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Packings and coverings in directed graphs

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Preface

I would like to thank András Frank, my supervisor, the enormous amount of work and time he expended on helping me. It would be hard to tell how grateful I am for the helpful discussions, his guidance and encouragement. It was inspiring to see his enthusiasm and extensive knowledge. I am really glad of working with him.

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Introduction

The starting point of this work is a proof of Lovász he gave for Edmonds' disjoint branchings theorem in [20]. Apart from the fact that his approach gives a great tool to simplify existing proofs it also suggests new results in subjects seemed to be examined once and for all. In this work we give some new applications of his technique. We deal with the problem of packing and covering of specified subgraphs in directed graphs, particulary when these subgraphs are arborescences, branchings, bibranchings, or directed cuts. In [25] Schrijver gave a monumental overview of the known results.

Chapter 1 is divided into three sections. In the first one we offer a brief survey of Edmonds' theorems and a recent generalization presented by Kamiyama, Katoh and Takizawa in [18]. This extension will be referred to as the Japanese arborescences theorem. Though the original proof of the theorem follows the main steps of Lovász it is a bit circuitous. We provide a new proof that completely relies on Lovász's proof and is simpler in many ways than the original. We also study an interesting special case which points to the fact that we can easily bump into hopelessly hard questions.

The second section of the chapter presents a nice application of Edmonds' strong theorem. This short digression leads us to planar graphs, more precisely to the so-called Schnyder labellings of triangular graphs. This notion was introduced by Schnyder in [24] and plays an important role in graph drawing problems, especially when we would like to embed a planar graph to a grid of size as small as possible. Many algorithms have been suggested to compute such a labelling in linear time. We propose a new algorithm that is simpler than the present ones. The section is closed with Schnyder's beautiful approach to construct compact embeddings with the help of barycentric representations.

The final section deals with coverings. It was observed in [13] that Vidyasankar's theorem can be derived from Edmonds' weak theorem, hence it can be considered as the covering variant of Edmonds' result. This observation motivated us to find such a variant of the Japanese arborescences theorem, too. Kamiyama and Katoh also study this question in [19] but from the viewpoint of algorithm, while we prove a theorem which is the direct analogue of Vidyasankar's.

Chapter 2 deals with more general covering questions. Theorems of Szegő and Frank show that there is a strong connection between the problem of covering special set-functions with digraphs and packing branchings or arborescences. In [6] Frank characterized those digraphs whose arc-set can be partitioned into k subsets such that each part covers a fixed intersecting family defined on the ground-set of nodes. Later Szegő improved this result to the case of k different intersecting families that also satisfy the so-called linking property. The first part of the chapter summarizes these results. As Edmonds' disjoint branchings theorem can be derived from Szegő's theorem the question offers itself that whether the Japanese arborescences theorem can be deduced similarly. However, the construction of families satisfying both the intersecting- and the linking property presents difficulty. Hence we invoke the notion of bi-sets, a structure introduced by Frank and Jordán in [8]. Bi-sets have been used successfully in node-connectivity augmentation problems and prove to be the remedy of our difficulties. We reformulate Szegő's theorem to bi-set families and give an equivalent form using T-intersecting families. As an immediate application of our extension, we show that it generalizes the Japanese arborescences theorem. The revelation of this connection is one of the main results of this work.

In the remaining part of the chapter we give a new proof for the extension of Vidyasankar's theorem and also show how the above mentioned results can be extended to directed hypergraphs -also called dypergraphs. We use a technique which was introduced in [9] and allows us to led back the problems concerning dypergraphs to simple digraphs by trimming dyperedges.

In Chapter 3 we turn to bibranchings, another well examined type of special subgraphs that can be considered as a generalization of branchings. Schrijver characterized the maximal number of disjoint bibranchings using an interesting coloring-type result. We present a slight extension of his theorem to intersecting families by combining Szegő's theorem with Schrijver's supermodular colorings.

As Edmonds' weak theorem can be derived from Schrijver's disjoint bibranchings theorem it would be natural to find a result about bibranchings corresponding to Edmonds' strong theorem. Surprisingly such a generalization does not exist. We show that we get NP-complete problems even in the case of bipartite graphs. However, some special case can be handled thanks to a conjecture of Evans about partial latin squares.

Chapter 4 presents a new approach to the Lucchesi-Younger theorem which is a central result in the theory of connectors. We fully reduce the problem to matroid intersection with the help of the Gröflin-Hoffman theorem [16]. The idea of using weighted matroid intersection was introduced by Frank and Tardos in [10] where they used it to find a minimum-length directed cut k-cover. However, the meaning of the dual solution was not read out. In [25] Schrijver studied the same question and also gave a proof based on this idea. Our approach still differs from the previous ones as it strictly refers to Gröflin and Hoffman's theorem.

The work closes with a collection of unsolved questions related to the above mentioned problems. We consider them for future research.

Preliminaries

Throughout the work we use the following notation. In a graph $G = (V, E)$, for a subset $X \subseteq V$ we denote by $\Delta(X)$ the set of edges going between X and $V - X$. If X contains at least two nodes then the *degree* $d(X)$ of X is equal to $|\Delta(X)|$. For single nodes this definition changes to the number of edges ending in v where the loops at v are counted twice. The set of edges having both ends in X is denoted by $I(X)$, while $E(X)$ is the set of edges having at least one end in X. Furthermore, $i(X) = |I(X)|$ and $e(X) = |E(X)|$. For $X, Y \subseteq V$ we denote by $d(X, Y)$ the number of edges going between X and Y. When we would like to indicate the graph on which the given function is defined we put its name in subscript.

In a digraph $D = (V, A)$, for a subset $X \subseteq V$ the *in-degree* $\varrho(X)$ denotes the number of arcs entering X. The set of these arcs is $\Delta^-(X)$. Similarly, the *out-degree* $\delta(X)$ denotes the number of arcs leaving X and their set is $\Delta^+(X)$. One of the most important properties of the in-degree and out-degree functions is their submodularity:

$$\varrho(X) + \varrho(Y) \geq \varrho(X \cap Y) + \varrho(X \cup Y)$$

and

$$\delta(X) + \delta(Y) \geq \delta(X \cap Y) + \delta(X \cup Y)$$

for each $X, Y \subseteq V$. The functions $i(X), e(X)$ and the sets $I(X), E(X)$ are defined in the same way as in the undirected case.

A *subgraph* of a graph or digraph $H = (V, F)$ is obtained by deleting some nodes and edges of H. If only nodes are deleted, then we call the arising subgraph *induced* by X where X is the set of remaining nodes. The *underlying graph* $G = (V, E)$ of a digraph $D = (V, A)$ is obtained from D by discarding the orientation of the arcs.

For simplicity we often use the signs $-, +$ for the difference and union of two sets. Also, if $H = (V, F)$ is a graph or a digraph then the subgraph obtained by deleting a subset $F^{'}$ of edges or a subset $V^{'}$ of nodes is denoted by $H - E^{'}$ and $H - V^{'}$.

Chapter 1

Branchings and arborescences

1.1 Packing branchings

The problem of packing arborescences can be considered as a special case of packing common bases of two matroids. The first question concerning this area was proposed by Edmonds. Since then his classical work became the starting point of several minimax results.

1.1.1 Edmonds' theorems

A directed tree is called an *arborescence rooted at* r if each node is reachable from r. A *branching with root set* R is a collection of $|R|$ edge-disjoint arborescences where R is the set of roots of the arborescences.

Let $D = (V + r, A)$ be a digraph. For $U \subseteq V$ with $\delta(U) = 0$ we call $\Delta^-(U)$ an *r-cut*. Edmonds gave in [3] the following characterization of the existence of disjoint spanning-arborescences with the same root r:

Theorem 1.1.1 (Edmonds' weak theorem, 1973)**.** *Let $D = (V, A)$ be a digraph and let $r \in V$. Then the maximum number of disjoint spanning r-arborescences is equal to the minimum size of an r-cut.*

A simple proof of the theorem was given by Lovász in 1976 [20] . His elegant approach also proved to be useful in generalizing Edmonds' result. In Chapter 2 we present several applications of Lovász's proof technique.

Edmonds actually proved a more general result by characterizing the existence of disjoint branchings with prescribed root sets:

Theorem 1.1.2 (Edmonds' strong theorem, 1973)**.** *Let $D = (V, A)$ be a digraph and let $R_1, ..., R_k \subseteq V$ be root sets. There exist k disjoint branchings of root sets $R_1, ..., R_k$, respectively, if and only if $\varrho(X) \geq p(X)$ for each $\emptyset \neq X \subseteq V$ where $p(X)$ denotes the number of R_i's disjoint from X.*

The theorem can be proved by applying Lovász's technique. It is interesting to reformulate Theorem 1.1.2 in an equivalent form [12]:

Theorem 1.1.3. *Let $D^{'} = (V + r, A)$ be a digraph, $F_1, ..., F_k$ be k disjoint r-arborescences and let D denote the subgraph of $D^{'}$ consisting of edges not used by the F_i's. The arborescences can be completed to k disjoint spanning r-arborescences if and only if $\varrho_D(U) \geq p(U)$ for each $\emptyset \neq U \subseteq V$ where $p(U)$ denotes the number of F_i's disjoint from U.*

1.1.2 Japanese arborescences theorem

Recently, in [18] N. Kamiyama, N. Katoh, and A. Takizawa proved yet another generalization of Edmonds' strong theorem. Since possibly not all of the nodes are reachable from a given root they broke with the concept of spanning arborescences and gave the following theorem:

Theorem 1.1.4 (Kamiyama-Katoh-Takizawa, 2008)**.** *Let $D = (V, A)$ be a digraph and $R = \{r_1, ..., r_k\}$ be a set of roots. Let S_i denote the set of nodes reachable from r_i. There exist disjoint arborescences $(S_1, A_1), ..., (S_k, A_k)$ rooted at $r_1, ..., r_k$, respectively, if and only if $\varrho(X) \geq p(X)$ for each $X \subseteq V$ where $p(X)$ denotes the number of r_i's such that $r_i \notin X$ and $S_i \cap X \neq \emptyset$.*

It is remarkable that the proof of the theorem is also based on Lovász's proof but it is more complicated because p is not supermodular in this case. The theorem can be reformulated in terms of arborescences already given:

