COBORDISMS OF FOLD MAPS OF $2K + 2$-MANIFOLDS INTO $\mathbb{R}^{3K+2}$

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Abstract. We will calculate $\text{Cob}^{1,0}(2k + 2, \mathbb{R}^{3k+2})$ for both oriented and unoriented cases.

1. Concept Given an "ideal" of singularities $\tau' = \{ \eta \} \cup \tau$ with a top singularity $\eta$ and a fixed codimension $k$ of the involved mappings, we can consider the classifying spaces $X_{\tau'}$ and $X_{\tau}$. It is known ([11]) that there is a fibration

$$X_{\tau'} \xrightarrow{\tau'} \Gamma T \xi^\eta,$$

where $\xi^\eta$ is the bundle associated to the universal $G_n$-bundle via the representation of $G_n$ in the image. Hence we have a long exact sequence

$$\cdots \to \pi_{N+1}(\Gamma T \xi^\eta) \to \pi_N(X_{\tau'}) \to \pi_N(X_{\tau}) \to \pi_{N-1}(X_{\tau'}) \to \cdots,$$

where the mapping $\pi_N(X_{\tau'}) \xrightarrow{\tau'} \pi_N(\Gamma T \xi^\eta)$ assigns to every map $f: M \to \mathbb{R}^N$ the map that classifies the immersion $f: \eta(f) \to \mathbb{R}^N$ with the $\xi^\eta$ normal structure added.

This will be applied to $\tau' = \{ \Sigma^{1,1,1,0} \} \cup \{ \Sigma^{1,0,0}, \Sigma^{1,1} \}$. Denoting the corresponding classifying spaces by $X_{1,1}$ and $X_{1,0}$, and denoting by $X_{\infty} = \Omega^{\infty} M(S) O(k + \infty)$ the classifying space for maps without restrictions on the singularities, we have

$$\cdots \to \pi_{N+1}(X_{1,1}) \xrightarrow{\tau'} \pi_{N+1}(T \xi^{1,1}) \to \pi_N(X_{1,0}) \to \pi_N(X_{1,1}) \to \pi_{N-1}(X_{1,1}) \to \cdots \approx \text{Cob}^{1,0}(N - k, \mathbb{R}^N)$$

and after calculating the involved groups and mappings we will be able to describe the groups $\text{Cob}^{1,0}(2k + 2, \mathbb{R}^{3k+2})$.

2. Calculations

Lemma 1. Given a vector bundle $\xi$ of rank $n \geq 1$ over a connected base $B$,

$$\pi_n(T \xi) = \begin{cases} 
\mathbb{Z} & \text{if } \xi \text{ is orientable} \\
\mathbb{Z}_2 & \text{if } \xi \text{ is not orientable} 
\end{cases}$$

The paper is in final form and no version of it will be published elsewhere.

[1]
and the mapping \([f] \to [f \cap B \xi]\) is an isomorphism.

Proof. Without loss of generality, we will assume all mappings and homotopies to be transversal to \(B\). Since both \(S^n\) and \(B\) are connected, we can assume that the intersection points are all the same on \(B\) and are in given positions on \(S^n\) after a deformation of the representative mapping. After selecting an orientation of the fiber \(F\) of \(\xi\) over the selected target point and an orientation of \(S^n\), we can assign a sign to every point of intersection depending on whether the two orientations of its small neighbourhood obtained this way coincide or not. If we have two points with opposite signs, they can easily be removed by a homotopy of the representative mapping even in \(F/\partial F \approx S^n\), so we can assume that all points have the same sign. Now, if \(\xi\) is orientable, then the pairs of points created or destroyed by a homotopy will always have different signs (we can produce a consistent sign everywhere on \(B\)) and so the sum of the signs will be a full invariant; and if \(\xi\) is not orientable, then any point will change sign after being pushed on a curve over which \(\xi\) is not orientable and hence only the mod 2 number of points is invariant, and it is indeed an invariant since homotopies can only create or destroy points in pairs.

Lemma 2. Let \(\xi\) be an arbitrary vector bundle of rank \(n \geq 5\) over a connected base \(B\). Then the mapping

\[
C: \pi_{n+1}(T\xi) \ni [f] \to [f \cap B] \in \begin{cases} 
\{[\gamma] \in \mathcal{H}_1(B) : \gamma^*\xi \text{ is orientable} \} 
\approx \ker \omega_1(\xi) \leq H_1(B; \mathbb{Z}_2) 
& \text{if } \xi \text{ is not orientable}, \\
\Omega_n(B) \approx H_1(B; \mathbb{Z}) & \text{if } \xi \text{ is orientable}. 
\end{cases}
\] (4)

is onto and if \(\pi_1(B)\) is commutative, then its kernel is either trivial or isomorphic to \(\mathbb{Z}_2\), depending on whether there is an element \([s] \in \pi_2(B)\) such that \(s^*\xi\) is an odd element in the space of all \(n\)-bundles over \(S^2\) (which is parametrized by \(\pi_1(O(n))\)).

