Four new upper bounds for the stability number of a graph

Miklós Ujvári *

Abstract. In 1979, L. Lovász defined the theta number, a spectral/semidefinite upper bound on the stability number of a graph, which has several remarkable properties (for example, it is exact for perfect graphs). A variant, the inverse theta number, defined recently by the author in a previous work, also constitutes an upper bound on the stability number. In the paper we will describe counterparts of theorems due to Wilf and Hoffman, four spectral upper bounds on the stability number, which differ from both the theta and the inverse theta numbers.

Keywords: stability number, spectral bound

1 Introduction

The earliest spectral bounds (upper, resp. lower bounds for the chromatic number of a graph) were derived in the late 1960s by H.S. Wilf and A.J. Hoffman (see e.g. Exercises 11.20, 21 in [4]). In 1979, L. Lovász applied the method of variables to the Wilf’s and Hoffman’s bounds, obtaining the theta number, which is “sandwiched” between the stability number of the graph and the chromatic number of the complementary graph, and, as the optimal value of a semidefinite program, is easily computable (see [5], [2]). In 1986 H.S. Wilf derived spectral lower bounds on the stability number (see [13]).

In this paper we will describe counterparts of Hoffman’s and Wilf’s bounds, four spectral upper bounds on the stability number. We begin the paper with stating the main results. First, we fix some notation.

Let \( n \in \mathbb{N} \), and let \( G = (V(G), E(G)) \) be an undirected graph, with vertex set \( V(G) = \{1, \ldots, n\} \), and with edge set \( E(G) \subseteq \{\{i,j\} : i \neq j\} \). Let \( A(G) \) be the 0-1 adjacency matrix of the graph \( G \), that is let

\[
A(G) = (a_{ij}) \in \{0, 1\}^{n \times n}, \text{ where } a_{ij} := \begin{cases} 
0, & \text{if } \{i,j\} \notin E(G), \\
1, & \text{if } \{i,j\} \in E(G). 
\end{cases}
\]

*H-2600 Vác, Szent János utca 1. HUNGARY
The set of \( n \times n \) real symmetric matrices \( A = A^T \in \mathbb{R}^{n \times n} \) satisfying \( |A| \leq A(G) \) will be denoted by \( \mathcal{A} \). (Here \( T \) stands for transpose, and \( \leq, |.| \) are meant elementwise.)

The complementary graph \( \overline{G} \) is the graph with adjacency matrix

\[
A(\overline{G}) := J - I - A(G),
\]

where \( I \) is the identity matrix, and \( J \) denotes the matrix with all elements equal to one. The disjoint union of the graphs \( G_1 \) and \( G_2 \) is the graph \( G_1 + G_2 \) with adjacency matrix

\[
A(G_1 + G_2) := \begin{pmatrix}
A(G_1) & 0 \\
0 & A(G_2)
\end{pmatrix}.
\]

We will use the notation \( K_n \) for the clique graph, and \( K_{s_1}, \ldots, s_k \) for the complete multipartite graph \( K_{s_1} + \ldots + K_{s_k} \).

Let \( (\delta_1, \ldots, \delta_n) \) be the sum of the row vectors of the adjacency matrix \( A(G) \). The elements of this vector are the degrees of the vertices of the graph \( G \). Let \( \delta_G, \Delta_G, \mu_G \) be the minimum, maximum, resp. the arithmetic mean of the degrees in the graph.

By Rayleigh’s theorem (see [8]) for a symmetric matrix \( M = M^T \in \mathbb{R}^{n \times n} \) the minimum and maximum eigenvalue, \( \lambda_M \) resp. \( \Lambda_M \), can be expressed as

\[
\lambda_M = \min_{||u||=1} u^T Mu, \quad \Lambda_M = \max_{||u||=1} u^T Mu.
\]

Attainment occurs if and only if \( u \) is a unit eigenvector corresponding to \( \lambda_M \) and \( \Lambda_M \), respectively. By the Perron-Frobenius theorem (see [7], Theorem 9.1.3) for an elementwise nonnegative symmetric matrix \( M = M^T \geq 0 \), we have

\[
-\lambda_M \leq \Lambda_M = u^T Mu
\]

for some \( u \geq 0 \), \( u^T u = 1 \).