Theorem 1.1.5. *Let $D^{'} = (V, A)$ be a directed graph and $\{r_1, ..., r_k\} \subseteq V$ be a set of specified nodes. Let S_i denote the set of nodes reachable from r_i. Assume that disjoint arborescences $F_1, ..., F_k$ are already given such that F_i is rooted at r_i. Let D denote the subgraph of $D^{'}$ consisting of edges not used by the F_i's. The arborescences can be extended to disjoint arborescences such that F_i spans S_i if and only if $\varrho^{'}(U) \geq p^{'}(U)$ for each $U \subseteq V$ where $\varrho^{'}(U)$ denotes the number of arcs entering U not used by the F_i's, and $p^{'}(U) = |\{i : \ S_i \cap U \neq \emptyset, \ V(F_i) \cap U = \emptyset|\}$.*

We will refer to this theorem as the *Japanese arborescences theorem*. In Chapter 2 we give a common extension of this result and a theorem of Szegő on covering intersecting families. Although it is simpler to derive Theorem 1.1.4 from our extension, now we give a direct proof just to present the difficulties that the lack of supermodularity causes:

Proof. If F_i spans S_i for each i then we are done. In other case there exists an F_i we would like to extend. Let F_1 be such an arborescence. There exists an arc e leaving $V_1 = V(F_1)$ since $\varrho^{'}(V - V_1) \geq p^{'}(V - V_1)$, and the right-hand side is positive because of S_1. We call a nonempty subset X of V *tight*, if $\varrho^{'}(X) = p^{'}(X)$ and $V_1 \cap X \neq \emptyset$. If e does not enter any tight set then we can add it to F_1 and we are done by induction. In other case e enters a tight set.

Let M be a minimal tight set. Our first observation is that there is an arc from $M \cap V_1$ to $M - V_1$. Otherwise

$$\varrho^{'}(M - V_1) = \varrho^{'}(M) - \overrightarrow{d}'(V - M, M \cap V_1),$$

where $\overrightarrow{d}'(X,Y)$ denotes the number of arcs from X to Y not used by the arborescences. At the same time

$$p^{'}(M - V_1) =$$

$$= p^{'}(M) + |\{i \in \{1,...,k\}: \ S_i \cap (M - V_1) \neq \emptyset, \ F_i \cap (M - V_1) = \emptyset, \ F_i \cap (M \cap V_1) \neq \emptyset\}| -$$

$$-|\{i \in \{1,...,k\}: \ S_i \cap (M - V_1) = \emptyset, \ S_i \cap M \neq \emptyset, \ F_i \cap M = \emptyset\}|.$$

The second member of the sum is greater than 1 because for $i = 1$ the proper conditions hold. Since we need

$$p^{'}(M - V_1) \leq \varrho^{'}(M - V_1)$$

hence

$$\mathcal{J} = \{j: \ S_j \cap (M - V_1) = \emptyset, \ S_j \cap M \neq \emptyset, \ F_j \cap M = \emptyset\} \neq \emptyset$$

holds. Let R denote the set of vertices $v \in V - (M \cap V_1)$ wherefrom $M - V_1$ is reachable in the directed graph. Then $R \neq V - (M \cap V_1)$ as for each $j \in \mathcal{J}$ we know that $s_j \notin R$. It can be seen easily from the definition of R that

$$|\mathcal{J}| \leq p^{'}(R \cup (M \cap V_1)) \leq \varrho^{'}(R \cup (M \cap V_1)) =$$

$$= \overrightarrow{d}'(V - R, M \cap V_1) \leq \overrightarrow{d}'(V - M, M \cap V_1).$$

From these inequalities we get

$$\varrho^{'}(M - V_1) \leq \varrho^{'}(M) - |\mathcal{J}| = p^{'}(M) - |\mathcal{J}| <$$

$$< p^{'}(M) - |\mathcal{J}| + 1 \leq p^{'}(M - V_1),$$

a contradiction.

According to the above, there is an edge f from $M \cap V_1$ to $M - V_1$. We will show that f does not enter any tight set so it can be added to F_1 and we are done. The following lemma is an easy observation about $p^{'}$ that proved by counting cases:

Lemma 1.1.1. *Let X, Y be subsets of V such that $X \cap Y \neq \emptyset$. Let*

$$d_\cap(X,Y) = |\{i \in \{1,...,k\}: \ F_i \cap X \neq \emptyset, \ F_i \cap Y \neq \emptyset, \ F_i \cap X \cap Y = \emptyset\}|$$

and

$$d_\cup(X,Y) = |\{i \in \{1,...,k\}: \ S_i \cap X \neq \emptyset, \ S_i \cap Y \neq \emptyset, \ S_i \cap X \cap Y = \emptyset, \ F_i \cap X = \emptyset \ or \ F_i \cap Y = \emptyset\}|.$$

Then

$$p^{'}(X) + p^{'}(Y) = p^{'}(X \cup Y) + p^{'}(X \cap Y) + d_\cup(X,Y) - d_\cap(X,Y).$$

Assume that f enters a tight set. Let X be a minimal tight set entered by f. We claim that $d_\cup(M,X)=0$. If not, then there is a subscript $j\in\{1,...,k\}$ such that $S_j\cap M\neq\emptyset,\ S_j\cap X\neq\emptyset,\ S_j\cap M\cap X=\emptyset$, and $F_j\cap M=\emptyset$ or $F_j\cap X=\emptyset$. We can assume that $F_j\cap M=\emptyset$. We will show that $M-S_j$ would be a smaller tight set contradicting to the minimality of M.

First of all, $\delta^{'}(S_j)=0$ implies that $(M-S_j)\cap V_1\neq\emptyset$ because the tail of f can not be in S_j. Also by this property of S_j

$$\varrho^{'}(M-S_j)=\varrho^{'}(M)-\overrightarrow{d}^{'}(V-M,M\cap S_j).$$

At the same time

$$p^{'}(M-S_j)=$$

$$=p^{'}(M)+|\{i\in\{1,...,k\}:\ S_i\cap(M-S_j)\neq\emptyset,\ F_i\cap(M-S_j)=\emptyset,\ F_i\cap(M\cap S_j)\neq\emptyset\}|-$$

$$-|\{i\in\{1,...,k\}:\ S_i\cap(M-S_j)=\emptyset,\ S_i\cap M\neq\emptyset,\ F_i\cap M=\emptyset\}|.$$

Let R denote the set of vertices $v\in V-(M\cap S_j)$ wherefrom $M-S_j$ is reachable in the directed graph. Then $R\neq V-(M\cap S_j)$ as $s_j\notin R$. Let

$$\mathcal{J}=\{l:S_l\cap(M-S_j)=\emptyset,\ S_l\cap M\neq\emptyset,\ F_l\cap M=\emptyset\}\neq\emptyset.$$

It can be seen easily from the definition of R that

$$|\mathcal{J}|\leq p^{'}(R\cup(M\cap S_j))\leq\varrho^{'}(R\cup(M\cap S_j))=$$

$$=\overrightarrow{d}^{'}(V-R,M\cap S_j)\leq\overrightarrow{d}^{'}(V-M,M\cap S_j).$$

From these inequalities we get

$$\varrho^{'}(M-S_j)=\varrho^{'}(M)-\overrightarrow{d}^{'}(V-M,M\cap S_j)\leq p^{'}(M)-|\mathcal{J}|\leq p^{'}(M-S_j),$$

contradicting to the minimality of M. Moreover, if $M\cap X\cap V_1=\emptyset$, then $d_\cap(M,X)\geq 1$ as S_1 increases $d_\cap(M,X)$. By Lemma 1.1.1

$$p^{'}(M)+p^{'}(X)=\varrho^{'}(M\cup X)+\varrho^{'}(M\cap X)\geq$$

$$\geq p^{'}(M\cup X)+p^{'}(M\cap X)>p^{'}(M)+p^{'}(X),$$

a contradiction. Hence $M\cap X\cap V_1\neq\emptyset$, so $M\cap X\subset M$ is a tight set, a contradiction. □

1.1.3 A special case

Let $D=(V,A)$ be a digraph whose node set is partitioned into a root-set $R=\{r_1,...,r_t\}$ and a terminal set T. Suppose that no edge of D enters any node of R. Let $m:R\to\mathbb{Z}_+$ be a function and let $k=m(R)$. When can we find k disjoint arborescences in D so that $m(r)$ of them are rooted at r and spanning $T+r$ for each $r\in R$? At first sight one could think that Edmonds' strong theorem would give the answer. But the truth is that Edmonds' theorem only can be applied when $m_i = 1$ for each i. Though the Japanese arborescences theorem could be used to characterize the existence of such arborescences, we will return to this problem in Chapter 2 when we give a generalization of Theorem 1.1.4.

Becoming enthusiastic about the new way of seeing things, another question comes up immediately. It is known that if $D = (V, A)$ is a digraph then we can characterize the existence of k spanning arborescences rooted at distinct nodes. Now let D be a digraph like above. When can we find k distinct nodes $r_{i_1}, ..., r_{i_k} \in R$ so that there exist k disjoint arborescences in D from which exactly one is rooted at r_{i_j} and spanning $V + r_{i_j}$? Actually, this problem is NP-complete!

To prove this, let D_T be the digraph induced by T. Suppose that $\varrho_{D_T}(v) = k - 2$ for each $v \in T$ and $\varrho_{D_T}(Z) \geq k - 1$ for each $Z \subset T,\ |Z| \geq 2$. We can easily construct a digraph with these properties, for example, let n be even and take the same directed Hamilton cycle on n nodes $k - 2$ times. Let $v_1, ..., v_n$ denote the nodes around this cycle. If we give the arcs $v_i v_{i+\frac{n}{2}}$ to the graph for each $i = 1, ..., n$ then the arising digraph satisfies the conditions. Let $R_i = \{v \in T :\ r_i v \in \Delta^+(r_i)\}$ and suppose that $|R_i| \geq 2$ holds for each i. Edmonds' strong theorem implies that for a choice of r_i's the requested arborescences exist if and only if $\varrho(Z) \geq p(Z)$ for each $Z \subseteq V$ where $p(Z)$ denotes the number of R_i's disjoint from Z. For a Z with $|Z| \geq 2$ the inequality holds automatically because of the structure of D. So we only have to care about the sets containing a single node. That means that our aim is to cover the set V with k sets from $\{R_1, ..., R_t\}$. This is a variant of the set-covering problem, which is NP-complete.