Proof. We will try to mimic the proof of Lemma 1. All mappings and homotopies can be assumed to be transversal to \(B\) without loss of generality. Since \(n \geq 5\), any chain of \(S^1\)'s can be deformed into some standard form and any homotopy can be assumed to produce standard, cylindrical intersection preimages and it is straightforward to see that we can assume that the preimage of \(B\) consists of a single circle. The mapping \([f] \to [f \cap B]\) is obviously a homotopy invariant of \(f\), and its range can only contain elements over which \(\xi\) is trivial, since the pullback \(f^*\xi\) is the normal bundle of the intersection preimage circle in \(S^n\), which can only be oriented (its sum with \(TS^1\) is trivial) and hence trivial. On the other hand, if \((\gamma^*)\xi\) is trivial, then we can construct a mapping with a single intersection circle over which the mapping is the same as \(\gamma^*\) up to homotopy trivially (any embedding of \(S^1\) in \(S^n\) will do, and any trivialization of the normal bundle will define a mapping into the one-point compactification of a fibre of \(\xi\)). We still have to determine the kernel of this mapping in the case when \(\pi_1(B)\) is commutative and hence \(H_1(B; \mathbb{Z}) \approx \pi_1(B)\). It is generated by circles which are mapped on contractible loops in \(B\). Every such homotopy class can be represented by a mapping for which the restriction on the intersection preimage circle is constant. For every representative of this form, we will therefore obtain a number in \(\pi_{n+1}(\xi_y/\partial \xi_y) \approx \pi_{n+1}(S^n) \approx \mathbb{Z}_2\), and if two representatives have the same sign, then they of course are homotopic in the less restrictive \(\pi_{n+1}(T\xi)\). This sign is well-defined for any homotopy class if and only if it does not depend on the choice of contracting the image loop, that is, if for all mappings \(s: S^2 \to B\) the change in the framed embedding of the circle, which is the pointwise application of the gluing mapping \(\tilde{s}: S^1 \to O(n)\) is trivial, that is, \(\tilde{s}\) is even, and this concludes the proof of Lemma 2.

Corollary 3. If the bundle \(\xi\) is associated to the universal \(G\)-bundle via the representation \(\lambda: G \to \text{Iso}(\mathbb{R}^n), n > 1\), then the mapping \(C\) from Lemma 2 is an isomorphism if and only if \(\lambda_*(\pi_1(G)) = \pi_1(O(n))\), that is, the image of the fundamental group of \(G\) under \(\lambda\) contains a non-contractible loop.
in $SO(n)$.

Proof. We will check the criterion of Lemma 2. $G$-bundles over $S^2$ correspond in a one-to-one fashion to their gluing maps, which can be identified with the elements of $\pi_1(G)$. For any $[s] \in \pi_2(BG)$ the pullback of the universal $G$-bundle on $S^2$ by $s$ has the gluing map $\partial[s] \in \pi_1(G)$ with $\partial$ being an isomorphism taken from the homotopic long exact sequence of the universal $G$-bundle. Indeed, when we lift $[s] : S^2 \setminus \text{point} \rightarrow BG$ as a homotopy of a trivial mapping of a circle to $EG$, we get the mapping $\partial[s]$ on the boundary (in the fibre over the excised point), and it is giving the difference between the trivialisations of the pullback bundle over the two hemispheres, i.e. the gluing map. Since $\xi$ is associated to the universal bundle via $\lambda$, the gluing map for the pullback of $\xi$ will be the image of the gluing map for the universal bundle under $\lambda$ and hence the degree of $s^*\xi$ can be regarded as $\lambda_*(\partial[s]) \in \pi_1(O(n))$. But as $[s]$ takes all values from $\pi_2(BG)$, $\partial[s]$ takes all values from $\pi_1(G)$, so we will obtain a pulled-back bundle of odd degree if and only if the whole image $\lambda_*(\pi_1(G))$ contains the generator of $\pi_1(O(n)) = \pi_1(SO(n))$, and that completes the proof. We will also need to know how does the symmetry group of the singularity $\Sigma^{1,0}$ look like. $G_{\Sigma^{1,1}}$ in the unoriented case has the homotopy type of the group $\mathbb{Z}_2 \times O(k)$ and the representations $\lambda_1$ (in the source) and $\lambda_2$ (in the image) are of the form

$$
\lambda_1(\varepsilon, A) = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & \varepsilon A \end{pmatrix} \quad \text{and} \quad \lambda_2(\varepsilon, A) = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & \varepsilon A \end{pmatrix}
$$