The minimum and maximum eigenvalue of the adjacency matrix \( A(G) \) will be denoted by \( \lambda_G \) resp. \( \Lambda_G \). It is a consequence of the Rayleigh and Perron-Frobenius theorems that for \( A \in \mathcal{A} \),

\[
\Lambda_A \leq \Lambda_{|A|} \leq \Lambda_G.
\]

Also, \( \lambda_A \) (resp. \( \Lambda_A \)) as a function of the symmetric matrix \( A \) is concave (resp. convex), specially the function \( \lambda_A + \Lambda_A \) is continuous, and attains its minimum and maximum on the compact convex set \( \mathcal{A} \).

The set of the \( n \times n \) real symmetric positive semidefinite matrices will be denoted by \( \mathcal{S}_n^+ \), that is

\[
\mathcal{S}_n^+ := \{ M \in \mathbb{R}^{n \times n} : M = M^T, u^T Mu \geq 0 (u \in \mathbb{R}^n) \}.
\]
It is well-known (see [8]), that the following statements are equivalent for a symmetric matrix $M = (m_{ij}) \in \mathbb{R}^{n \times n}$: a) $M \in S_n^+$; b) $\lambda_M \geq 0$; c) $M$ is Gram matrix, that is $m_{ij} = v_i^T v_j$ ($i, j = 1, \ldots, n$) for some vectors $v_1, \ldots, v_n$.

Furthermore, by Lemma 2.1 in [10], the set $S_n^+$ can be described as

$$S_n^+ = \left\{ \frac{a_i^T a_j}{(a_i a_j^T)_{11}} - 1 \right\}_{i,j=1}^n \left| m \in \mathcal{N}, \ a_i \in \mathbb{R}^m \ (1 \leq i \leq n) \right.$$ (3)

For example, diagonally dominant matrices (that is $M = (m_{ij}) \in \mathbb{R}^{n \times n}$ with $m_{ii} \geq \sum_{i \neq j} |m_{ij}|$ for $i = 1, \ldots, n$) are positive semidefinite by the Gershgorin’s disc theorem, see [8]. Hence,

$$F_{s_1, \ldots, s_k} := k(J - A(K_{s_1, \ldots, s_k})) - J \in S_n^+.$$ (Note that if $F_{1, \ldots, 1}$ is Gram matrix then so is $F_{s_1, \ldots, s_k}$.)

The stability number, $\alpha(G)$, is the maximum cardinality of the (so-called stable) sets $S \subseteq V(G)$ such that $\{i, j\} \subseteq S$ implies $\{i, j\} \notin E(G)$. The chromatic number, $\chi(G)$, is the minimum number of stable sets covering the vertex set $V(G)$.

Let us define an orthonormal representation of the graph $G$ (shortly, o.r. of $G$) as a system of vectors $a_1, \ldots, a_n \in \mathbb{R}^m$ for some $m \in \mathbb{N}$, satisfying

$$a_i^T a_i = 1 \ (i = 1, \ldots, n), \ a_i^T a_j = 0 \ (\{i, j\} \notin E(G)).$$

In the seminal paper [5] L. Lovász proved the following result, now popularly called sandwich theorem, see [3]:

$$\alpha(G) \leq \vartheta(G) \leq \chi(G),$$ (4)

where $\vartheta(G)$ is the Lovász number of the graph $G$, defined as

$$\vartheta(G) := \inf \left\{ \frac{1}{\max_{1 \leq i \leq n} (a_i a_i^T)_{11}} : a_1, \ldots, a_n \text{ o.r. of } G \right\}.$$ 

The Lovász number has several equivalent descriptions, see [5]. For example, by (3) and standard semidefinite duality theory (see e.g. [9]), it is the common optimal value of the Slater-regular primal-dual semidefinite programs

$$(TP) \quad \min \lambda, \left\{ \begin{array}{l} x_{ii} = \lambda - 1 \ (i \in V(G)), \\
x_{ij} = -1 \ (\{i, j\} \in E(G)), \\
X = (x_{ij}) \in S_n^+, \ \lambda \in \mathbb{R} \end{array} \right.$$ and