1.2 Planar graphs

We call a graph G *planar* if it can be drawn so that the edges only meet in nodes. A *plane graph* is an abstract graph with a given embedding in the plane. This embedding is a *straight line embedding* if the edges are represented by straight line segments. A *triangular graph* is a maximal plane graph with at least three nodes (Figure 1.1). The nodes and edges on the exterior face of G are called *exterior nodes and edges*. We can define *interior nodes and edges* similarly.

The problem of embedding graphs compactly is related to the drawing of graphs on finite display devices. Fáry showed [5] that each plane graph has straight line embeddings, and many algorithms were suggested to construct one. Most of these algorithms had several drawbacks as they concentrate on the node-positions only, and do not care about the size of the output embedding. So embeddings were at hand, but their view on a terminal was impossible because of their huge size.

Embedding a planar graph on the n by m grid means that the nodes have integer valued coordinates in the range $0 \leq v_1 \leq n$ and $0 \leq v_2 \leq m$. In [22] Rosenstiehl and Tarjan asked whether or not every planar graph of size n has a straight line embedding on a grid of side length bounded by n^k for some fixed k. Fraysseix, Pach and Pollack showed [14] that every plane graph with n vertices has such an embedding on the $2n-4$ by $n-2$ grid and provided an $\mathcal{O}(n)$ space, $\mathcal{O}(nlogn)$ time algorithm to effect this embedding. Schnyder improved this bound to $n-2$ by $n-2$ by using Schnyder-labellings. His interesting approach provides embeddings in which the vertex-coordinates have a purely combinatorial meaning [24].

1.2.1 3-orientations of triangular graphs

Let us show another application of Edmonds' strong theorem. While studying the embeddings of planar graphs on grids of "small" size, Schnyder observed the following. Let $G = (V, E)$ be a triangular graph with exterior nodes $r_{1,}, r_2$, and r_3. After dropping out the exterior edges of G the remaining edges can be oriented in such a way that the arising directed graph is the union of three arborescences F_1, F_2, and F_3 where F_i is rooted at r_i and spans $V - \{r_{i+1}, r_{i+2}\}$ (indeces are modulo 3). In fact every orientation of the interior edges such that $\varrho(r_i) = 0$ for $i = 1, 2, 3$ and $\varrho(v) = 3$ for other nodes (Figure 1.2), have this decomposition property, as we will show by using Edmonds' strong theorem. The orientations satisfying these in-degree conditions are called 3-*orientations*. First of all we need the following lemma:

Lemma 1.2.1. *Let G be a triangular graph with exterior nodes r_1, r_2, r_3. There is an orientation of the interior edges such that $\varrho(r_i) = 0$ for $i = 1, 2, 3$ and $\varrho(v) = 3$ for other nodes.*

We will use a corollary of Hakimi's theorem about orientations with upper bounds on the indegrees [17]. Originally Hakimi considered lower bounds, but we need the following form:

Theorem 1.2.1 (Hakimi, 1965)**.** *Let $G = (V, E)$ be an undirected graph and let $g : V \to \mathbb{Z}_+$. Then G has an orientation $D = (V, A)$ with $\varrho(v) \le g(v)$ for each $v \in V$ if and only if $i(Z) \le g(Z)$ for each $Z \subseteq V$, where $g(Z)$ denotes $\sum_{v \in Z} g(v)$.*

An easy consequence of the theorem is:

Corollary 1.2.1. *Let $G = (V, E)$ be an undirected graph and let $g : V \to \mathbb{Z}_+$. Then G has an orientation $D = (V, A)$ with $\varrho(v) = g(v)$ for each $v \in V$ if and only if $g(V) = |E|$ and $i(Z) \le g(Z)$ for each $Z \subseteq V$.*

Proof of Lemma 1.2.1. Let $G = (V + r_1 + r_2 + r_3, E)$ be a triangular graph with exterior nodes r_1, r_2, r_3 and with exterior edges dropped out. According to Corollary 1.2.1, we only have to show that $|E| = 3n - 9$ and $i_E(Z) \le 3(|Z| - |\{r_1, r_2, r_3\} \cap Z|)$. By the fact that a triangular graph with n nodes has $3n - 6$ edges the equality is clear since the exterior edges are dropped out. The inequalities can be proved as follows:

(a) $|\{r_1, r_2, r_3\} \cap Z| = 0, 1$ *or* 2**:** Z induces a planar graph with at most $3|Z| - 6$ edges, so the inequality holds in all cases.

(b) $|\{r_1, r_2, r_3\} \cap Z| = 3$**:** Z induces a planar graph. Adding the edges r_1r_2, r_2r_3, r_3r_1 to this graph we get a planar graph again with at most $3|Z| - 6$ edges. Hence $i_E(Z)$ is at most $3|Z| - 9$.

□

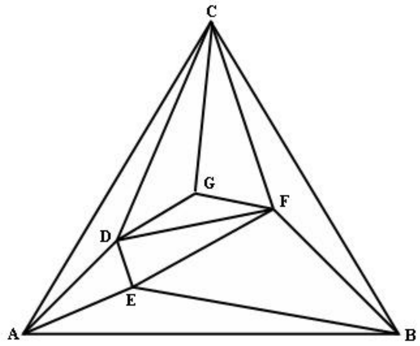

Figure 1.1: *A triangular graph G with 7 nodes*

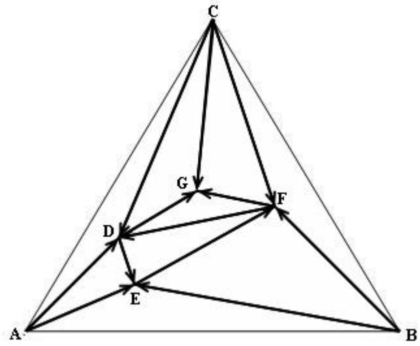

Figure 1.2: *A 3-orientation of G*

Now we show that any orientation given by Lemma 1.2.1 has the decomposition property described earlier:

Theorem 1.2.2. *Let $G = (V + r_1 + r_2 + r_3, E)$ be a triangular graph with exterior nodes r_1, r_2, r_3 and with exterior edges dropped out. Orientate the interior edges in such a way that $\varrho(r_i) = 0$ for $i = 1, 2, 3$ and $\varrho(v) = 3$ for $v \in V$. Let $D_0 = (V + r_1 + r_2 + r_3, A_0)$ denote the arising directed graph. Then A_0 can be partitioned into subsets A_1, A_2, A_3 such that A_i is an arborescence rooted at r_i and spanning $V + r_i$.*

Proof. Let R_i denote the neighbors of r_i and let $D = (V, A)$ be the subgraph of D_0 induced by V. Then, by Edmonds' strong theorem, we have to show that $\varrho_D(X) \geq p_D(X)$ for each $\emptyset \neq X \subseteq V$ where $p_D(X)$ denotes the number of R_i's disjoint from X. We have the following cases:

(a) $p_D(X) = 0$**:** The inequality obviously holds.

(b) $p_D(X) = 1$**:** Assume that $X \cap R_1 \neq \emptyset$ and $X \cap R_2 \neq \emptyset$. Notice that $X' = X + r_1 + r_2$ induces a subgraph in D_0 in which each $v \in X$ has in-degree 3 as $R_3 \cap X = \emptyset$. Then

$$3|X| = 3|X'| - 6 = \sum_{v \in X'} \varrho_{D_0}(v) = \varrho_{D_0}(X') + i_{D_0}(X') = \varrho_D(X) + i_{D_0}(X') \leq$$

$$\leq \varrho_D(X) + 3|X'| - 7 = \varrho_D(X) + 3|X| - 1.$$

The inequality $i_{D_0}(X') \leq 3|X'| - 7$ holds since giving the arc r_1r_2 to the subgraph induced by X' in D_0 we also get a planar graph. Hence $\varrho(X) \geq 1$, and we are done.

(c) $p_D(X) = 2$**:** Assume that $X \cap R_1 \neq \emptyset$. Let $X' = X + r_1$. Notice that $X' = X + r_1$ induces a subgraph in D_0 in which each $v \in X$ has in-degree 3 as $R_i \cap X = \emptyset$ for $i = 2, 3$. Then

$$3|X| = 3|X'| - 3 = \sum_{v \in X'} \varrho_{D_0}(v) = \varrho_{D_0}(X') + i_{D_0}(X') = \varrho_D(X) + i_{D_0}(X') \leq$$

$$\leq \varrho_D(X) + 3|X^{'}| - 6 = \varrho_D(X) + 3|X| - 3.$$

Hence $\varrho(X) \geq 2$, and we are done.

(d) $p_D(X) = 3$**:** X induces a planar subgraph in D, so $3|X| = \sum_{v \in X} \varrho_D(v) = \varrho_D(X) + i(X) \leq \varrho(X) + 3|X| - 6$, hence $\varrho(X) \geq 1$ clearly holds.

□

Theorem 1.2.2 also follows from the existence of Schnyder-labellings. In the following section we present a new algorithm to construct such labellings and hence give a new proof of the Theorem.

1.2.2 Schnyder labellings

Let $G = (V, E)$ be a triangular graph. A *realizer* or *Schnyder labelling* is a 3-orientation of G plus a coloring of the interior edges with three colors, such that

1. every interior node has one incoming edge in each color,
2. the colors of the incoming edges appear always in counterclockwise order at every interior node,
3. outgoing edges of one color appear exactly between the incoming edges of the two remaining colors.

Schnyder showed that each triangular graph has a Schnyder labelling, and also presented an algorithm to construct such labellings in linear time. This observation proved to be useful in applications concerning planar graph drawings. The base of his algorithm is an operation called *contraction*. An edge e is *contractible* if it is incident to an exterior and an interior node and these nodes have exactly two common neighbors. It is part of his work [24] that in every triangular graph there exists such a contractible edge. It is easy to see that after contraction the graph remains triangular, hence everything is at hand for induction. We can contract edges until we get a single triangle and then expand the edges again in reverse order, while taking care of the colors and directions to the reappearing edges. A great summary and some interesting result on Schnyder labellings can be found in [2]. We present an algorithm here which is not based on edge contractions. While Schnyder's algorithm assigns colors and orientations simultaneously, our algorithm starts with an arbitrary 3-orientation and only colors the arcs properly.