in an appropriate local coordinate system. Hence the symmetry group in the oriented case, that is, the subgroup of $\mathbb{Z}_2 \times O(k)$ forming the kernel of the orientation mapping of the virtual normal bundle $(\varepsilon, A) \mapsto \det \varepsilon A = \varepsilon^k \det A$, is $\mathbb{Z}_2 \times SO(k)$ for even $k$ and $\{1\} \times SO(k) \cup \{-1\} \times SO(k)$ for odd $k$. This implies that the connected components of $G_{\Sigma^{1,1}}$ are in all cases separated by the projections $p_1(\varepsilon, A) = \varepsilon$ and $p_2(\varepsilon, A) = \det \varepsilon A$. When interpreted as projections from $\pi_1(BG_{\Sigma^{1,1}})$, $p_1$ is returning the orientability of the the kernel bundle over $\Sigma^{1,1}$ (on every loop in $\Sigma^{1,1}(f)$), and $p_2$ is returning the orientability of the virtual normal bundle of $f$ over $\Sigma^{1,1}$ (on every loop in $\Sigma^{1,1}(f)$). We will express these projections in terms of the Stiefel-Whitney characteristic classes of the underlying manifold $M$ defined by the Pontryagin-Thom construction from a representative mapping $f$ of $[f] \in \pi_{3k+3}(T\xi)$ (and hence additive notation will be used for convenience). $p_2$ is obviously evaluating $w_7$. $T_{\Sigma^{1,1}} = [w_{k+1}, w_1 + w_{k+2}, w_k w_1]$, on the fundamental class of $M$, $[M]$, since $w_7$ gives the orientability of all restrictions of the virtual normal bundle, in particular, the restriction to the dual of $T\Sigma^{1,1}$, represented by $\Sigma^{1,1}(f)$. As to $p_1$, a direct adaptation of [5] gives us the characteristic number $w_{k+3} w_k + w_{k+2} w_{k+1}$.

2.1. Calculating $\pi_{3k+2}(X_1,0)$ Extraction of $\pi_{3k+2}(X_1,0)$ from the long exact sequence 2 gives a short exact sequence

$$
0 \rightarrow \text{coker } T \rightarrow \pi_{3k+2}(X_1,0) \rightarrow \ker T^{1,0}_{3k+2} \rightarrow 0
$$

where $\ker T^{1,0}_{3k+2}$ is calculated in [3], so we need to determine $\text{coker } T$.

First, we claim that Corollary 3 is applicable and the kernel of $C$ is always trivial. Indeed, in all cases the component of unity of the symmetry group $G_{\Sigma^{1,1}}$ is the group $SO(k)$ and the bundle $\xi^{1,1}$ is associated to the universal $G_{\Sigma^{1,1}}$-bundle via the image representation. Hence, it is sufficient to check whether the image of a non-contractible loop $\gamma$ in $SO(k)$ under the image representation $\lambda_2$ is non-contractible as well. The representation $\lambda_2$ has the form $(\varepsilon, A) \mapsto \text{diag}(1,1,1,\varepsilon A,\varepsilon A)$, and it is easy to check that the mapping $[\gamma] \mapsto [\text{diag}(1,1,1,\varepsilon,\varepsilon)]$ is an isomorphism between $\pi_1(SO(k))$ and $\pi_1(SO(3k+2))$. It follows by applying Corollary 3 that $C$ is indeed an isomorphism.
This fact implies that \( \text{coker } T = \text{coker } C \circ T \). We will prove that \( C \circ T([f]) \) only depends on the (oriented) cobordism class of the manifold \( M \) in the mapping \( M \to \mathbb{R}^{3k+2} \) obtained from \( f \) by the generalized Pontryagin-Thom construction. Indeed, if we have an arbitrary cobordism of \( M \) and represent it with a generic mapping into \( \mathbb{R}^{3k+2} \), it will only have isolated \( H_{0,2} \)-points apart from cusps and folds, so \( C \circ T([f]) \) is well-defined up to the subgroup generated by the mapping on the boundary of a normal form of an \( H_{0,2} \) point. This subgroup is however trivial, because both the kernel bundle and the virtual normal bundle over the cusp-circle are trivial. The virtual normal bundle is trivial because both the source and the image bundles are trivial as normal bundles of a circle in a \( 2k+1 \)-sphere and a \( 3k+1 \)-sphere, respectively, and the kernel bundle is easily checked to contain the multiples of \((\sin \alpha, -\cos \alpha, 0, 0, 0, 0, 0, 0)\) over the cone of cusps with the base \((\sin^2 \alpha \cos \alpha, \sin \alpha \cos^2 \alpha, -3 \sin^2 \alpha \cos \alpha, -3 \sin \alpha \cos^2 \alpha, 0, 0, 0, 0)\) in the canonical form of the \( H_{0,2} \) singularity, \((x, y, u, v, w, z, s, t) \mapsto (xy, x^2 + ux + vy, y^2 + wx + sy, xz + yt, u, v, w, z, s, t)\) (see [4]).

