CALCULATION OF THE AVOIDING IDEAL FOR $\Sigma^{1,1}$

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Abstract. We calculate the mapping $H^*(BO;\mathbb{Z}_2) \to H^*(K^{1,0};\mathbb{Z}_2)$ and obtain a generating system of its kernel. As a corollary, bounds on the codimension of fold maps from real projective spaces to Euclidean space are calculated and the rank of a singular bordism group is determined.

1. Introduction and definitions. We work with homologies and cohomologies with $\mathbb{Z}_2$ coefficients, even when the coefficient ring is not indicated. We will investigate the spaces of the Kazarian construction (see [5]) and the maps in cohomology induced by the natural embeddings of these spaces into one another. In Section 2 these maps are calculated explicitly, this result is then used in Section 3 to provide bounds on the codimension of fold maps of real projective spaces into Euclidean spaces. Another application is demonstrated in Section 4, where we obtain a description of the rank of the unoriented right-left fold bordism group ($C^{1,0}(n,k)$ in the notation of [3]).

To reach these goals, the Kazarian construction will be considered for immersions, locally stable maps without $\Sigma^{1,1}$ singularities and maps without any constraints on their singularities. The fine details of the construction are exposed in [5]; we briefly recall its properties relevant to the aims of this paper. The Kazarian spaces of the classes of maps defined above, denoted here by $K^0 \approx BO(k)$, $K^{1,0}$ and $K^\infty \approx BO$ respectively, can be thought about as subspaces of the bundle of jets over $BO$ cut out by the appropriate restrictions, so we have natural embeddings $K^0 \supseteq K^{1,0} \supseteq K^\infty$, the composition of which will be denoted by $\overline{\nu} : K^0 \to K^\infty$; it is known to be homotopic to the standard embedding $BO(k) \to BO$. Whenever we have a mapping $f : M^n \to P^{n+k}$ with all singularities in a class $\tau$ (in our case: regular points only; regular points and folds only; and any singularities), the mapping inducing the stable normal bundle of $f$, $\nu_f \oplus \varepsilon^N : M \to BO(N+k) \to BO$ can be chosen to lie in $K^\tau (K^0, K^{1,0}, K^\infty$ respectively). Using

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an alternative (equivalent up to homotopic equivalence) way of constructing these spaces, we can obtain $K^{1,0}$ as the total space of the vector bundle $\xi$ over the base $B = BG_{\Sigma^1,0} \approx \mathbb{R}P^\infty \times BO(k)$, which has the form

$$\xi = l \oplus \gamma$$

with $l$ and $\gamma$ being the pullbacks of the tautological bundle over $\mathbb{R}P^\infty$ and $BO(k)$, respectively, glued to $K^0$. This gives us an embedding $b : B \to K^{1,0}$, and after factoring out $K^0$ by the projection $p : K^{1,0} \to (K^{1,0}, K^0)$ we obtain an embedding $\tilde{b} : B \to (K^{1,0}, K^0)$. By excision, for all cohomological purposes $\tilde{b}$ is the embedding of $B$ into the pair of the unit ball and unit sphere bundles of $\xi$ for a suitable metric, $(D\xi, S\xi)$. For the calculations, we will need to be able to identify the restrictions of the elements of $H^*(K^\infty)$ to $B$, which are the corresponding characteristic classes of the restriction of the virtual normal bundle $\nu$ over $K^\infty$ to $B$; it can be shown that stably, $\nu|_B \approx l \otimes \gamma \otimes l$.

The mappings defined above commute in a natural manner, implying the commutativity of the corresponding diagram of cohomological groups in all dimensions. The elements of those groups will be expressed in the terms of the usual generators $w_I \in H^*(BO)$ in case of $K^0$ and $K^\infty$, while the elements of $H^*(B)$ (and subsequently $H^*(D\xi, S\xi)$) will be expressed in terms of the generators $c \in H^1(\mathbb{R}P^\infty)$ and $v_I = w_I(\gamma) \in H^*(BO(k))$.