$$(TD) \quad \max \text{tr} (Y), \left\{ \begin{array}{l} \text{tr} (Y) = 1, \\
y_{ij} = 0 \ (\{i, j\} \in E(G)), \\
Y = (y_{ij}) \in S_n^+ \right.$$
(Here tr stands for trace.) Note that for appropriately chosen $s_1, \ldots, s_k$ and $s$, the matrices

$$X = F_{s_1, \ldots, s_k}, \quad Y = \begin{pmatrix} J/s & 0 \\ 0 & 0 \end{pmatrix}$$

are feasible solutions of programs $(TP)$ and $(TD)$, respectively, with values $\lambda = k = \chi(G)$ and $\text{tr}(JY) = s = \alpha(G)$, resulting in a nice proof of the sandwich theorem (see [2]).

Analogously, the inverse theta number, $\iota(G)$, satisfies the inverse sandwich inequality,

$$\left(\alpha(G)\right)^2 + n - \alpha(G) \leq \iota(G) \leq n\vartheta(G),$$

(5) see [12]. Here the inverse theta number, defined as

$$\iota(G) := \inf \left\{ \sum_{i=1}^{n} \frac{1}{\langle a_i, a_i^T \rangle_{11}} : a_1, \ldots, a_n \text{ o.r. of } G \right\},$$

equals the common attained optimal value of the primal-dual semidefinite programs

$$(TP^-) \quad \inf \text{tr}(W) + n, \quad w_{ij} = -1 \left( \{i, j\} \in E(G) \right), \quad W = (w_{ij}) \in S^n_+,$$

$$(TD^-) \quad \sup \text{tr}(JM), \quad \begin{cases} m_{ii} = 1 \ (i = 1, \ldots, n), \\ m_{ij} = 0 \ (\{i, j\} \in E(G)), \end{cases} \quad M = (m_{ij}) \in S^n_+.$$

Both bounds can be obtained via convex spectral optimization: obviously (compare with $(TP)$),

$$\vartheta(G) = \min \left\{ \Lambda \left| \begin{array}{l} z_{ii} = 1 \ (i \in V(G)), \\ Z = (z_{ij}) = (z_{ji}) \in R^{n \times n} \end{array} \right. \right\},$$

and, similarly (compare with $(TP^-)$),

$$\iota(G) = \min \left\{ n - \text{tr}U + n\Lambda_U \left| \begin{array}{l} u_{ij} = 1 \ (\{i, j\} \in E(G)), \\ U = (u_{ij}) \in S^n_+ \end{array} \right. \right\}.$$
where
\[ \Sigma := (u_1 + \ldots + u_n)^2 \quad \text{with} \quad \begin{cases} u = (u_i) \in \mathbb{R}^n, \ u \geq 0, \\ u^T u = 1, \ u^T A(G) u = \Lambda_G. \end{cases} \]
(Note that \( \Lambda_G + 1 \leq \Sigma \leq n \) by the Cauchy-Schwarz inequality, hence each bound is at least \( n - \Lambda_G \).)

We will prove in Sections 2, 3, 4, and 4, respectively, that the inequalities
\[ \alpha(G) \leq \iota_1(G), \iota_2(G), \iota_3(G), \iota_4(G) \quad (6) \]
hold. These upper bounds are efficiently computable via methods in [8]. (For lower bounds on \( \alpha(G) \), see e.g. [11].) Several open problems arise: Can the four bounds give better results than other upper bounds (see e.g. [11]), such as \( \Lambda_G + 1 \) (Wilf’s upper bound for the chromatic number of the complementary graph), or \( \vartheta(G), n + 1 - \vartheta(G) \)? How do they relate to \( \chi(G), n + 1 - \chi(G) \)? How do they relate to each other? These questions (partially answered in the paper) need further investigation.

\section{The counterpart of Wilf’s bound}

In this section we will describe the counterpart of Wilf’s lower bound on the stability number. The spectral upper bound \( \iota_1(G) \) is derived via estimating from above the maximum eigenvalue of the adjacency matrix.