We start with a triangular graph $G = (V + r_R + r_B + r_G, E)$ where r_R, r_B, r_G are the exterior nodes in counterclockwise order, and take an arbitrary 3-orientation of G provided by Lemma 1.2.1. The arising digraph will be denoted by D. Our algorithm will assign colors to the arcs - say RED, $BLUE$ and $GREEN$- where the colors also have a cyclic order: $BLUE$ comes after RED, $GREEN$ after $BLUE$, and RED after $GREEN$. The algorithm is really simple: we build up arborescences F_R, F_B and F_G rooted at the exterior nodes where F_i denotes the arborescence rooted at r_i. Initially let $F_i = r_i$. Firstly we assign the color RED to the outgoing edges from r_R and give them to F_R. Then we take an interior node v reached by one of these RED arcs. We know that v has in-degree 3, and one of the three incoming edges is RED. So we color the outgoing edges appearing exactly between the two remaining incoming edges to RED, give them to F_R, and iterate the procedure until there is no arc we can color. After that we apply the same to r_B with color $BLUE$, and finally to r_G with color $GREEN$.

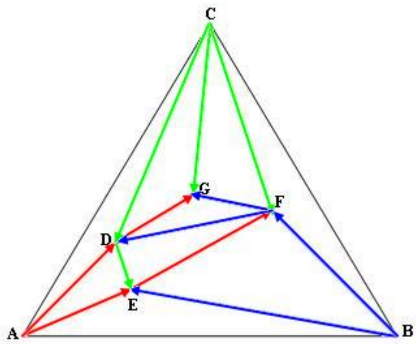

Figure 1.3: *The labelling belonging to the* 3*-orientation presented on Figure 1.2*

To prove that our algorithm really provides a Schnyder labelling we have to show that the steps are well-defined, which means that at each step we only have to color uncolored edges. We also have to prove that the coloring arising from the algorithm has the properties described above.

Claim 1.2.1. *The steps are well-defined as each edge gets at most one color.*

Proof. The only problem could be that we arrive to an interior node v and one of the arcs lying between the two other incoming edges is already colored. But this is a contradiction, since at any interior node an outgoing edge can be colored only if the proper incoming edge is already colored. □

Claim 1.2.1 implies that the F_i's are edge disjoint subgraphs. Our next observation is:

Claim 1.2.2. *F_i is an arborescence rooted at r_i for $i \in \{R, B, G\}$.*

Proof. We prove this for $i = R$, the other two cases are similar. It is clearly enough to show that the underlying graph of F_R does not contain a cycle. Suppose to the contrary that the claim does not hold, and let C be the first cycle that arises while the algorithm is building up F_R (if more than one such cycle exist then let C be one of them). Obviously, just before C appears F_R was an arborescence. Hence we have two cases: C is a directed cycle or the union of two internally node-disjoint directed path. Let H be the subgraph of D induced by C and its interior. We will get a contradiction by double-counting the arcs in H. Let $c = |C|$ and let t and l denote the number of nodes and arcs in H. We know that H is triangular except the outer face which is bounded by

C. With an extra node and c additional edges we can triangulate the outer face, hence

$$l = 3(t+1) - 6 - c = 3t - c - 3.$$

If C is a directed cycle then our algorithm yields $\varrho_H(v) = 3$ for each node in $H - C$, $\varrho_H(v) = 2$ for each $v \in C$ except at most one $v \in C$, for which $\varrho_H(v) \geq 1$. But that would mean

$$l \geq 3t - c - 1,$$

a contradiction. If C is the union of two paths the proof is similar, the only difference is that we get $l \geq 3t - c - 2$, which is still a contradiction. □

Notice that we still know nothing about the node-set of F_i, whether it spans $V + r_i$ or not. Actually this will be true:

Claim 1.2.3. *F_i spans $V + r_i$ for $i \in \{R, B, G\}$.*

Proof. The node-set of F_i can not be bigger as the exterior nodes have in-degree 0. So we only have to show that each interior node is reachable from r_i in F_i. Let v be an interior node. As D is a 3-orientation of G, $\varrho(v) = 3$. We take one of the incoming edges at v and go backward on it. If we arrive to another interior node then it also has three incoming edges and the edge we came on lays between two of them. We continue our way backward on the third one. This hall procedure can be considered as the reverse of our algorithm. Clearly, we only can get stuck in an exterior node. Furthermore, we arrive to different exterior nodes if at the first step we choose different incoming edges to go on backward. Hence v is in F_i for each $i \in \{R, B, G\}$. □

According to the claims above every interior node has one incoming edge in each color. The algorithm assures that outgoing edges of one color appear exactly between the incoming edges of the two remaining colors. It only remains to show that colors of the incoming edges appear always in counterclockwise order at every interior node.

Claim 1.2.4. *Let $v \in V$ be an interior node and let $P_i(v)$ denote the unique directed path in F_i from r_i to v for each $i \in \{R, B, G\}$. Then any two of these paths are node-disjoint apart from v.*

Proof. Assume that $P_R(v)$ and $P_B(v)$ has some common nodes different from v. Let u be the first such node on the paths seen from v. The parts of $P_R(v)$ and $P_B(v)$ lying between u and v form a cycle C. With the same argument as in Claim 1.2.2 we get a contradiction by double-counting the edges in the subgraph induced by C and its interior. □

Form Claim 1.2.4 it easily follows that incoming edges appear in counterclockwise order at every interior node. Otherwise there would be a $v \in V$ for which the order is RED, $GREEN$ and $BLUE$, but then two of the paths $P_i(V)$ should have a common node different from v, a contradiction.

Hence our algorithm works and really gives a Schnyder labelling. The fact that every 3-oriented triangular graph has the decomposition property described in Section 1.2.1 now clearly follows from the foregoing (without using Edmonds' theorem). Indeed, the arborescences F_R, F_B and F_G -provided by the algorithm- define such a decomposition.

Notes 1.2.1. Claim 1.2.4 is interesting: the unique paths in the arborescences to an interior node are not just edge- but also node-disjoint! It is easy to show that this condition holds for every Schnyder-labellings.

1.2.3 Embedding planar graphs on the grid

Schnyder used these labellings to construct straight line embeddings on the $n-2$ by $n-2$ grid. A *weak barycentric representation* of a graph G is an injective function $v \in V(G) \to (v_1, v_2, v_3) \in \mathbb{R}^3$ with the following two properties:

1. $v_1 + v_2 + v_3 = 1$ for all nodes v,
2. for each edge uv and each node $w \neq u, v$, there is some $k \in \{1, 2, 3\}$ such that $(u_k, u_{k+1}) <_{lex} (w_k, w_{k+1})$ and $(v_k, v_{k+1}) <_{lex} (w_k, w_{k+1})$.

The following lemma shows the importance of weak barycentric representations:

Lemma 1.2.2. *Let $v \in V(G) \to (v_1, v_2, v_3)$ be a weak barycentric representation of a graph G. Then given any three noncolinear points α, β and γ, the mapping $f : v \in V(G) \to v_1\alpha + v_2\beta + v_3\gamma$ is a straight line embedding of G in the plane spanned by α, β and γ.*

Proof. It is easy to see that f is injective, since α, β and γ are noncolinear. Moreover, for an edge uv and a node $w \neq u, v$ the condition $(u_k, u_{k+1}) <_{lex} (w_k, w_{k+1})$ and $(v_k, v_{k+1}) <_{lex} (w_k, w_{k+1})$ must hold for some k, hence the point $f(w)$ does not lie on the segment $f(u)f(v)$.

If xy and uv are disjoint edges, then there exist indices $i, j, k, l \in \{1, 2, 3\}$ such that

$$(x_i, x_{i+1}) >_{lex} (u_i, u_{i+1}),\ (v_i, v_{i+1}),$$

$$(y_j, y_{j+1}) >_{lex} (u_j, u_{j+1}),\ (v_j, v_{j+1}),$$

$$(u_k, u_{k+1}) > (x_k, x_{k+1}),\ (y_k, y_{k+1}),$$

$$(v_l, v_{l+1}) > (x_l, x_{l+1}),\ (y_l, y_{l+1}).$$

These inequalities clearly imply $\{i, j\} \cap \{k, l\} = \emptyset$. As $i, j, k, l \in \{1, 2, 3\}$ there holds $i = j$ or $k = l$. Therefore the segments $f(x)f(y)$ and $f(u)f(v)$ are separated by a straight line parallel to $\alpha\beta, \alpha\gamma$ or $\beta\gamma$, hence do not intersect. $\square$

Lemma 1.2.2 implies that only planar graphs can have weak barycentric representations. Furthermore, if such a representation of G is given then straight line embeddings of G can be constructed easily. Schnyder's idea was to find a "nice" weak barycentric representation of planar graphs with the help of Schnyder labellings.

Let G be a triangular graph with a Schnyder labelling provided by our algorithm described in the previous section. In Claim 1.2.4 we showed that for each interior node $v \in V$ the paths $P_R(v), P_B(v)$ and $P_G(v)$ are pairwise node-disjoint apart from v. However, it can be showed that this condition holds for any Schnyder labellings. Therefore, $P_R(v), P_B(v)$ and $P_G(v)$ divide G in three regions denoted by $R_R(v), R_B(v)$ and $R_G(v)$ where $R_i(v)$ denotes the closed region opposite to the root of F_i. From now we use indices $1, 2$ and 3 instead of R, B and G, respectively. From the definition of Schnyder labellings it easily follows that:

Lemma 1.2.3. *For any two distinct interior nodes u and v*

$$u \in R_i(v) \Rightarrow R_i(u) \subset R_i(v)$$

holds. The inclusion is proper.

Proof. Suppose that $u \in R_3(v)$ and u does not lie on the boundary of $r_3(v)$ (the other case is similar). Let x denote the first node of $P_1(u)$ that belongs to the boundary of $R_3(v)$. From the definition of Schnyder labellings it follows that $x \notin P_2(v)$, hence $x \in P_1(v) - v$. Similarly, the first node y of $P_2(u)$ belonging to the boundary of $R_3(v)$ must lie on $P_2(v) - v$. Hence $R_3(u) \subset R_3(v)$, and the inclusion is proper as $v \in R_3(v) - R_3(u)$. □

For an interior node v of G let $v_i = |R_i(v)| - |P_{i-1}(v)|$ -so v_i denotes the number of nodes in region $R_i(v)$ from which the path $P_{i-1}(v)$ has been removed. For the root r of F_i we extend this definition by setting $r_i = n-2, r_{i+1} = 1, r_{i+2} = 0$. Then we get $v_1 + v_2 + v_3 = n - 1$ for each node and $0 \leq v_1, v_2, v_3 \leq n-2$ (with $1 \leq v_1, v_2, v_3 \leq n-3$ for interior nodes).