2. Calculation. The Stiefel-Whitney characteristic classes of the tensor product $l \otimes \gamma$ are easily calculated using the splitting lemma to be

$$w_i(l \otimes \gamma) = \sum_{j=0}^{i} \binom{k-j}{i-j} v_j c^{i-j}.$$ 

Inverting the total Stiefel-Whitney class of $l$ we have

$$w(-l) = w(l)^{-1} = 1 + c + c^2 + ..., \quad w_i(-l) = c^i,$$
so the characteristic classes of the sum are

\[ w_i(\nu|_B) = \sum_{s=0}^{i} w_s(l \otimes \gamma)v_{i-s}(-l) = \sum_{s=0}^{i} \sum_{j=0}^{s} \binom{k-j}{s-j} v_j e^{s-j} c^{i-j-s} = \sum_{j=0}^{i} v_j e^{i-j} \sum_{s=j}^{i} \binom{k-j}{s-j}. \]

If additionally \( i \geq k \), then the inner sum takes the form

\[ \sum_{s=j}^{i} \binom{k-j}{s-j} = \binom{k-j}{0} + \cdots + \binom{k-j}{k-j} + 0 + \cdots + 0 = 2^{k-j}, \]

if \( j \leq k \) and is 0 otherwise, so for these values of \( i \) we have

\[ w_i(\nu|_B) = \sum_{j=0}^{k} 2^{k-j} v_j e^{i-j} = v_k e^{i-k}. \]

Consider now the mapping \( b^* \circ g^* : w_I \mapsto w_I(\nu|_B) \) on monomials with \( \text{max } I > k \). By the formula derived above, the image will be divisible by \( w_{k+1}(\xi) = v_k c \), let \( a_I \in H^*(B) \) be such that \( b^* g^* w_I = v_k c a_I \) (if \( I = I^+ \cup I^- \) with \( \text{max } I^- \leq k \), \( \text{min } I^+ \geq k + 1 \) and \( \sum_{i \in I, (i-k) = S} a_I = w_I(\nu|_B) = v_k c b_I \). Since \( H^*(B) \) has no zero divisors, this implies \( a_I = b_I \) and hence

\[ g^* w_I = p^*(U a_I). \]

The mapping \( p^* \) is injective given that even \( \overline{p}^* = b^* \circ p^* \) is injective, so

\[ g^* \sum_{i \in I} w_I = p^* \sum_{i \in I} U a_I = 0 \iff \sum_{i \in I} a_I = 0 \]

if all of the index sets \( I \) satisfied \( \text{max } I > k \) to begin with. However, for \( \text{max } I \leq k \) we have \( u^* g^* w_I = \overline{u}^* w_I = w_I \), so if a class in \( H^*(K^\infty) \) lies in the kernel of \( g^* \), then all of its monomials (with non-zero coefficients) have to satisfy \( \text{max } I > k \).

**Theorem 1.** The avoiding ideal \( \mathcal{A} \) for the singularity \( \Sigma^{1,1} \) is generated as an \( H^*(K^\infty) \) ideal by the set

\[ \{ w_{k+l} w_{k+m} + w_{k+q} w_{k+r} | l, m, q, r \geq 0 \text{ and } l + m = q + r \geq 2 \}. \]

**Proof.** Denote by

\[ \mathcal{B} = (w_{k+l} w_{k+m} + w_{k+q} w_{k+r} | l, m, q, r \geq 0 \text{ and } l + m = q + r \geq 2)_{H^*(K^\infty)} \]

the ideal generated by the elements given in the statement of the theorem. It is easy to see that \( \mathcal{B} \subseteq \ker \overline{p}^* \) and

\[ a_{k+l,k+m} = v_k^2 c^{l+m} = v_k^2 c^{q+r} = a_{k+l,k+m} \]

holds for all the involved quartuples \((l, m, q, r)\), so by equality (1)

\[ \mathcal{B} \subseteq \mathcal{A}. \]
To finish the proof, it is sufficient to verify that
\[ \text{rank } A^n \leq \text{rank } B^n \quad \text{for all } n. \tag{2} \]

The left hand side can be calculated from the facts that \( \ker g^* = \text{ker } b^* g^* \).