In [13] Wilf proved, as a consequence of a theorem of Motzkin-Straus, the relation
\[ \frac{n}{n - \Lambda_G} \leq \alpha(G). \quad (7) \]
The next proposition describes a weaker form of (7).

\begin{proposition}
For any graph \( G \), \( \chi(G) \geq n/(n - \Lambda_G) \).
\end{proposition}

\begin{proof}
Let \( S_1, \ldots, S_k \) be stable sets in \( G \) with cardinality \( s_1, \ldots, s_k \), respectively, so that \( s_1 + \ldots + s_k = n \). Then, \( G \) is a subgraph of \( H = K_{s_1, \ldots, s_k} \). In other words,
\[ 0 \leq A(G) \leq A(H) = J - \frac{F_{s_1, \ldots, s_k} + J}{k}, \]
implying, by \( F_{s_1, \ldots, s_k} \in S^n \), that
\[ \Lambda_G \leq \Lambda_H \leq \frac{(k - 1)n}{k}. \]
As here \( k = \chi(G) \) can be chosen, so the statement follows.
\end{proof}

The counterpart of Proposition 2.1 can be proved similarly, and leads us to the bound \( \iota_1(G) \).
PROPOSITION 2.2. For any graph $G$, the inequality
\[ \alpha(G) \leq n - \frac{\Lambda_G}{n - \Lambda_G} =: i_1(G) \]
holds.

Proof. Let $\{1, \ldots, s\}$ be a stable set in $G$. Then, $G$ is a subgraph of the graph $H = K_{s,1,\ldots,1}$. In other words,
\[ 0 \leq A(G) \leq A(H) = J - \frac{F_{s,1,\ldots,1}}{n - s + 1} + J. \]
Thus, for the maximal eigenvalues the inequalities
\[ \Lambda_G \leq \Lambda_H \leq \frac{(n - s)n}{n - s + 1} \]
hold, by $F_{s,1,\ldots,1} \in S_n$. The statement follows with $s = \alpha(G)$.

In [13] Wilf used the method of variables to strengthen the bound in (7): he proved the relation
\[ \Sigma - \Lambda_G \leq \alpha(\overline{G}). \] (8)
Analogously, the proof of Proposition 2.2 can easily be adapted to imply

THEOREM 2.1. For any graph $G$, $\alpha(G) \leq i_1(G)$ holds.

Note that (8) implies the stronger relation
\[ \alpha(G) \leq \chi(\overline{G}) \leq n + 1 - \alpha(\overline{G}) \leq i_1(G) \] (9)
also, but the proof of Theorem 2.1 does not use the Motzkin-Straus theorem.

3 The counterpart of Hoffman’s bound

In this section we describe the counterpart of a spectral lower bound for the chromatic number due to Hoffman. The proof relies on estimating from above the minimum eigenvalue of the adjacency matrix.

Hoffman’s theorem (see e.g. [4]) states that for any graph $G$,
\[ \chi(G) \geq 1 + \frac{\Lambda_G}{-\Lambda_G}. \] (10)
The proof remains valid for arbitrary matrix $A \in \mathcal{A}$ instead of $A(G)$, and the strongest bound obtained this way (by the so-called method of variables) is the Lovász number $\vartheta(\overline{G})$ (see [5], [6]).

The proof of the counterpart closely follows the proof of (10).
THEOREM 3.1. For any graph $G$, $\alpha(G) \leq \iota_2(G)$.

Proof. Let $A := A(G)$, and suppose that $\{1, \ldots, s\}$ is a stable set in $G$ for some $1 \leq s \leq n - 1$. Then, the matrix $A$ can be partitioned as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11} = 0 \in \mathbb{R}^{s \times s}$, $A_{12} = A_{21}^T$, $A_{22} = A_{22}^T \in \mathbb{R}^{(n-s) \times (n-s)}$.

Let $x \in \mathbb{R}^n$ be an eigenvector corresponding to the eigenvalue $\Lambda_A$. Let $x = (x_1^T, x_2^T)^T$, where $x_1 \in \mathbb{R}^s$, $x_2 \in \mathbb{R}^{n-s}$. Let us denote by $y_1 \in \mathbb{R}^n$ the vector with first element $||x_1||$, otherwise zero, and let us define similarly the vector $y_2 \in \mathbb{R}^{n-s}$, too. Let $y \in \mathbb{R}^n$ be the vector obtained by stacking the vectors $y_1, y_2$ on the top of each other.