So, \( C \circ T \) can be expressed in terms of Stiefel-Whitney characteristic numbers (and Pontryagin numbers in the oriented case) of the underlying manifold. Hence we have the following cases:

- **Unoriented case**: \( \pi_1(G_{S^1,1}) \approx \mathbb{Z}_2 \times \mathbb{Z}_2 \), and \( \hat{T} \) can be identified with the pair
  \[
  \left( w^2_{k+1} w_1 + w_{k+2} w_k w_1, w_{k+2} w_{k+1} + w_{k+3} w_k \right).
  \]
  However, the characteristic number
  \[
  (Sq^1 + w_1)(w^2_{k+1} + w_{k+2} w_k),
  \]
  which always evaluates to 0 according to [2], is the first element of the given pair when \( k \) is odd and the sum of the two elements of the pair when \( k \) is even. Therefore it is enough to check whether the second element of the pair always evaluates to 0 or not; it is an easy computation to see that \( Y^0 \) evaluates to 1 and multiplying by \( \mathbb{R}P^2 \) does not change this value, whereas if \( k = 0 \), then \( \mathcal{N}_{2k+3} \approx \mathbb{N} \approx 0 \) and no characteristic number can be non-zero.

So, if \( k > 0 \), then \( \pi_{2k+2}(X,0) \) is an extension of \( \text{ker}(T): \pi_{2k+2}(X,1) \to \pi_{2k+2}(TT^1) \), which is an index 2 subgroup of \( \mathcal{N}_{2k+2} \), by \( \mathbb{Z}_2 \), and if \( k = 0 \), then \( \pi_{2k+2}(X,0) \approx \pi_{2k+3}(TT^1) \approx \mathbb{Z}_2 \times \mathbb{Z}_2 \).

- **Oriented case, \( k \) is odd**: \( \xi^{1,1} \) is orientable, \( \pi_1(G_{S^1,1}) \approx \mathbb{Z}_2 \) and the mapping \( \hat{T} \) is the characteristic number
  \[
  w_{k+2} w_{k+1} + w_{k+3} w_k.
  \]
  Now, \( Y^0 \times (\mathbb{R}P^2)^{k-1} \approx \mathcal{N}_{k} \times (CP)^{(k-1)/2} \) evaluates to 1, so \( T \) is always onto and \( \pi_{2k+2}(X,0) \approx \text{ker}(T): \pi_{2k+2}(X,1) \to \pi_{2k+2}(TT^1) \) is an index 3 subgroup of \( \mathcal{N}_{2k+2} \) with an appropriate \( v \) defined in [6].

- **Oriented case, \( k \) is even**: \( \xi^{1,1} \) changes orientation over all noncontractible loops in \( B\xi^{1,1} \), so \( T \) is onto and \( \pi_{2k+2}(X,0) \approx \text{ker}(T): \pi_{2k+2}(X,1) \to \pi_{2k+2}(TT^1) \) is the whole \( \mathcal{N}_{2k+2} \approx 0 \) when \( k \) is either 0 or 2 and is an index 2 subgroup of \( \mathcal{N}_{2k+2} \) when \( k \geq 4 \).

As a reformulation of this result, we have the following theorem:

**Theorem A**. There is an exact sequence
\[
0 \to \mathbb{Z}_2 \to \text{Cob}^\oplus_{2k+2}(2k + 2, \mathbb{R}^{3k+2}) \to G \to 0,
\]
where \( G \) is an index 2 subgroup of \( \mathcal{N}_{2k+2} \), for \( k > 0 \). For \( k = 0 \), \( \text{Cob}^\oplus_{2k+2}(2, \mathbb{R}^{3k+2}) \approx \mathbb{Z}_2 \times \mathbb{Z}_2 \).

b1) If \( k \) is odd, then \( \text{Cob}^\oplus_{2k+2}(2k + 2, \mathbb{R}^{3k+2}) \) is isomorphic to the kernel of the epimorphic mapping \( p_{(k+1)/2}[1]: \mathcal{N}_{2k+2} \to \mathbb{Z}_2 \).

b2) If \( k = 0 \) or \( k = 2 \), then \( \text{Cob}^\oplus_{2k+2}(2k + 2, \mathbb{R}^{3k+2}) \approx \mathbb{Z}_2 \). If \( k \geq 4 \) is even, then \( \text{Cob}^\oplus_{2k+2}(2k + 2, \mathbb{R}^{3k+2}) \) is an index 2 subgroup of \( \mathcal{N}_{2k+2} \).
References
[3] T. Ekholm, A. Szücs, T. Terpai: Cobordisms of fold maps and maps with prescribed number of cusps, submitted