Indeed, if \( b^* g^* \alpha = 0 \) for some \( \alpha \in H^*(BO) \), then set \( \alpha^- = \pi^* \alpha \in H^*(BO(k)) \subset H^*(BO) \) and \( \alpha^+ = \alpha - \alpha^- \). We have \( b^* g^* \alpha = \alpha^- (\nu) + \alpha^+ (\nu) \). Observe that the mapping \( H^*(BO(k)) \ni \alpha \mapsto \alpha^- (\nu) + \alpha^+ (\nu) \in H^*(B) \) is the sum of coordinate maps \( w_I \mapsto w_I (l \odot \gamma - l) = w_I (\gamma) + c \cdot (\ldots) \), so \( \alpha^- (\nu) \) written in the basis we use will contain every \( v_I \) for which \( \alpha^- \) contains \( w_I \). On the other hand, all of the monomials of \( \alpha^+ (\nu) \) contain \( c \) (since all \( w_{k+1} (\nu) = a^c v_k \) do), so if \( b^* g^* \alpha = 0 \), then \( \alpha^- = 0 \). By (1) we then have \( g^* \alpha = g^* \alpha^+ = p^* (U\alpha (\nu)/v_k c) = 0 \) and \( \alpha \in \ker g^* \).

To calculate the image of \( b^* g^* \), we know that \( b^* g^* w_I = w_I (\nu) \), in particular,
\[ b^* g^* w_{k+1} w_I = v_k c^a w_I (\nu). \]

If we choose any \( I \) with \( \text{max } I \leq k \), then \( w_I (\nu) = v_I + c \cdot (\ldots) \) shows that \( b^* g^* \) is onto the factor ring \( H^*(BO(k)) = H^*(B)/(c) \), and \( w_{k+1} w_I (\nu) = v_k v_I c^a + c^{a+1} \cdot (\ldots) \) shows that the image of \( b^* g^* \) in the slice \( c^a H^*(BO(k)) = c^a H^*(B)/(c^{a+1}) \) contains exactly the elements divisible by \( v_k \). Thus the image of \( b^* g^* \) is spanned by \( w_I \), \( \text{max } I \leq k \) and \( c^a w_I \), \( \text{max } I = k \).

Therefore
\[
\text{rank } A^n = \text{rank } \ker g^n = \text{rank } b^n g^n = \text{rank } H^n (K^\infty) - \text{rank } \text{im } b^n \circ g^n =
\begin{align*}
&= \{|a_0 \geq \cdots \geq a_m \geq 0| n = a_0 + \cdots + a_m| \\
&\quad - |\{k \geq a_0 \geq \cdots \geq a_m \geq 0| n = a_0 + \cdots + a_m| \\
&\quad - |\{k = a_0 \geq \cdots \geq a_m \geq 0| n = a_0 + \cdots + a_m| \\
&\quad - |\{a_0 > k \geq a_1 \geq \cdots \geq a_m \geq 0| n = a_0 + a_1 + \cdots + a_m| \\
&\quad = \{|a_0 \geq \cdots \geq a_m \geq 0| a_0 > k \text{ and } n = a_0 + \cdots + a_m|.
\end{align*}
\]

The right hand side of (2) can be estimated similarly, once we observe that the elements of \( H^n (BO)/B^n \) can be represented as sums of monomials \( w_I \) or \( w_{k+1} w_I \) with \( \text{max } I \leq k \); indeed, if a monomial is of form \( w_{k+1} w_{k+1} \cdot \ldots \cdot w_k \cdot \ldots \cdot w_1 \), we can change it by \( (w_{k+1} w_{k+1} + w_{k+1} w_{k+1} \cdot \ldots \cdot w_k \cdot \ldots \cdot w_1) \hat{w} \in B \) to get an equivalent representation \( w_{k+1} w_{k+1} \cdot \ldots \cdot w_k \hat{w} \) with less indices larger than \( k \). Thus we have \( \text{rank } H^n (K^\infty)/B^n \leq \text{rank } b^n \circ g^n \) (the number of words \( c^a w_k w_I \) with \( \text{max } I \leq k \) is the same as the number of words \( w_{k+1} w_I \) with \( \text{max } I \leq k \), implying \( \text{rank } B^n = \text{rank } H^n (BO) - \text{rank } H^n (BO)/B^n \geq \text{rank } H^n (K^\infty) - \text{rank } b^n \circ g^n = \text{rank } A^n \) and (2) holds, completing the proof. \( \blacksquare \)