Lemma 1.2.4. *Let u and b be distinct nodes of G. If v is an interior node and $u \in R_i(v)$ there holds $(u_i, u_{i+1}) <_{lex} (v_i, v_{i+1})$.*

Proof. Firstly we show that the implication $u \in R_k(v) - P_{k-1}(v) \Rightarrow u_k < v_k$ holds. If u is an exterior node then $u_k = 0$ while $v_k \geq 0$. In other case the inequality follows from Lemma 1.2.3.

If $u \in R_i(v)$ then Lemma 1.2.3 implies $u_i \leq v_i$. We have two cases: if $u \notin P_{i-1}(v)$ then $u_i < v_i$, else $u \in P_{i-1}(v)$ thus $u \in R_{i+1}(v) - P_i(v)$ and $u_{i+1} < v_{i+1}$, by the previous observation with $k = i+1$. □

The function $f : v \in V(G) \to (v_1, v_2, v_3)$ is clearly injective. In addition, the following lemma holds:

Lemma 1.2.5. *The function $v \in V(G) \to \frac{1}{n-1}(v_1, v_2, v_3)$ is a weak barycentric representation of G.*

Proof. The first condition of the definition of barycentric representations is clearly satisfied. Let uv be an edge and $w \neq u, v$. If w is an exterior node, the root of F_i, then $w_i = n - 1 > u_i, v_i$. Else, w is an interior node and $u, v \in R_i(w)$ for some i. By Lemma 1.2.3 this implies $w_i > u_i, v_i$ again. □

By applying Lemma 1.2.2 with choice $\alpha = (n-1, 0), \beta = (0, n-1), \gamma = (0, 0)$ we get Schnyder's theorem:

Theorem 1.2.3. *The mapping $v \in V(G) \to (v_1, v_2)$ is a straight line embedding of G on the $n-2$ by $n-2$ grid.*

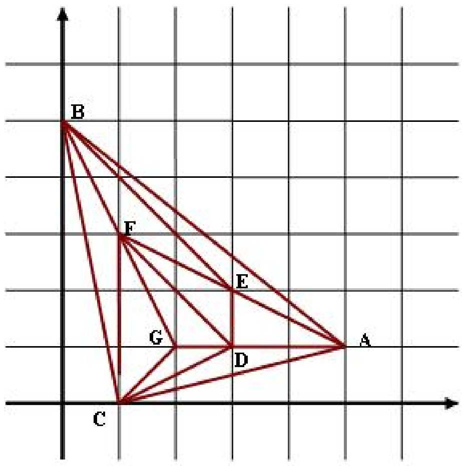

Figure 1.4: *An embedding of G on the 5 by 5 grid*

The embeddings induced by this beautiful approach have many advantages, such as nice separation properties. For example, it follows from the foregoing, that:

Theorem 1.2.4. *Let $\lambda_1, \lambda_2, \lambda_3$ be three pairwise non parallel straight line in the plane. Then each plane graph has a straight line embedding in which any two disjoint edges are separated by a straight line parallel to λ_1, λ_2 or λ_3.*

1.3 Covering by arborescences

The problem of covering the edge-set of a graph by subgraphs with specified properties have been extensively studied because of its practical applications such as vehicle routing problems, fire station location problems or evacuation planning problems.

1.3.1 Vidyasankar's theorem

Let $D = (V, A)$ be a digraph and $r \in V$. It is a natural question that when can A be covered by k r-rooted arborescences. Vidyasankar proved the following theorem, which can be considered as the covering analogue of Edmonds' strong theorem. For a subset $U \subseteq V$ let $H(U)$ denote the set of the heads of the arcs entering U. We call $H(U)$ the *door* of U.

Theorem 1.3.1 (Vidyasankar, 1978)**.** *Let $D = (V + r, A)$ be a digraph with $\varrho(r) = 0$ and k be a positive integer. Then A can be covered by k r-arborescences if and only if*

$$(i)\ \varrho(v) \leq k$$

for each $v \in V$ and

$$(ii) \sum_{v \in H(U)} (k - \varrho(v)) \geq k - \varrho(U),$$

for each $\emptyset \neq U \subseteq (V - r)$.

It has been showed in [13] that Vidyasankar's theorem can be derived from Edmonds' weak theorem. With the help of this observation we give an extension of Theorem 1.3.1 by using the Japanese arborescences theorem. In [19] Kamiyama and Katoh also studied this question from algorithmic aspects. Our result is the following:

Theorem 1.3.2. *Let $D = (V, A)$ be a digraph and $\{r_1, ..., r_k\} = R \subseteq V$ a set of specified nodes. Let S_i denote the set of nodes reachable from r_i. There exist arborescences $F_1, ..., F_k$ rooted at $r_1, ..., r_k$, respectively, and covering A if and only if*

$$(i)\ \varrho(v) \leq p(v)$$

for each $v \in V$ and

$$(ii)\ p(U) - \varrho(U) \leq \sum [p(v) - \varrho(v) :\ v \in H(U)]$$

for each $\emptyset \neq U \subseteq V$, where $H(U)$ denotes the door of U and $p(U) = |\{i \in \{1, ..., k\} :\ S_i \cap U \neq \emptyset,\ r_i \notin U\}|$.

Proof. Assume that $F_1, ..., F_k$ are proper arborescences. We can suppose that F_i spans S_i for each $i \in \{1, ..., k\}$. If $v \notin R$ then each arborescence can only contain one arc from $\Delta^-(v)$, while if $v \in R$ then those arborescences that are rooted at v contains no arcs from $\Delta^-(v)$. From the definition of p one can see that (i) is necessary.

Necessity of (ii) can be seen as follows. For each $e \in A$ let $z'(e)$ denote the number of arborescences that contain the arc e. Let $z(e) = z'(e) - 1$. Then for each $e \in A :\ z(e) \geq 0$. Moreover, since the arborescences cover A: $\varrho_z(U) + \varrho(U) \geq p(U)$ for each $\emptyset \neq U \subseteq V$ and $\varrho_z(v) + \varrho(v) = p(v)$ for each $v \in V$. Hence for each $\emptyset \neq U \subseteq V$:

$$p(U) - \varrho(U) \leq \varrho_z(U) \leq \sum [\varrho_z(v) :\ v \in H(U)] = \sum [p(v) - \varrho(v) :\ v \in H(U)].$$

To see sufficiency, we extend D as follows. For each $v \in V$ we give a copy of v denoted by v' to D. Moreover, we give $p(v)$ parallel arcs from v to $v^{'}$, $p(v) - \varrho(v)$ parallel arcs from $v^{'}$ to v, and finally $p(v)$ parallel arcs from u to $v^{'}$ for each $uv \in A$. Let $D^{'}$ denote the directed graph thus arising.

If there exist $F_1^{'}, ..., F_k^{'}$ disjoint arborescences in $D^{'}$ such that $F_i^{'}$ is rooted at r_i and $F_i^{'}$ is spanning $S_i \cup S_i^{'}$ (where $S_i^{'}$ denotes the copy of S_i), then these arborescences cover A. Hence restricting them to the arcs of the original graph D we obtain k proper arborescences covering A.

In other case, by Theorem 1.1.4 there is a subset U of $V \cup V^{'}$ such that $p_{D'}(U) > \varrho_{D'}(U)$ where $p_{D'}(U) = |\{i \in \{1, ..., k\} :\ (S_i \cup S_i^{'}) \cap U \neq \emptyset,\ r_i \notin U\}|$. From now ϱ and p denote the functions defined in the original graph while $\varrho_{D'}$ and $p_{D'}$ denote the new ones. We define the following subsets of U:

$$X = \{v \in V :\ v \in U\},$$

$$Y = \{v \in V :\ v^{'} \notin U\}\ (\subseteq X),$$

and

$$Z = \{v^{'} \in U :\ v \notin U\}\ (\subseteq U - X).$$

Using these definitions we get

$$p_{D'}(U) \leq p(X) + \sum[p(v) :\ v^{'} \in Z].$$

On the other hand

$$\varrho_{D'}(U) \geq \varrho(X) + \sum[p(v) - \varrho(v) :\ v \in Y] + \sum[p(v) :\ v \in H(X) - Y] + \sum[p(v) :\ v^{'} \in Z].$$

The explanation of the second sum is that if $v \in H(X) - Y$ then $v^{'} \in U$ also holds. Moreover, there exists $u \notin U$ such that $uv \in A$ -since v is in the door- and so there are $p(v)$ arcs from u to $v^{'}$.

From these inequalities we get

$$p(X) > \varrho(X) + \sum[p(v) - \varrho(v) :\ v \in Y] + \sum[p(v) :\ v \in H(X) - Y] \geq$$

$$\geq \varrho(X) + \sum[p(v) - \varrho(v) :\ v \in H(X)],$$

which contradicts (ii). □

In Chapter 2 we give another proof of this theorem using a more general result on covering positively intersecting set functions.

Chapter 2

Covering set-functions

2.1 Intersecting families

In this section we describe the problem of covering intersecting families. In some ways this problem can be considered as a generalization of coverings by arborescences. We call a family $\mathcal{F} \subseteq 2^V$ of sets *intersecting* if

$$X, Y \in \mathcal{F},\ X \cap Y \neq \emptyset \ \Rightarrow\ X \cap Y,\ X \cup Y \in \mathcal{F}$$

holds. We say that $D = (V, F)$ *covers* $\mathcal{F}$ if $\varrho_F(X) \geq 1$ holds for each $X \in \mathcal{F}$ where $\varrho_F(X)$ denotes the number of arcs in F entering X.

