A new approach to alternating paths

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The Hungarian Method is 50

Outline

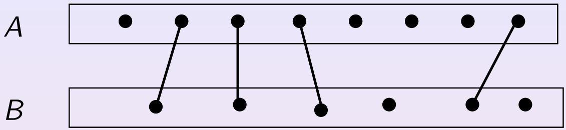
"Relax – Reduce"

A black-box approach to combinatorial algorithms

"3-Way Lemma" and an "Equivalent Reduction"

- Kőnig's Theorem
- Square-free simple 2-matchings
- Further applications:
  — non-bipartite matching
  — path-matching, even factor
  — packing node-disjoint $\mathcal{A}$-paths
  — hypo-matching in directed graphs

Bipartite Matching



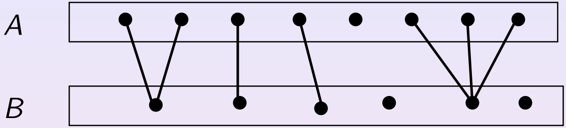
Kőnig's Theorem

If $G = (A, B; E)$ is a bipartite graph, then

the maximum cardinality of a matching
=
the minimum total capacity of a cover.

capacity:

The "Relax – Reduce" approach for bipartite matching

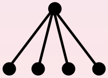


Observation

If $G = (A, B; E)$ is a bipartite graph, then

$$\text{the maximum cardinality of an } A\text{-matching} = \text{the minimum total capacity of a cover.}$$

capacity:

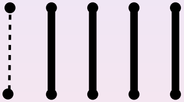


≤ 1

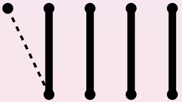
The "Relax – Reduce" approach for bipartite matching

Given a matching M. Is M a max A-matching?

YES $\Rightarrow$ we are done.

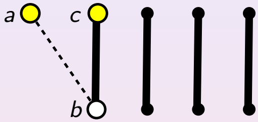
NO $\Rightarrow$ $\exists$  $\Rightarrow$ **augment**

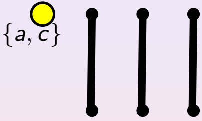
or

$\exists$  $\Rightarrow$ **reduce ...**

The "Relax – Reduce" approach for bipartite matching

...the reduction:





$G, M \quad \Longrightarrow \quad (G-b)/\{a,c\}, M - bc$

$N + ab$ or $N + bc \quad \Longleftarrow \quad N$ a larger matching

$X + b \quad \Longleftarrow \quad X$ a verifying cover

The "Relax – Reduce" approach for bipartite matching

"3-Way Lemma"

Given a matching M in bipartite graph $G = (A, B; E)$, then at least one of the following alternatives holds:

- **a)** M is a maximum A-matching.
- **b)** There is a matching N with $|N| = |M| + 1$.
- **c)** There are nodes $a, c \in A, b \in B$ with $bc \in M$ and $a \notin V(M)$.

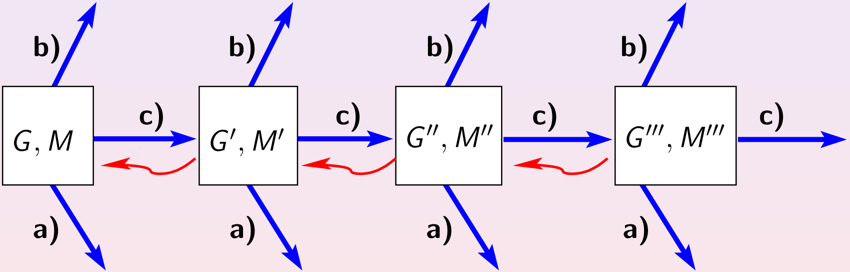
"Equivalent Reduction"

Suppose $a, c \in A, b \in B$ with $bc \in M$ and $a \notin V(M)$. Then

$$M \text{ is max in } G \iff M - bc \text{ is max in } (G - b)/\{a, c\}.$$

The "Relax – Reduce" approach for bipartite matching

The algorithmic scheme:

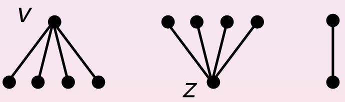


Simple 2-matchings

Theorem

$G = (A, B; E)$ a simple bipartite graph with $b : A \cup B \to \{0, 1, 2\}$. Then

$$\text{the maximum cardinality of a simple } b\text{-matching} = \text{the minimum total capacity of a cover.}$$



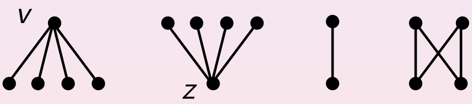
capacity: $b(v)$ $b(z)$ 1

Square-free simple 2-matchings

Theorem — Z. Király, 1999

$G = (A, B; E)$ a simple bipartite graph with $b : A \cup B \to \{0, 1, 2\}$.
Then

$$\text{the maximum cardinality of a square-free simple } b\text{-matching} = \text{the minimum total capacity of a cover.}$$



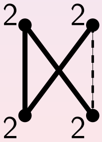
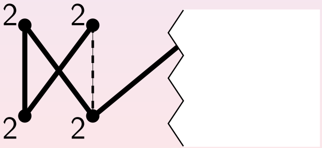
capacity: $b(v)$ $b(z)$ 1 3

"Relax – Reduce" for square-free simple 2-matchings

"3-Way Lemma"

Given a square-free b-matching M in G, then at least one of the following alternatives holds:

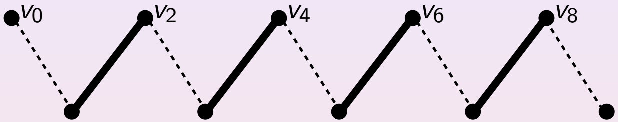
- **a)** M is a maximum b-matching.
- **b)** There is a square-free b-matching N with $|N| = |M| + 1$.
- **c)** There is a square-free b-matching N with $|N| = |M|$ s.t. there is square which is "nice with N".

nice square:  **or** 

Proof.

If M a maximum b-matching $\Longrightarrow$ **a)**

Otherwise there is an augmenting M-alternating path P:

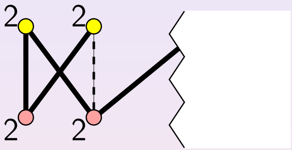


If $M \Delta P$ is square-free $\Longrightarrow$ **b)**

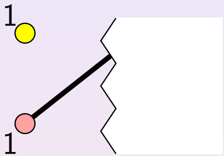
Otherwise, $P_{2i} :=$ the sub-path from v_0 to v_{2i}. $M_i := M \Delta P_{2i}$
Some M_i is square-free, but M_{i+1} is not square-free.
Then $N := M_i$ is nice with the square in M_{i+1}. $\Longrightarrow$ **c)**

"Relax – Reduce" for square-free simple 2-matchings

G, b, N



G', b', N'



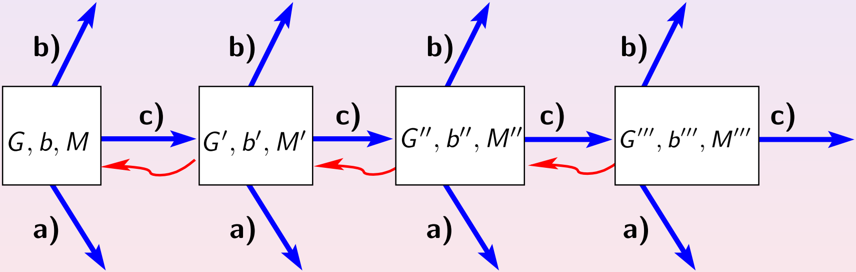
"Equivalent Reduction"

N is a maximum □-free b-matching in G

$$\iff$$

N' is a maximum □-free b'-matching in G'.