As an immediate consequence, we obtain the following corollary which allows us to efficiently decide whether a characteristic number lies in the avoiding ideal or not:

**Corollary 2.** The avoiding ideal for the singularity \( \Sigma_{1,1} \) consists of elements
\[ \sum_{i \in I} w_I \text{ such that } \sum_{i \in I} c^S w_{k}^{I^+} w_I \cdot I^+ = 0, \]
where \( I \) contains only index sets \( I \) with \( \text{max } I > k \), \( I^+ \) denotes \( \bigcup \{J \subseteq I | \min J > k \} \) and \( S = \sum_{i \in I^+} (i - k) \).
3. Fold maps of projective spaces. As an application of Theorem 1, we will consider maps of projective spaces into Euclidean space. If we have a mapping \( f : M^* \to \mathbb{R}^n \) with only regular points and folds, then the classifying map of its stable normal bundle \( \nu_f : M \to \mathbb{R}^n \) is homotopic to a composition of a suitable \( \tilde{\nu}_f : M \to \mathbb{R}^n \) and the canonical embedding \( g : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \). Hence the induced mapping in cohomology \(-\nu_f : H^*(\mathbb{R}^n) \to H^*(M)\) decomposes as \( -\nu_f = (\tilde{\nu}_f)^* \circ g^* \) and consequently \( \ker (\nu_f) \supseteq \ker g^* \). In particular, all elements \( \alpha(l, m, q, r) = \sum_{j=0}^{n} \binom{2s+1}{j} \alpha^j \) must evaluate to 0 on \( \nu_f \). When we choose \( M = \mathbb{R}P^n \), then this evaluation is particularly easy to compute since if we denote by \( a \) the generator of \( H^1(\mathbb{R}P^n) \) and \( n = 2^t + s \) is the unique decomposition of \( n \) such that \( s \) and \( m < 2^s \) are nonnegative integers, then \( (a + 1)^{n-1} = \sum_{j=0}^{n} \binom{2s+1}{j} \alpha^j \).

Therefore \( \alpha(l, m, q, r)(\nu_f) = \sum_{j=0}^{n} \binom{2s+1}{j} \alpha^j \).

Start listing the values
\[
\binom{2^s-t-1}{\left\lfloor \frac{t}{2} \right\rfloor}, \binom{2^s-t-1}{\left\lfloor \frac{t}{2} \right\rfloor - 1}, \binom{2^s-t-1}{\left\lfloor \frac{t}{2} \right\rfloor - 2}, \ldots, \binom{2^s-t-1}{\left\lfloor \frac{t}{2} \right\rfloor - h}
\]

and assume that the first \( h \) elements of this sequence have the same parity while the next one has the opposite parity. If the sequence starts with even elements, then it is clear that any term \( w_{\alpha}w_{\beta} \) which does not evaluate to zero has \( \min \{b, c\} \leq \left\lfloor \frac{t}{2} \right\rfloor - h \), and an optimal \( \alpha \) is either \( \alpha(j, j+1, j+1, j+1) \) with \( i \) such that \( h < i \leq n-j \) and \( \left( \frac{2^s-t-1}{2^s-t-1} \right) \) is odd, or \( \alpha(j+1, j+1, j, j) \) with \( j = \left\lfloor \frac{t}{2} \right\rfloor - h \) if there is no such \( i \). If the sequence starts with odd elements, then the same argument with the roles of 0 and \( a^{s+t} \) reversed gives us that an optimal \( \alpha \) is either \( \alpha(j, j+i, j+1, j+1) \) with \( i \) such that \( h < i \leq n-j \) and \( \left( \frac{2^s-t-1}{2^s-t-1} \right) \) is even, or \( \alpha(j-1, j+1, j, j) \) with \( j = \left\lfloor \frac{t}{2} \right\rfloor - h \) if there is no such \( i \).