2.1.1 An extension of Szegő's theorem

Frank observed in [6] that, by using Lovász's proof technique, Edmonds' weak theorem can be extended as follows:

Theorem 2.1.1 (Frank, 1979)**.** *Let $D = (V, A)$ be a digraph and $\mathcal{F} \subseteq 2^V$ an intersecting family. Then A can be partitioned into k coverings of $\mathcal{F}$ if and only if $\varrho(Z) \geq k$ for each $Z \in \mathcal{F}$.*

By choosing $\mathcal{F} = 2^{V-r} - \{\emptyset\}$ one obtains Edmonds' theorem. Also on the ground of Lovász's approach Szegő gave a common generalization of Edmonds' strong theorem and Theorem 2.1.1 in [23]:

Theorem 2.1.2 (Szegő, 2001)**.** *Let $\mathcal{F}_1, ..., \mathcal{F}_k$ be intersecting families with the following linking property:*

$$X \in \mathcal{F}_i,\ Y \in \mathcal{F}_j,\ X \cap Y \neq \emptyset \ \Rightarrow\ X \cap Y \in \mathcal{F}_i \cap \mathcal{F}_j.$$

Then A can be partitioned into $A_1, ..., A_k$ such that A_i covers $\mathcal{F}_i$ if and only if $\varrho(X) \geq p(X)$ for each $X \subseteq V$ where $p(X)$ denotes the number of $\mathcal{F}_i$'s containing X.

The proof is based on the observation that the mixed intersecting property implies that p is positively intersecting supermodular and hence Lovász's approach works again. It is easy to see that with the choice $\mathcal{F}_1 = ... = \mathcal{F}_k = \mathcal{F}$ we get Theorem 2.1.1, while choosing $\mathcal{F}_i = 2^{V-R_i} - \{\emptyset\}$ gives Theorem 1.1.2.

Now we give a slight generalization of Szegő's theorem using *bi-sets*. The idea of bi-sets was firstly introduced by Frank and Jordán [8]. They successfully used this framework in node-connectivity augmentation problems.

We call a pair $X = (X_O, X_I)$ a *bi-set* if $X_I \subseteq X_O \subseteq V$. For $X = (X_O, X_I),\ Y = (Y_O, Y_I)$ let:

$$X \cap Y = (X_O \cap Y_O, X_I \cap Y_I),$$

$$X \cup Y = (X_O \cup Y_O, X_I \cup Y_I),$$

$$X - Y = (X_O - Y_O, X_I - Y_I).$$

We say that $X \subseteq Y$ if $X_I \subseteq Y_I$ and $X_O \subseteq Y_O$. An arc e *enters* X if e enters both X_O and X_I. Now we prove the following theorem that can be considered as the extension of Szegő's theorem to bi-set-systems:

Theorem 2.1.3 (Covering bi-set families)**.** *Let $\mathcal{F}_1, ..., \mathcal{F}_k$ be families of bi-sets on the ground set V with the intersecting- and linking properties, i.e.,*

$$X, Y \in \mathcal{F}_i,\ X_I \cap Y_I \neq \emptyset\ \Rightarrow\ X \cap Y,\ X \cup Y \in \mathcal{F}_i$$

for each $i \in \{1, ..., k\}$ and

$$X \in \mathcal{F}_i,\ Y \in \mathcal{F}_j,\ X_I \cap Y_I \neq \emptyset\ \Rightarrow\ X \cap Y \in \mathcal{F}_i \cap \mathcal{F}_j.$$

The edge set of a digraph $D = (V, A)$ can be partitioned into $A_1, ..., A_k$ such that A_i covers $\mathcal{F}_i$ if and only if $(*)$ $\varrho(X) \geq p(X)$ for each bi-set X where $p(X)$ denotes the number of $\mathcal{F}_i$'s containing X.

The proof will use the following lemma which is an easy corollary of the linking property:

Lemma 2.1.1. *If $X \in \mathcal{F}_i$ and $Y \in \mathcal{F}_j$ for some i and j and $X_I \cap Y_I \neq \emptyset$ then $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$. Moreover, equality holds if and only if $X \cap Y \in \mathcal{F}_i$ implies that X or $Y \in \mathcal{F}_i$.*

Proof of the theorem. The necessity is clear. We prove sufficiency by induction on $\sum_i |\mathcal{F}_i|$. If the sum is 0 then there is nothing to prove. If the sum is greater than 0 then $\mathcal{F}_i$ is not empty for some i. We can assume that $i = 1$.

Let F be a maximal member of $\mathcal{F}_1$. By $(*)$ there is an arc $e \in A$ entering F. Let $\mathcal{F}_1^{'}$ be the collection of bi-sets $Z \in \mathcal{F}_1$ not covered by e. We claim that $\mathcal{F}_1^{'}$ is intersecting and the linking property holds for $\mathcal{F}_1^{'}, \mathcal{F}_2, ..., \mathcal{F}_k$.

The first statement is obvious. Suppose that the linking property does not hold. Then there exist $X \in \mathcal{F}_i$ and $Y \in \mathcal{F}_j$ for some i and j that $X_I \cap Y_I \cap \neq \emptyset$ but $X \cap Y \notin \mathcal{F}_i \cap \mathcal{F}_j$. Hence, by the linking property, $X \in \mathcal{F}_1^{'}$, $Y \in \mathcal{F}_j$ for some $(j \neq 1)$ and $X \cap Y \in \mathcal{F}_1 - \mathcal{F}_1^{'}$. But that implies $F \cup X \in \mathcal{F}_1$ contradicting the maximality of F.

We call a bi-set X *tight* if $\varrho(X) = p(X) > 0$ and $X \notin \mathcal{F}_1$. If e does not enter any tight bi-set then we are done by induction. Otherwise let M be a minimal tight bi-set. Then $M - F$ can not be empty otherwise $M \subseteq F$ and so $M \in \mathcal{F}_1$ because of the linking property. But M is tight, thus $M \notin \mathcal{F}_1$, a contradiction. There exists an arc f from $M - F$ to $M \cap F$ because of the linking property and $(*)$. We claim that f does not enter any tight bi-set. Assume that f enters a tight bi-set N. By $(*)$

$$p(M) + p(N) = \varrho(M) + \varrho(N) \geq \varrho(M \cup N) + \varrho(M \cap N) \geq$$

$$\geq p(M \cup N) + p(M \cap N).$$

By Lemma 2.1.1 equality holds everywhere so $M \cap N$ is a tight bi-set contradicting the minimality of M. □

Theorem 2.1.3 actually can be reformulated in terms of *T-intersecting families*. Let $\mathcal{F}$ be a family on the ground-set V and let $T \subseteq V$. We call $\mathcal{F}$ *T-intersecting* if $X, Y \in \mathcal{F},\ X \cap Y \cap T \neq \emptyset \Rightarrow X \cap Y, X \cup Y \in \mathcal{F}$ holds. The inking property can be modified in a similar way.

The reformulated theorem is the following:

Theorem 2.1.4 (Covering T-intersecting families)**.** *Let $D = (V, A)$ be a directed graph and $T \subseteq V$ be a specified subset of V such that each arc has its head in T. Let $\mathcal{F}_1, ..., \mathcal{F}_k$ be T-intersecting families on the ground-set V with the linking property:*

$$X \in \mathcal{F}_i,\ Y \in \mathcal{F}_j,\ X \cap Y \cap T \neq \emptyset \Rightarrow X \cap Y \in \mathcal{F}_i \cap \mathcal{F}_j.$$

There exists a partition $A_1, ..., A_k$ of A such that A_i covers $\mathcal{F}_i$ if and only if $\varrho(U) \geq p(U)$ for each $U \subseteq V$, where $p(U)$ denotes the number of $\mathcal{F}_i$'s containing U.

Now we show the equivalence of Theorem 2.1.3 and Theorem 2.1.4:

Proposition 2.1.1. *The (i) Theorem 2.1.3 and (ii) Theorem 2.1.4 are equivalent.*

Proof. Among the proof we use the notation $p^{'}, \varrho^{'}$ when bi-sets and p, ϱ when sets are studied.

$(i) \Rightarrow (ii)$

Let $\mathcal{F}_1, ..., \mathcal{F}_k$ be T-intersecting families. We define the bi-set families $\mathcal{F}_i^{'} = \{(U, U \cap T) : U \in \mathcal{F}_i\}$. It is easy to see that the bi-set families $\mathcal{F}_1^{'}, ..., \mathcal{F}_k^{'}$ are intersecting and satisfy the linking property. Hence there exists $A_1^{'}, ..., A_k^{'}$ partitioning A such that $A_i^{'}$ covers $\mathcal{F}_i^{'}$ if and only if $p^{'}(X) \leq \varrho^{'}(X)$ for each bi-set X, where $p^{'}(X)$ denotes the number of $\mathcal{F}_i^{'}$'s containing X. If $U \subseteq V$ then $p(U) = p^{'}(\widetilde{U})$ holds where $\widetilde{U}_O = U$ and $\widetilde{U}_I = U \cap T$. Moreover, $\varrho(U) = \varrho^{'}(\widetilde{U})$ since each arc has its head in T. If $A_i^{'}$ covers $\mathcal{F}_i^{'}$ then $A_i = A_i^{'}$ covers $\mathcal{F}_i$, so we are done.

$(ii) \Rightarrow (i)$

Let $\mathcal{F}_1^{'}, ..., \mathcal{F}_k^{'}$ be bi-set families and define $D^{'}$ as follows. We take a copy $v^{'}$ for each $v \in V$. For every $e = uv \in A$ we give $uv^{'}$ to $D^{'}$. Finally we get a bipartite directed graph $D^{'} = (V \cup V^{'}, A^{'})$ with arcs directed from V to $V^{'}$. Take $\mathcal{F}_i = \{X_I^{'} \cup X_O : X \in \mathcal{F}_i^{'}\}$ where $X_I^{'} = \{v^{'} : v \in X_I\}$. Let $T = V^{'}$. Obviously, $\mathcal{F}_i$ is a T-intersecting family for each $i \in \{1, ..., k\}$. Moreover, each $e \in A^{'}$ has its head in T. By (ii), $A^{'}$ can be partitioned into subsets $A_1^{'}, ..., A_k^{'}$ so that $A_i^{'}$ covers $\mathcal{F}_i$ if and only if $p(U) \leq \varrho_{D'}(U)$ holds for all $U \subseteq (V \cup V^{'})$, where $p(U)$ denotes the number of $\mathcal{F}_i$'s containing U. But $p^{'}(\widetilde{U}) = p(U)$ holds for each $\widetilde{U} \in \mathcal{F}_i{'}$ where $U = (\widetilde{U}_O - \widetilde{U}_I) \cup \widetilde{U}_I^{'}$. Moreover, $\varrho^{'}(\widetilde{U}) = \varrho_{D'}(U)$ by the construction of $D^{'}$. If $A_i^{'}$ covers $\mathcal{F}_i$ then $A_i = \{uv \in A : uv^{'} \in A_i^{'}\}$ covers $\mathcal{F}_i^{'}$, so we are done. $\square$

Notes 2.1.1. In Section 1.1.3 we proposed the following problem. Let $D = (V, A)$ be a digraph whose node set is partitioned into a root-set $R = \{r_1, ..., r_t\}$ and a terminal set T. Suppose that no edge of D enters any node of R. Let $m : R \to \mathbb{Z}_+$ be a function and let $k = m(R)$. When can we find k disjoint arborescences in D so that $m(r)$ of them are rooted at r and spanning $T + r$ for each $r \in R$?

