A bound on the number of points of a plane curve

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Abstract: A conjecture is formulated for an upper bound on the number of points in \( \text{PG}(2, q) \) of a plane curve without linear components, defined over \( \text{GF}(q) \). We prove a new bound which is half way from the known bound to the conjectured one. The conjecture is true for curves of low or high degree, or with rational singularity.

Let \( C \) be a plane curve of degree \( n \), defined over \( \text{GF}(q) \), without (rational) linear components. Let \( M_q \) denote the number of points of the projective plane \( \text{PG}(2, q) \) satisfying the equation of \( C \), counted without multiplicity. In this short note we discuss upper bounds on \( M_q \).

For the number of rational points the well-known bound is \( N \leq q + 1 + (n - 1)(n - 2)\sqrt{q} \) if \( C \) is absolutely irreducible ([Hasse-Weil] [9]); and we also have the combinatorial \( M_q \leq (n - 1)q + n \) (Barlotti [2], Thas [7]). Thas proved \( M_q \leq (n - 1)q + n - 2 \) (if \( n > 2 \)) and there were other improvements on this bound but under strong additional conditions only. [1, 4] and [8] use the assumption \( nq \leq q \) while [6] either gives a small improvement on the bound or uses an assumption \( n \ll q \). In fact these (more general) results give bounds for the size of a \((k, n)\)-arc, a point set intersecting every line in \( \leq n \) points; the set of points in \( \text{PG}(2, q) \) on \( C \) is obviously a special \((k, n)\)-arc. Here the following conjecture is made.

**Conjecture 1** A curve of degree \( n \) defined over \( \text{GF}(q) \), without linear components, has always \( M_q \leq (n - 1)q + 1 \) points in \( \text{PG}(2, q) \).

If true then for \( n = 1, 2, \sqrt{q} + 1, q - 1 \) it would be sharp as the curves \( X^2 - YZ, X\sqrt{q} + 1 + Y\sqrt{q} + 1 + Z\sqrt{q} + 1 \) and \( \alpha X^q - 1 + \beta Y^q - 1 - (\alpha + \beta)Z^q - 1 \) (where \( \alpha, \beta, \alpha + \beta \neq 0 \)) show. Note that Lunelli and Seč conjectured the similar bound for \((k, n)\)-arcs (and that conjecture was false). The conjecture is true if there exists a line skew to the curve and \((q, n) = 1 \), see Blokhuis [3]; also if there exists a line with 1 rational point of \( C \), see below; or if \( n \geq q + 2 \). If \( C \) has a rational singular point \( P \) then each line through \( P \) contains \( \leq n - 2 \) further points of \( C \) so \( M_q \leq (n - 2)(q + 1) + 1 \). (So from now on \( M_q = N_q \) can be assumed.)

We also remark that it is enough to prove the conjecture for absolutely irreducible curves; then for general \( C \) it can be proved by induction: let \( C \) split to the absolutely irreducible components \( C_1 \cup C_2 \cup ... \cup C_k \) with degrees \( n_1, ..., n_k \); if each \( C_i \) had \( \leq (n_i - 1)q + 1 \) points then in total \( C \) would have \( \leq \sum_{i=1}^{k} (n_i q - q + 1) = nq - k(q - 1) < nq - q + 1 \) points. So at least one of them, \( C_j \), say, has more than \( n_j q - q + 1 \) points, so more than \( n_j^2 \) (if \( n_j < q \), which can be supposed); so by [5], Lemma 2.24(i), \( C_j \) can be defined over \( \text{GF}(q) \) and then the induction hypothesis finishes the proof.

As a corollary we immediately see that if \( n \leq \sqrt{q} + 1 \) then \( q + 1 + (n - 1)(n - 2)\sqrt{q} \leq (n - 1)q + 1 \) proves the conjecture by Weil’s bound. Note that by the reasoning above, if \( C \) cannot be defined over \( \text{GF}(q) \) and \( n \neq q, q + 1 \) then the bound in the conjecture is true.

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The truth of the conjecture would also mean that the counterexamples for the Lunelli-Sce conjecture for \((k, n)\)-arcs are not pointsets of curves of the expected degree \(n\).

Here we go half way to proving the Conjecture:

**Theorem 2** A plane curve \(C\) of degree \(n\), defined over \(GF(q)\), without rational linear components, has always \(M_q \leq (n - 1)q + \lfloor \frac{n}{2} \rfloor\) points in \(PG(2, q)\).

**Proof:** For each rational point \(P\) of \(C\) choose a line \(t_P\) through \(P\) in such a way that \(|C \cap t_P| < n\): if \(C\) is smooth at \(P\) then let \(t_P\) be the tangent, if not then any line through \(P\) can be chosen. Let \(k\) be minimal such that every line \(t_P\) contains at least \(k\) points of the the curve (we always count without multiplicity, in \(PG(2, q)\)). Note that \(1 \leq k \leq n - 1\).

(i) Take a \(k\)-secant line \(t_P\), then counting the points on the lines through \(P\) we get the bound \(M_q \leq (n - 1)q + k\).

(ii) We say that a point \(P\) can see a tangency at \(Q\) if \(Q \in C\) and \(t_Q\) goes through \(P\); the possibility that \(P = Q\) is allowed. Now the points of the curve can see at least \(M_q k\) tangencies, so there is at least one point \(P\) of the curve seeing at least \(k\) tangencies. Counting the points of the curve looking around from \(P\) we “lose” at least one (from the total number) at each tangency that \(P\) can see, which gives \(M_q \leq (n - 1)q + n - k\) as an upper bound. Finally \(\min\{ (n - 1)q + k, (n - 1)q + n - k \} \leq (n - 1)q + \frac{n}{2}\).

Finally we remark that the estimates on the number of rational points on a hypersurface can be substantially improved using the new bound of the theorem above.

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**References**