Let us choose orthogonal matrices $B_1 \in \mathbb{R}^{s \times s}$, $B_2 \in \mathbb{R}^{(n-s) \times (n-s)}$ such that $B_1 y_1 = x_1$ and $B_2 y_2 = x_2$ hold. Let $B$ be the block-diagonal matrix formed by the matrices $B_1, B_2$. Then, $B \in \mathbb{R}^{n \times n}$ is an orthogonal matrix, $By = x$, and

$$B^{-1} A By = B^{-1} A x = \Lambda_A B^{-1} x = \Lambda_A y.$$ 

Hence, the vector $y$ is an eigenvector (with eigenvalue $\Lambda_A$) of the matrix

$$B^{-1} AB = (B_1^{-1} A_{ij} B_j)_{i,j=1,2}.$$ 

Let us consider the submatrix $C = (c_{ij}) \in \mathbb{R}^{2 \times 2}$,

$$C = ((B_1^{-1} A_{ij} B_j)_{11})_{i,j=1,2}.$$ 

As $B^{-1} A B y = \Lambda_A y$, so $C z = \Lambda_A z$ with the 2-vector $z := (||x_1||, ||x_2||)^T$, implying $\Lambda_A \leq \Lambda_C$. By $A_{11} = 0$, we have $c_{11} = 0$, thus the trace of the matrix $C$ equals $\Lambda_C + \lambda_C = c_{22}$. Furthermore, as $|A_{22}| \leq J - I$, so the matrix $(n - s - 1)I - A_{22}$ is diagonally dominant, necessarily positive semidefinite. Hence, the inequalities

$$c_{22} \leq \Lambda_B^{-1} A_{22} B_2 = \Lambda_{A_{22}} \leq n - s - 1$$

hold. Cauchy's theorem on interlacing eigenvalues (see [8]) gives $\lambda_A \leq \lambda_C$ and $\Lambda_C \leq \Lambda_A$.

Summarizing, we have

$$\lambda_A \leq \lambda_C = c_{22} - \lambda_C = c_{22} - \Lambda_A \leq n - s - 1 - \Lambda_A,$$

where $s$ can be chosen to be the stability number $\alpha(G)$. This completes the proof of the theorem. \(\square\)

We already have mentioned that $\iota_2(G) \geq n - \Lambda_G$, but more can be claimed:
PROPOSITION 3.1. For any graph $G$,

$$\iota_2(G) \geq n - \Lambda_G + \frac{\Lambda_G}{\vartheta(G) - 1} - 1 =: i_2(G)$$

holds.

Proof. By the remark preceding Theorem 3.1, we have

$$-\lambda_A \geq \frac{\Lambda_A}{\vartheta(G) - 1},$$

where $A := A(G)$. Hence,

$$\iota_2(G) \geq n - 1 - \Lambda_A \cdot \left(1 - \frac{1}{\vartheta(G) - 1}\right),$$

and the statement follows from (2). \qed

Both proofs can be carried through with $A \in A$ instead of $A = A(G)$, which means that

$$\iota_2(G, A) \geq \alpha(G), i_2(G)$$

for $A \in A$, where

$$\iota_2(G, A) := n - 1 - \Lambda_A - \lambda_A.$$

By compactness of the set $A := n - 1 - \Lambda_A - \lambda_A$.

As concluding remarks in this section, we will show examples when $A_* \neq A(G)$, and when $A_* = A(G)$ meets the requirements.

First, note that there exists a matrix $B \in A$ such that

$$\iota_2(G, B) \leq n + 1 - \vartheta(G).$$

In fact, let us choose a matrix $B = (b_{ij}) \in A$ satisfying

$$\vartheta(G) = 1 + \frac{\Lambda_B}{-\lambda_B}$$

(by the remark preceding Theorem 3.1 there exists such an optimal matrix $B$).

We can assume that for some indices $i \neq j$, $|b_{ij}| = 1$. Then, from Rayleigh’s theorem, $\lambda_B \leq -1$ follows. Moreover, we have

$$\Lambda_B + \lambda_B = -\lambda_B (\vartheta(G) - 2) \geq 0.$$

Summarizing, we obtain

$$n + 1 - \vartheta(G) = n - 1 - \frac{\Lambda_B + \lambda_B}{-\lambda_B} \geq \iota_2(G, B).$$
On the other hand, for the perfect graph $G_0 := K_3 + K_{2,2}$ we have

$$n + 1 - \vartheta(G_0) = 5 < 6 = n - 1 - \Lambda G_0 - \lambda G_0;$$

we can see that in this case $A_* \neq A(G_0)$.