Let $p(X) = \sum[m_i : r_i \notin X]$ if $X \cap T \neq \emptyset$, and $p(X) = 0$ in other cases. Let $\mathcal{F}_i^j = \{Z : Z_O \subseteq V - r_i,\ Z_O \cap T \neq \emptyset,\ Z_I = Z_O \cap V\}$ be bi-set families for $i \in \{1, ..., t\}$ and $j \in \{1, ..., m_i\}$. One can easily check that these families are intersecting and satisfy the linking property. It also easy to see that there exist arborescences with the required properties if and only if A can be partitioned into subsets A_i^j for $i \in \{1, ..., t\},\ j \in \{1, ..., m_i\}$) so that A_i^j covers $\mathcal{F}_i^j$. By Theorem 2.1.3, this can be done if and only if $\varrho(Z) \geq p^{'}(Z)$ for each bi-set Z where $p^{'}(Z)$ denotes the number of $\mathcal{F}_i^j$'s containing Z. But the structure of D implies $p^{'}(Z) = p(Z_O)$ and $\varrho(Z) = \varrho(Z_O)$. Hence

$$\varrho(Z) \geq p^{'}(Z) \text{ for every bi-set } Z \Leftrightarrow \varrho(X) \geq p(X) \text{ for every } X \subseteq V,$$

which is exactly the necessary and sufficient condition that Theorem 1.1.4 would give. So this simple but interesting special case can be easily handled with Theorem 2.1.3 -without using atoms or even the Japanese arborescences theorem.

2.1.2 Proof of Japanese arborescences theorem

In the previous section we presented an extension of Szegő's theorem to bi-set families. A conspicuous parallelism between Theorem 2.1.3 and Theorem 1.1.4 is that both results are extensions of Edmonds' disjoint branchings theorem and also both proofs use Lovász's technique [23],[18]. Hence the question naturally emerge: whether there is a common generalization or any connection between them? In the followings we will show that the theorem of Kamiyama, Katoh and Takizawa is actually a consequence of our theorem about bi-set families.

Theorem 2.1.5 (Kamiyama-Katoh-Takizawa, 2008)**.** *Let $D = (V, A)$ be a digraph and a $R = \{r_1, ..., r_k\}$ set of roots. Let S_i denote the set of nodes reachable from r_i. There exist disjoint arborescences $(S_1, A_1), ..., (S_k, A_k)$ rooted at $r_1, ..., r_k$, respectively, if and only if $\varrho(X) \geq p(X)$ for each $X \subseteq V$ where $p(X)$ denotes the number of r_i's for which $r_i \notin X$ and $S_i \cap X \neq \emptyset$.*

Proof. Necessity being trivial, we prove sufficiency. It can be seen easily that $\cup_{i=1}^{k} S_i = V$ can be supposed. We will define proper $\mathcal{F}_1, ..., \mathcal{F}_k$ bi-set families on the ground-set $V \times V$ and show that if a subset of arcs $A_i \subseteq A$ covers $\mathcal{F}_i$, then A_i includes an arborecence F_i rooted at r_i such that F_i spans S_i.

The sets $S_1, ..., S_k$ define a partition of V into *atoms* in which two nodes u and v belong to the same *atom* if there is no S_i with $|\{u, v\} \cap S_i| = 1$. Since $\delta(S_i) = 0$ the atoms arising from S_i can be arranged in a topological order in which there is no edge from an atom to an earlier one. So we can take for each i an order $S_i^1, ..., S_i^{k_i}$ of the atoms arising from S_i for that there is no arc from $S_i^{j_1}$ to $S_i^{j_2}$ if $j_1 > j_2$.

These atoms also could be defined as follows: we call a subset $X \subseteq V$ *separable* if there exists an $i \in \{1, ..., k\}$ such that $X \cap S_i \neq \emptyset$ and $X - S_i \neq \emptyset$. If there is no such i we call X *non-separable.* Then the maximal non-separable subsets are just the atoms.

Let $\mathcal{F}_1, ..., \mathcal{F}_k$ be defined as follows:

$$\mathcal{F}_i = \{(X_O, X_I) : \ \emptyset \neq X_I \subseteq S_i - \{r_i\},\ X_I \text{ is non-separable, } X_O - X_I \subseteq V - S_i\}.$$

It is easy to see that the bi-set families defined above satisfy the conditions of Theorem 2.1.3, namely:

Claim 2.1.1. *The $\mathcal{F}_i$'s are intersecting bi-set families, and also satisfy the linking property.*

So we can apply Theorem 2.1.3 to the bi-set families $\mathcal{F}_1, ..., \mathcal{F}_k$. Hence A can be partitioned into subsets $A_1, ..., A_k$ such that A_i covers $\mathcal{F}_i$ if and only if $p^{'}(Z) \leq \varrho(Z)$ holds for each bi-sets Z where $p^{'}(Z)$ denotes the number of $\mathcal{F}_i$'s containing Z.

Our next observation is the following:

Claim 2.1.2. *If $\varrho(X) \geq p(X)$ for all $X \subseteq V$, then $\varrho(Z) \geq p^{'}(Z)$ also holds for each bi-set Z, where $p(X)$ is the same as in the theorem.*

Proof. Let Z be a bi-set and $\{i_1, ..., i_t\}$ be the set of subscripts such that $Z \in \mathcal{F}_i$. Then by Menger's theorem there is t edge-disjoint directed path from the set $\{r_{i_1}, ..., r_{i_t}\}$ to Z_I. But these paths can not leave the set $S_{i_1} \cup ... \cup S_{i_t}$ because $\varrho(S_i) = 0$ for each i. Hence there are t arcs entering Z_I such that each of them also enters the bi-set Z. □

Hence there exists a proper partition of A. So what remains is to prove that each A_i includes an arborescence rooted at r_i which spans S_i. The next claim ensures this:

Claim 2.1.3. *If $A_i \subseteq A$ covers $\mathcal{F}_i$ then it includes an arborescence F_i rooted at r_i that spans S_i.*

Proof. Suppose to the contrary that there is an $i \in \{1, ..., k\}$ violating the claim. Assume that $i = 1$. Let T_1 denote the set of nodes in S_1 not reachable from r_1 in A_1. Then $\varrho_{A_1}(T_1) = 0$. Let t be the smallest subscript for which $T_1 \cap S_1^t$ is not empty and $Z_I = T_1 \cap S_1^t,\ Z_O = Z_I \cup (V - S_1)$. Then $Z \in \mathcal{F}_1$, but $\varrho_{A_1}(Z) = 0$. Indeed, no arc can enter this bi-set neither from $V - S_1$ since $(V - S_1) \subseteq Z_O$, nor from T_1 because of the topological order. That means that no arc covers Z in A_1, a contradiction. □

□

Since Theorem 2.1.3 generalizes the theorem of Kamiyama, Katoh and Takizawa and it also extends Szegő's theorem, it can be considered as a generalization of all previous theorems about packings. The atomic structure described above also proved to be useful in other applications. With the help of this model we give a new proof of Theorem 1.3.2 and also extend the Japanese arborescences theorem to hypergraphs.

2.2 Positively intersecting supermodular bi-set functions

So far, we dealt with problems concerning covering special families of sets. By way of digression now we turn to the problem of coverings of bi-set functions. For a digraph $D = (V, A)$ and a bi-set Z we define the *door* of Z similarly to the case of traditional sets: $H(Z) = \{v \in Z_I :$ *there is an arc* $uv \in A$ *with* $u \in V - Z_O\}$.

2.2.1 In-degree constraints

Let $g : V \to \mathcal{Z}_+$ be a function on V. We use the notation $\beta_g(Z) = \sum[g(v) :\ v \in H(Z)]$ for each bi-set Z. Frank studied the problem of in-degree constrained coverings of positively intersecting supermodular bi-set functions and proved the following theorem in [13]:

Theorem 2.2.1 (In-degree constraints)**.** *Let $D = (V, A)$ be a digraph and $g : V \to \mathcal{Z}_+$ a function on its node set. Let p be a positively intersecting supermodular bi-set function. There exists an integer-valued function $x : A \to \mathcal{Z}_+$ covering p so that $\varrho_x(v) \leq g(v)$ for every node $v \in V$ if and only if*

$$p(Z) \leq \beta_g(Z)$$

holds for every bi-set Z.

The proof of the theorem is based on the following lemma:

Lemma 2.2.1. *β_g is a fully submodular bi-set function, that is, $\beta_g(X) + \beta_g(Y) \geq \beta_g(X \cap Y) + \beta(X \cup Y)$. If equality holds for bi-sets X and Y, then $g(v) > 0$ and $v \in H(X) \cap H(Y)$ imply $v \in H(X \cup Y)$.*

Notes 2.2.1. A natural question is that what happens if in Theorem 2.2.1 we change the in-degree to out-degree constraints. Surprisingly, we get NP-complete problems even in the special case of sets [13]. Let $D = (V, A)$ be a digraph and $s \in V$. Define p as follows: $p(X) = 1$ for $\emptyset \neq X \subseteq V - s$ and $p(X) = 0$ in other case. Let $g \equiv 1$. If x is an integer vector so that $\delta_x(v) \leq 1$ for each $v \in V$ and $\varrho_x(Z) \geq 1$ for each $\emptyset \neq Z \subseteq V$, then x is the incidence vector of the arc set of a spanning s-arborescence (V, F) in which each node has out-degree at most 1. Hence F is a Hamilton path starting at s. However, deciding the existence of a Hamilton path is NP-complete.