Therefore, we have to investigate the parity of \( F(j) = \binom{2^s-t-1}{j} \) for values of \( j \) close to \( n/2 \), and for that, we need to look at the binary expansions of \( 2^s-t-1 \) and \( j \); by [4], a binomial coefficient \( \binom{2^s-t-1}{j} \) is odd precisely when the binary expansion of \( j \) starts with digits 1 at all the places where the binary expansion of \( c \) has digits 1. Given the binary expansion
of \( n = 2^s + t \) we can obtain the binary expansion of \( 2^{s+1} - n = 1\ldots12 - n \) by bitwise negation, and the binary expansion of \( \lfloor \frac{n}{2} \rfloor \) is obtained by shifting right by one position. This implies that \( F([n/2]) \) is odd precisely when the binary expansion of \( n \) does not contain the substring \( 11\ldots \), and according to this we have two cases.

If \( n \) contains \( \ldots11\ldots \), first at position \( u: n = 2^u(8a+3)+b \) with \( u \) maximal and \( 0 \leq b < 2^u \). Then decreasing \( j \) starting from \([n/2] \) we will get even values of \( F(j) \) until the decrease does not affect the \( u^{th} \) digit since this is the highest 1 at a place where \( 2^s - t - 1 \) has a 0; once that location is reached, the highest value of \( j \) for which \( F(j) \) is odd has to copy the rest of the string from \( 2^s - t - 1 \), that is, \( j = 2^{u+2}a + 2^u - 1 - b > \frac{n}{2} - 2^{u+1} \). Increasing \( j \), on the other hand, does not change the parity of \( F(j) \) as long as \( j < 2^{u+2}(a+1) \) due to either the \( u^{th} \) or the \( u+1^{st} \) digit, which is more than \( 2^{u+1} \) steps so we don’t get a better estimate on \( k \) than \( k \geq j - 1 = 2^{u+2}a + 2^u - b - 2 \).

If \( n \) does not contain \( \ldots11\ldots \). In this case we first deal with the case of \( n \) odd; \( 2^s - t - 1 \) is even and both \([n/2] - 1 \) and \([n/2] + 1 \) are odd, so the sharpest possible estimate holds, \( k \geq \frac{n-3}{2} \). And if \( n \in 2^{p+1}\mathbb{Z} + 2^p \), \( p > 0 \), it is easy to see that increasing \( j \) first produces a parity change after \( 2^{p-1} \) steps and decreasing \( j \) does the same after \( 2^{p-1} + 1 \) steps, so the estimate is \( k \leq \frac{n}{2} - 2^{p-1} - 1 \).

We have thus proved the following result:

**Theorem 3.** If there exists a fold map \( \mathbb{R}P^n \to \mathbb{R}^{n+k} \), then

\[
\begin{align*}
k \geq & \begin{cases} 2^{s+1} - n - 2 & \text{if } \frac{5}{3}2^s < n < 2^{s+1} \\ \lfloor \frac{n}{2} \rfloor - 1 & \text{if } 2^s < n < \frac{5}{3}2^s \text{ is odd and } \forall u \lfloor \frac{n}{2^{u+1}} \rfloor \neq 3 \mod 4 \\ \frac{n}{2} - 2^{p-1} & \text{if } 2^{s-1} < n = 2^pm \leq 2^{s-1}2^{p} \text{ with } p > 0, \text{ odd } m \\ \text{and } \forall u \lfloor \frac{n}{2^{u+1}} \rfloor \neq 3 \mod 4 & \\ 2^{u+2}a + 2^u - b - 2 & \text{if } n = 2^u(8a+3)+b \text{ with } 0 \leq b < 2^u \text{ and } u \text{ maximal.} \end{cases}
\end{align*}
\]