Finally, note that $\iota_2(G) = n + 1 - \vartheta(G)$ if and only if $\vartheta(G) = 2$ (e.g. for a bipartite graph) or $\vartheta(G) = \Lambda G + 1$ (e.g. when $G = K_{s_1} + \ldots + K_{s_k}$). Hence, for bipartite graphs and for disjoint unions of cliques we have $A_* = A(G)$ with $\iota_2(G) = n - 1$ and $\iota_2(G) = n - \Lambda G$, respectively.

## 4 Variants

In this section we describe two further spectral upper bounds on the stability number, derived via similar methods, and hence considered as variants of $\iota_1(G)$.

In order to derive the bound $\iota_3(G)$ we will use the following technical lemma.

**Lemma 4.1.** Let $1 \leq s \leq n - 1$, and let

$$M := I + A(K_s + K_{n-s}).$$

Then,

$$\Lambda_M = \frac{1}{2} \left( n - s - 1 + \sqrt{(n - s - 1)^2 + 4(s - 1)(n - s)} \right)$$

is the maximum eigenvalue of the matrix $M$.

**Proof.** The eigenvalue $\Lambda_M$ can be rewritten as

$$\Lambda_M = \min\{\lambda \in \mathbb{R} : \lambda I - M \in S^+_n\}.$$

For $\lambda \leq 1$, $\lambda I - M \not\in S^+_n$, as then the diagonal elements of $\lambda I - M$ are nonpositive. For $\lambda > 1$, the matrix $\lambda I - M \in \mathbb{R}^{n \times n}$ is positive semidefinite if and only if the Schur complement of its positive definite principal submatrix $(\lambda - 1)I \in \mathbb{R}^{s \times s}$ is positive semidefinite (see [7]). In other words, for $\lambda > 1$, $\lambda I - M \in S^+_n$ if and only if

$$\lambda I - J - (-J) \cdot ((\lambda - 1)I)^{-1} \cdot (-J) \in S^+_n.$$

Here (11) can easily be seen to be equivalent to the inequality

$$\lambda - \left( 1 + \frac{s}{\lambda - 1} \right) \cdot (n - s) \geq 0;$$

(12)

$\Lambda_M$ will be the least positive solution of (12), as stated. \hfill \Box
If the set \( \{1, \ldots, s\} \) is a stable set in \( G \), then \( 0 \leq A(G) \leq M - I \), where \( M \) is the same matrix as in Lemma 4.1. Hence, for the maximum eigenvalues \( \Lambda_G \leq \Lambda_M - 1 \) holds, and we can derive easily, from Lemma 4.1, the following

**THEOREM 4.1.** For any graph \( G \), \( \alpha(G) \leq \iota_3(G) \) holds. \( \square \)

The bound \( \iota_3(G) \) is exact e.g. for complete bipartite graphs \( G = K_{1,s} \).

We remark that the proof of Theorem 4.1 can be carried through for any matrix \( A \in \mathcal{A} \) instead of \( A(G) \), but this way we obtain weaker bounds than \( \iota_3(G) \). In fact, \( \iota_3(G) \), as a function of \( \Lambda_A \), is strictly monotone decreasing on the interval \( 0 \leq \Lambda_A \leq n - 1 \) (the first derivative of the function is negative on this interval). This means that we get the strongest bound when \( \Lambda_A \) is maximal, that is when \( A = A(G) \).

The next proposition, too, is immediate from the fact that for any graph \( G \), \( 0 \leq \mu_G \leq \Lambda_G \leq n - 1 \), see Exercise 11.14 in [4].

**PROPOSITION 4.1.** With

\[
\iota_3(\Lambda) := \frac{1}{2} \left( n - \Lambda + \sqrt{(n - \Lambda)^2 + 4\Lambda(n - 1 - \Lambda)} \right)
\]

for \( \Lambda \in \mathbb{R} \), we have

\[
n - \Lambda_G \leq \iota_3(\Lambda_G) \leq \iota_3(\mu_G) \leq n,
\]

for any graph \( G \). \( \square \)

We remark also that

\[
\iota_3(G) \leq \iota_1(G) \tag{13}
\]

(as it can easily be verified), but it is an open problem whether \( \iota_3(G) \leq \iota_3(G) \) holds or not, generally.

