoud–Goulden theorems [4], conjectures of Kadell [13, 14], the q-Morris constant term identity [10, 12, 21, 26] and its far reaching generalizations conjectured by Forrester [3, 7]; see [16, 17, 18]. We decided to publish this proof separately because of its sheer simplicity.

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2. The proof

Note that if $a_i = 0$, then we may omit all factors that include the variable x_i without affecting the constant term of f_q. Accordingly, we may assume that each a_i is a positive integer. Consider the homogeneous polynomial

$$F(x_1, x_2, \dots, x_n) = \prod_{1\le i<j\le n} \left(\prod_{t=0}^{a_i-1} (x_j - x_i q^t) \cdot \prod_{t=1}^{a_j} (x_i - x_j q^t) \right) \in \mathbb{Q}(q)[\boldsymbol{x}].$$

Clearly, the constant term of $f_q(\boldsymbol{x})$ is equal to the coefficient of $\prod_i x_i^{\sigma - a_i}$ in $F(\boldsymbol{x})$, where $\sigma = \sum_i a_i$. To express this coefficient we apply the following effective version of the Combinatorial Nullstellensatz [1] observed independently by Lasoń [19] and by Karasev and Petrov [15]. A sketch of the proof is included for the sake of completeness.

Lemma 2.1. *Let $\mathbb{F}$ be an arbitrary field and $F \in \mathbb{F}[x_1, x_2, \dots, x_n]$ a polynomial of degree $\deg(F) \le d_1 + d_2 + \cdots + d_n$. For arbitrary subsets $A_1, A_2, \dots, A_n$ of $\mathbb{F}$ with $|A_i| = d_i + 1$, the coefficient of $\prod x_i^{d_i}$ in F is*

$$\sum_{c_1\in A_1} \sum_{c_2\in A_2} \cdots \sum_{c_n \in A_n} \frac{F(c_1, c_2, \dots, c_n)}{\phi_1'(c_1)\phi_2'(c_2)\dots\phi_n'(c_n)},$$

where $\phi_i(z) = \prod_{a\in A_i}(z-a)$.

Proof. Construct a sequence of polynomials $F_0 := F, F_1, \dots, F_n \in \mathbb{F}[\boldsymbol{x}]$ recursively as follows. For $i = 1, \dots, n$, let $F_i = F_i(\boldsymbol{x})$ denote the remainder obtained after dividing $F_{i-1}(\boldsymbol{x})$ by $\phi_i(x_i)$ over the ring $\mathbb{F}[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$. This process does not affect the coefficient of $\prod x_i^{d_i}$. The polynomial F_n satisfies $F_n(\boldsymbol{c}) = F(\boldsymbol{c})$ for all $\boldsymbol{c} \in A_1 \times \cdots \times A_n$ and its degree in x_i is at most d_i for every i. The unique polynomial with that property is expressed in the form

$$F_n(\boldsymbol{x}) = \sum_{\boldsymbol{c}\in A_1\times\cdots\times A_n} F(\boldsymbol{c}) \prod_{i=1}^{n} \prod_{\substack{\gamma\in A_i \\ \gamma\ne c_i}} \frac{x_i - \gamma}{c_i - \gamma}$$

by the Lagrange interpolation formula, hence the result. □

The idea is to apply this lemma taking $\mathbb{F} = \mathbb{Q}(q)$ with a suitable choice of the sets A_i such that $F(\boldsymbol{c}) = 0$ for all but one element $\boldsymbol{c} \in A_1 \times \cdots \times A_n$. Put $A_i = \{1, q, \dots, q^{\sigma - a_i}\}$, then $|A_i| = \sigma - a_i + 1$; and introduce $\sigma_i = \sum_{j=1}^{i-1} a_j$. Thus, $\sigma_1 = 0$ and $\sigma_{n+1} = \sigma$.

Claim 2.2. *For $\boldsymbol{c} \in A_1 \times \cdots \times A_n$ we have $F(\boldsymbol{c}) = 0$, unless $c_i = q^{\sigma_i}$ for all i.*

Proof. Suppose that $F(\boldsymbol{c}) \ne 0$ for the numbers $c_i = q^{\alpha_i} \in A_i$. Here α_i is an integer satisfying $0 \le \alpha_i \le \sigma - a_i$. Then for each pair $j > i$, either $\alpha_j - \alpha_i \ge a_i$, or $\alpha_i - \alpha_j \ge a_j + 1$. In other words, $\alpha_j - \alpha_i \ge a_i$ holds for every pair $j \ne i$, with strict inequality if $j < i$. In particular, all of the α_i are distinct. Consider the unique permutation π satisfying $\alpha_{\pi(1)} < \alpha_{\pi(2)} < \cdots < \alpha_{\pi(n)}$. Adding up the inequalities $\alpha_{\pi(i+1)} - \alpha_{\pi(i)} \ge a_{\pi(i)}$ for $i = 1, 2\dots, n-1$ we obtain

$$\alpha_{\pi(n)} - \alpha_{\pi(1)} \ge \sum_{i=1}^{n-1} a_{\pi(i)} = \sigma - a_{\pi(n)}.$$