**Remark.** When \( t > 2^s/3 \), our estimate on the codimension is one less than the geometric dimension of the stable normal bundle of \( \mathbb{R}P^n \), so allowing fold singularities does not decrease the necessary codimension for a mapping to \( \mathbb{R}^{n+k} \) to exist by more than 1 compared to the analogous estimate for immersions. In the case \( 0 < t < 2^s/3 \) this is no longer the case, but our estimate stays close to the sharpest possible value \( k = [n/2] \), for which any generic mapping can only have fold singularities anyway: the restriction on \( t \) implies \( p \leq s - 2 \Rightarrow k > 3n/8 \) in the second and third cases as well as \( a \geq 2 \Rightarrow 2^{u+2}a + 2^u - b - 2 = \frac{n}{2} - (2^{u+1} + \frac{3}{2}) + 2 \geq \frac{n}{2} - (2^{u+1} + \frac{3}{2}) > \frac{3n}{8} \) in the last case with the sole exception of \( n = 19 \) (alternatively, \( k \geq 7n/19 \) for all \( n \)).

4. Fold bordism groups. We will simply combine several previously known results. The trivial adaptation of [6, Theorem 14] to the case of unoriented manifolds reduces the calculation of the group \( C^{1,0}(n,k) \) to the calculation of the bordism group \( \mathcal{M}_{n+k}(X_{\Sigma^{1,0}}) \). By [1, Theorem 1.9],

\[
\mathcal{M}_{n+k}(X_{\Sigma^{1,0}}) \approx (H_\ast(X_{\Sigma^{1,0}};\mathbb{Z}_2) \otimes \mathcal{M}_\ast)_{n+k}.
\]

[6, Corollary 72] expresses \( X_{\Sigma^{1,0}} \) as \( \Gamma T\nu^k \) with \( \nu^k \) a bundle over \( K^{1,0} \), so we can apply the results of [2] to calculate the ranks of \( H_\ast(X_{\Sigma^{1,0}};\mathbb{Z}_2) \) from an additive basis of \( \overline{H}_\ast(T\nu^k) \approx \overline{H}_\ast(\nu^k) \).
Theorem 4. \( C^{1,0}(n, k) \approx Z_2 r_{n,k} \), where \( r_{n,k} \) is the number of different sets of multiindex pairs and a partition \( \{(I_1, J_1), \ldots, (I_s, J_s), d_1, \ldots, d_u\} \) such that

- Each \( J_m = (j_{m,0} \geq j_{m,1} \geq \ldots \geq j_{m,s_m}) \) consists of positive indices with \( j_{m,1} \leq k \).
- Each \( I_m \) is either empty or \( I_m = (i_{m,1} \geq \ldots \geq i_{m,t_m} > 0) \) with \( i_{m,1} \leq 2i_{m,2}, \ldots, i_{m,t_m-1} \leq 2i_{m,t_m} \).
- If \( I_m \) is not empty, then \( i_{m,1} - i_{m,2} - \ldots - i_{m,t_m} > j_{m,0} + \ldots + j_{m,s_m} + k \).
- \( \sum_m (i_{m,1} + \ldots + i_{m,t_m} + j_{m,0} + \ldots + j_{m,s_m}) = n - d_1 - \ldots - d_u \).
- \( d_1 \geq d_2 \geq \ldots \geq d_u > 0 \) do not contain numbers of form \( 2^v - 1 \).

Proof. We have

\[
C^{1,0}(n, k) \approx \mathfrak{N}_{n+k}(X_{\Sigma^{1,0}}) \approx \bigoplus \mathfrak{N}_d \otimes H_{n+k-d}(\Gamma T \nu).
\]

\( \mathfrak{N}_d \) is a free \( Z_2 \)-module with a basis enumerated by partitions of \( d \) into natural numbers not of form \( 2^v - 1 \), and \( \Pi_*(\Gamma T \nu) \) has an additive basis consisting of products of admissible elements of form \( Q^I(\Phi a) \) with \( \Phi \) the Thom isomorphism of \( \nu \) and \( a \) chosen from an additive basis of \( H_*(K^{1,0}; Z_2) \approx H_*(K_{1,0}; Z_2) \) with all \( a \) homogeneous. Choosing this basis to be the elements of form \( g^* w_j \) with \( \max(J \setminus \{\max J\}) \leq k \) as in the proof of Theorem 1, we obtain exactly our claim.

Remark. While this rank is clearly calculable for every \( n \) and \( k \), it does not seem to have a closed form that would ease its handling.

References