Now, we turn to the bound \( \iota_4(G) \). With minor modification of the proof of Proposition 2.2, we obtain a close variant, \( \iota_4(G) \).

**PROPOSITION 4.2.** For any graph \( G \), the inequality

\[
\alpha(G) \leq \frac{3n}{2} - 1 - \Lambda_G =: \iota_4(G)
\]

holds.

**Proof.** Let \( \{1, \ldots, s\} \) be a stable set in \( G \). Then, \( G \) is a subgraph of the graph \( H = K_s + K_{n-s} \). In other words,

\[
0 \leq A(G) \leq A(H) = A(K_{s,n-s}) + A(K_s + K_{n-s}).
\]

Thus, for the maximal eigenvalues the inequalities

\[
\Lambda_G \leq \Lambda_H \leq \Lambda_{K_{s,n-s}} + \Lambda_{K_s + K_{n-s}}
\]
hold. Here $\Lambda_{K_s+K_{n-s}} = n - s - 1$, and, by
\[
F_{s,n-s} = \frac{J}{2} - A(K_{s,n-s}) \in S_n^+, \nonumber
\]
we have $\Lambda_{K_{s,n-s}} \leq n/2$. Hence,
\[
\Lambda_G \leq \frac{n}{2} + n - s - 1, \nonumber
\]
from which with $s = \alpha(G)$ the statement follows. \hfill $\Box$

As in the case of Proposition 2.2 and Theorem 2.1, the proof of Proposition 4.2 can easily be adapted to imply

THEOREM 4.2. For any graph $G$, $\alpha(G) \leq \iota_4(G)$ holds. \hfill $\Box$

The following proposition is the analogue of Propositions 3.1 and 4.1.

PROPOSITION 4.3. For any graph $G$, $\iota_4(G) \geq n/2$.

Proof. Let $A := A(G)$. Then, the matrix
\[
B := \left(\frac{n}{2} - 1\right)I + \frac{J}{2} - A
\]
is diagonally dominant, implying $B \in S_n^+$. Consequently, we have
\[
n - 1 + \frac{u^TJu}{2} - \Lambda_A = n - 1 + u^T\left(\frac{J}{2} - A\right)u \geq n - 1 - \left(\frac{n}{2} - 1\right) = \frac{n}{2}
\]
for all nonnegative unit eigenvectors $u$ corresponding to $\Lambda_G$, which was to be proved. \hfill $\Box$

Finally, we mention an open problem. The minimum eigenvalue $\lambda_G$ of a graph $G$ is negative, specially the corresponding unit eigenvector $v \in \mathbb{R}^n$ has both positive and negative coordinates. Writing the eigenvector $v$ as the difference of its positive and negative part (i.e. $v = v_+ - v_-$, where $v_+, v_- \text{ are }$nonnegative, orthogonal $n$-vectors), we have
\[
0 > \lambda_G = v^T A(G)v \geq -2v_+^T A(G)v_- \geq -2v_+^T J v_-,
\]
and it is not hard to conclude that $-\lambda_G \leq n/2$. This result with a different proof is due to Constantine, see [1], and implies in particular, the relation
\[
\iota_2(G) \leq \iota_4(G). \quad (14)
\]
It would be interesting to see a similar proof of the conjecture $\iota_2(G) \leq \iota_4(G)$ via showing the inequality $-\lambda_G \leq (u^TJu)/2$ for all nonnegative unit eigenvectors $u$ corresponding to $\Lambda_G$. (The relation $\iota_2(G) \leq \iota_4(G)$ can easily be verified e.g. for bipartite graphs or for disjoint unions of cliques.)
5 Conclusion

In this paper we studied spectral upper bounds on the stability number of a graph, counterparts of classical bounds due to Wilf and Hoffman. Several questions arised and were partially answered: for example concerning the relation of the spectral bounds introduced in the paper among each other and with the chromatic number of the complementary graph.

Acknowledgements. I thank Mihály Hujter for the $K_3 + K_{2,2}$ example in Section 3.

References