The algorithmic scheme:



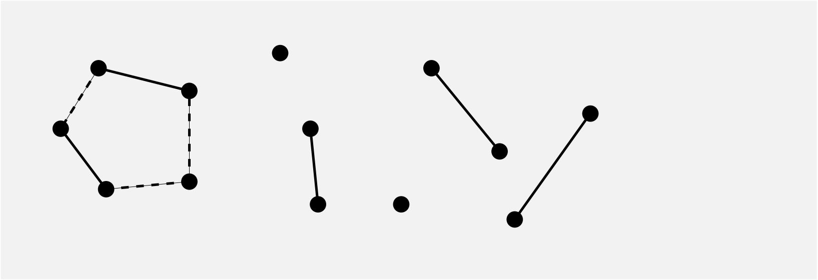
3-Way Lemma for Matching

Given a matching M in graph G. Then at least one of the following alternatives holds:

a) M is a maximum fractional matching.

b) There is a matching N with $|N| = |M| + 1$.

c) There is a matching N with $|N| = |M|$ s.t. there is an N-alternating odd cycle C in G.



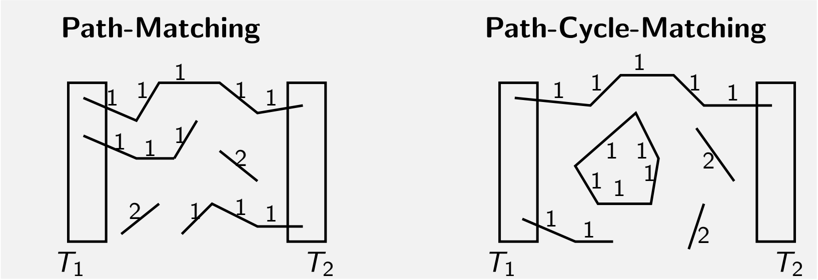
3-Way Lemma for Path-Matching

Given a path-matching x with respect to G, T_1, T_2. Then at least one of the following alternatives holds:

a) x is a maximum path-cycle-matching.

b) There is a path-matching x' with $||x'|| = ||x|| + 1$.

c) There is a path-matching x' with $||x'|| = ||x||$ s.t. there is an odd cycle C which is nice with x'.



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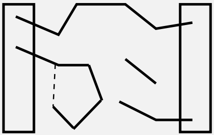
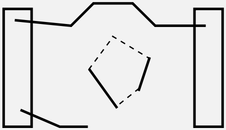
a nice odd cycle:



or



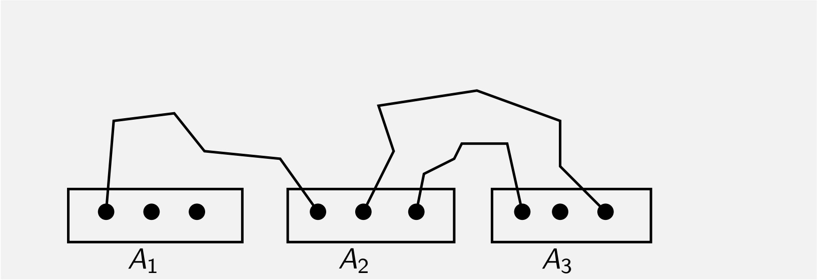
3-Way Lemma for $\mathcal{A}$-paths

Given a packing $\mathcal{P}$ of $\mathcal{A}$-paths in G. Then at least one of the following alternatives holds:

a) $\mathcal{P}$ is a maximum fractional packing of $\mathcal{A}$-paths.

b) There is a packing $\mathcal{P}'$ with $|\mathcal{P}'| = |\mathcal{P}| + 1$.

c) There is a packing $\mathcal{R}$ with $|\mathcal{R}| = |\mathcal{P}|$ s.t. there is an odd cycle, or a rod which is nice with $\mathcal{R}$.



3-Way Lemma for $\mathcal{A}$-paths

Given a packing $\mathcal{P}$ of $\mathcal{A}$-paths in G. Then at least one of the following alternatives holds:

a) There is a packing $\mathcal{P}'$ with $|\mathcal{P}'| = |\mathcal{P}| + 1$.

b) $\mathcal{P}$ is a maximum fractional packing of $\mathcal{A}$-paths.

c) There is a packing $\mathcal{R}$ with $|\mathcal{R}| = |\mathcal{P}|$ s.t. there is an odd cycle, or a rod which is nice with $\mathcal{R}$.