Given that $\alpha_{\pi(1)} \ge 0$ and $\alpha_{\pi(n)} \le \sigma - a_{\pi(n)}$, strict inequality is excluded in all of these inequalities. It follows that π must be the identity permutation and $\alpha_i = \alpha_{\pi(i)} = \sum_{j=1}^{i-1} a_{\pi(j)} = \sigma_i$ must hold for every $i = 1, 2, \dots, n$. This proves the claim. □

This way finding the constant term of f_q is reduced to the evaluation of

$$\frac{F(q^{\sigma_1}, q^{\sigma_2}, \dots, q^{\sigma_n})}{\phi_1'(q^{\sigma_1})\phi_2'(q^{\sigma_2})\dots\phi_n'(q^{\sigma_n})},$$

where $\phi_i(z) = (z-1)(z-q)\dots(z-q^{\sigma-a_i})$. Here

$$\begin{aligned}\phi_i'(q^{\sigma_i}) &= \prod_{t=0}^{\sigma_i-1}(q^{\sigma_i}-q^t)\cdot\prod_{t=\sigma_i+1}^{\sigma-a_i}(q^{\sigma_i}-q^t)\\ &= \prod_{t=0}^{\sigma_i-1}q^t(q^{\sigma_i-t}-1)\cdot\prod_{t=1}^{\sigma-\sigma_{i+1}}q^{\sigma_i}(1-q^t)\\ &= (-1)^{\sigma_i}q^{\tau_i}\,(q)_{\sigma_i}\,(q)_{\sigma-\sigma_{i+1}}\end{aligned}$$

with $\tau_i=\binom{\sigma_i}{2}+\sigma_i(\sigma-\sigma_{i+1})$, whereas

$$\begin{aligned}F(q^{\sigma_1},q^{\sigma_2},\ldots,q^{\sigma_n}) &= \prod_{1\le i<j\le n}\left(\prod_{t=0}^{a_i-1}q^{\sigma_i+t}(q^{\sigma_j-\sigma_i-t}-1)\cdot\prod_{t=1}^{a_j}q^{\sigma_i}(1-q^{\sigma_j-\sigma_i+t})\right)\\ &= (-1)^u q^v\prod_{1\le i<j\le n}\left(\frac{(q)_{\sigma_j-\sigma_i}}{(q)_{\sigma_j-\sigma_{i+1}}}\cdot\frac{(q)_{\sigma_{j+1}-\sigma_i}}{(q)_{\sigma_j-\sigma_i}}\right)\\ &= (-1)^u q^v\prod_{i=1}^{n}\frac{(q)_{\sigma_i}\,(q)_{\sigma-\sigma_i}}{(q)_{\sigma_{i+1}-\sigma_i}}\end{aligned}$$

with $u=\sum_i(n-i)a_i$ and $v=\sum_i\left((n-i)a_i\sigma_i+(n-i)\binom{a_i}{2}+\sigma_i(\sigma-\sigma_{i+1})\right)$.

In view of the simple identity $\sum_i(n-i)a_i=\sum_i\sigma_i$, we have $u=\sum_i\sigma_i$, thus the powers of -1 cancel out. The same happens with the powers of q due to the following observation, which implies $v=\sum_i\tau_i$.

Claim 2.3. $\sum_i(n-i)\left(a_i\sigma_i+\binom{a_i}{2}\right)=\sum_i\binom{\sigma_i}{2}$.

Proof. We proceed by a routine induction on n. When $n=0$, both expressions are 0, and one readily checks the relation

$$\sum_{i=1}^{n}\left(a_i\sigma_i+\binom{a_i}{2}\right)=\binom{\sigma_{n+1}}{2},$$

which completes the induction. ☐

Putting everything together we obtain that the constant term of f_q is indeed

$$\begin{aligned}\frac{F(q^{\sigma_1},q^{\sigma_2},\ldots,q^{\sigma_n})}{\phi_1'(q^{\sigma_1})\phi_2'(q^{\sigma_2})\ldots\phi_n'(q^{\sigma_n})} &= \prod_{i=1}^{n}\frac{(q)_{\sigma_i}\,(q)_{\sigma-\sigma_i}}{(q)_{\sigma_i}\,(q)_{\sigma-\sigma_{i+1}}\,(q)_{\sigma_{i+1}-\sigma_i}}\\ &= \frac{(q)_\sigma}{\prod_{i=1}^{n}(q)_{\sigma_{i+1}-\sigma_i}}\\ &= \frac{(q)_{a_1+a_2+\cdots+a_n}}{(q)_{a_1}\,(q)_{a_2}\ldots(q)_{a_n}}.\end{aligned}$$

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