2.2.2 Vidyasankar's theorem revisited

At first Frank formulated Theorem 2.2.1 to positively intersecting set functions and showed that Vidyasankar's theorem is a special case of this simpler variant [13]. As a common application of Frank's result and the atomic structure described in the proof of Theorem 2.1.5 we give a new proof of Theorem 1.3.2:

Theorem 2.2.2. *Let $D = (V, A)$ be a digraph and $\{r_1, ..., r_k\} = R \subseteq V$ a set of specified nodes. Let S_i denote the set of nodes reachable from r_i. There exist arborescences $F_1, ..., F_k$ rooted at $r_1, ..., r_k$ respectively, and covering A if and only if*

$$(i)\ \varrho(v) \leq p(v)$$

for each $v \in V$ and

$$(ii)\ p(U) - \varrho(U) \leq \sum[p(v) - \varrho(v):\ v \in H(U)]$$

for each $\emptyset \neq U \subseteq V$, where $H(U)$ denotes the door of U and $p(U) = |\{i \in \{1, ..., k\}:\ S_i \cap U \neq \emptyset,\ r_i \notin U\}$.

Proof. Necessity as earlier, we prove sufficiency. Let $\mathcal{F}_1, ..., \mathcal{F}_k$ be defined as follows:

$$\mathcal{F}_i = \{(X_O, X_I):\ \emptyset \neq X_I \subseteq S_i - \{r_i\},\ X_I \text{ is non-separable},\ X_O - X_I \subseteq V - S_i\}.$$

We already showed that these bi-set families are intersecting and satisfy the so called linking property. Let $p_2^{'}(Z) = |\{i:\ Z \in \mathcal{F}_i\}|$ for each bi-set Z. Then Lemma 2.1.1 and the submodularity of the in-degree function imply that $p_2(Z) = p_2^{'}(Z) - \varrho(Z)$ is a positively intersecting supermodular bi-set function. Let $g(v) = p(v) - \varrho(v)$ where p is the same as in the theorem.

At first we show that if (i) and (ii) holds then p_2 and g satisfy the inequality in Theorem 2.2.1, and so there exists a proper integer vector x. Let Z be a bi-set. We call $Z_D = Z_O - Z_I$ the *difference-set* of the bi-set. If $Z \notin \mathcal{F}_i$ for each i then the inequality is clearly holds since the left-hand side is 0, and the right-hand side is always non-negative by (i). So assume that $Z \in \mathcal{F}_i$ for some i. The most significant point of the proof is the following observation: taking $Z^{'} = (Z_O^{'}, Z_I^{'})$ instead of Z, where $Z_I^{'} = Z_I$ and

$$Z_D^{'} = \bigcup(S_i:\ Z_I \subseteq S_i,\ S_i \cap Z_D \neq \emptyset) - \big(\bigcup(S_i:\ Z_I \not\subseteq S_i) \cup \bigcup(S_i:\ Z_I \subseteq S_i,\ Z_D \cap S_i = \emptyset)\big),$$

would not decrease the left-hand side and would not increase the right-hand side of the inequality. Hence if it holds for Z' then it also holds for Z.

So we can assume that Z has a structure described above. Let $\overrightarrow{d}(X,Y)$ denote the number of arcs going from X to Y. The special structure of Z implies the followings:

$$p_2'(Z) = p(Z_I) - p(Z_D) - |\{i:\ r_i \in Z_D\}|,$$

$$p(Z_I) = p(Z_I \cup Z_D) + |\{i:\ r_i \in Z_D\}|,$$

$$\varrho(Z) = \varrho(Z_I) - \overrightarrow{d}(Z_D, Z_I).$$

Furthermore:

$$\varrho(Z_D) = 0,$$

from what

$$H(Z_D) = \emptyset$$

clearly holds, so

$$H(Z_I \cup Z_D) = H(Z),$$

and

$$\overrightarrow{d}(Z_D, Z_I) = \varrho(Z_I) + \varrho(Z_D) - \varrho(Z_I \cup Z_D).$$

By these observations and (ii):

$$p_2(Z) = p_2'(Z) - \varrho(Z) = p(Z_I) - p(Z_D) - |\{i:\ r_i \in Z_D\}| - \varrho(Z_I) + \overrightarrow{d}(Z_D, Z_I) =$$

$$= p(Z_I \cup Z_D) + |\{i:\ r_i \in Z_D\}| - p(Z_D) - |\{i:\ r_i \in Z_D\}| + \varrho(Z_D) - \varrho(Z_I \cup Z_D) \leq$$

$$\leq \sum[p(v) - \varrho(v):\ v \in H(Z_I \cup Z_D)] - p(Z_D) \leq \sum[p(v) - \varrho(v):\ v \in H(Z)],$$

the required inequality holds. Hence Theorem 2.2.1 implies that there exists an integer vector so that $\varrho_x(Z) \geq p_2(Z)$ for every bi-set Z and $\varrho_x(v) = p(v) - \varrho(v)$ for each $v \in V$. Extend D with $x(e)$ copies of each $e \in A$ and let D^+ denote the digraph arising. In D^+ each node v has in-degree exactly $p(v)$ and every bi-set Z has in-degree at least $p_2(Z)$. Thus, by Theorem 1.1.4, we can choose $F_1, ..., F_k$ arc-disjoint arborescences in D^+ so that F_i is rooted at r_i and spans S_i. Furthermore, these arborescences give a partition of the arc-set of D^+ because of $\varrho_{D^+}(v) = p(v)$. So the corresponding arborescences in D are suffice for the purpose. $\square$

2.3 Hypergraphs

One way to extend the notion of graphs is the following. Let V be a set of nodes. The pair $H = (V, \mathcal{E})$ is called a *hypergraph* if $\mathcal{E}$ is a family of subsets of V. The members of $\mathcal{E}$ are called *hyperedges*. Note that each hyperedge may occur in more than one copy. When each hyperedge contains two nodes we are back at undirected graphs. The *degree* $d_H(X)$ of a bi-set X is the number of hyperedges intersecting both X_I and $V - X_O$. One can easily check that d_H is submodular on bi-sets:

$$d_H(X) + d_H(Y) \geq d_H(X \cap Y) + d_H(X \cup Y)$$

for each bi-sets X and Y.

Digraphs can also be extended to hypergraphs in various ways. We present one which allows us to generalize the previous results to hypergraphs. Let V be the set of nodes again. By a *dyperedge* we mean a pair (X, v) where X is a subset of V and $v \in X$. We call v the *head* of X. A *dypergraph* is a pair $D = (V, \mathcal{D})$ where $\mathcal{D}$ is a set of dyperedges. We say that a dyperedge (X, v) *enters* a bi-set Z if $v \in Z_I$ and $X - Z_O \neq \emptyset$; and leaves a bi-set Z if $v \in V - Z_O$ and $X \cap Z_I \neq \emptyset$. Hence the in-degree and out-degree functions of a dypergraph can be defined easily. By easy case checking we get that the in-degree function on bi-sets is submodular. Moreover:

Lemma 2.3.1.

$$\varrho_D(X) + \varrho_D(Y) = \varrho_D(X \cap Y) + \varrho_D(X \cup Y) + d_D(X, Y),$$

where $d_D(X, Y)$ denotes the number of dyperedges (Z, v) for which

$$v \in X_I \cup Y_I,\ Z \subseteq X_O \cup Y_O,\ (X_O - Y_O) \cap Z \neq \emptyset,\ (Y_O - X_O) \cap Z \neq \emptyset.$$

2.3.1 Trimming dyperedges

The following extension of Edmonds' weak theorem was noticed in [9]:

Theorem 2.3.1. *Suppose every dyperedge of the dypergraph $D = (V, \mathcal{D})$ has at least two elements. Let r be a given root node. Then D can be decomposed into k disjoint spanning rooted k-edge-connected dypergraphs if and only if*

$$(i)\ \varrho_D(X) \geq k$$

for every non-empty $X \subseteq V - r$.

Proof. Necessity being trivial, we prove sufficiency by induction on $\sum[|Z| - 2 :\ Z \in \mathcal{D}]$. If the sum is zero, then the dypergraph is actually a digraph and by Theorem 1.1.1 we are done. Suppose that there exists a dyperedge (Z, z) with $|Z| \geq 3$. Let u and v be two elements of $Z - z$. We call a set $X \subseteq V - r$ *tight* if $\varrho_D(X) = k$. If after replacing Z by $Z - u$ inequality (i) still holds then we are done by induction. In other case there is a tight set X for which $u \notin X$ and $Z - u \subseteq X$. Similarly for v, we may assume that there is a tight set Y for which $v \notin Y$ and $Z - v \subseteq Y$.

But then the hyperedge Z shows that $d_D(X,Y) \geq 1$, and hence by Lemma 2.3.1 and (i): $k+k = \varrho_D(X) + \varrho_D(Y) = \varrho_D(X \cap Y) + \varrho_D(X \cup Y) + d_D(X,Y) \geq k+k+1$, contradiction. $\square$

So the problem can be traced back to Edmonds' theorem by trimming the hyperedges one by one until we get a directed graph. It is remarkable that the same approach works in the case of Edmonds' strong theorem. We use this technique to extend Theorem 2.1.3 -and so Theorem 1.1.4- to dypergraphs.

2.3.2 Covering bi-set families by dypergraphs

At first sight one could think that the Japanese arborescences theorem could be extended to dypergraphs in the same way as in the previous section -without using bi-sets. That approach in fact would not work because the function p appearing in Theorem 1.1.4 is not supermodular at all. However, using bi-sets will solve this problem. Firstly we prove the extension of Szegő's theorem to dypergraphs:

Theorem 2.3.2 (Covering bi-set families by dypergraphs)**.** *Suppose every dyperedge of the dypergraph $D=(V,\mathcal{D})$ has at least two elements. Let $\mathcal{F}_1,...,\mathcal{F}_k$ be intersecting bi-set families with the linking property. Then $\mathcal{D}$ can be partitioned into subsets $\mathcal{D}_1,...,\mathcal{D}_k$ such that $\mathcal{D}_i$ covers $\mathcal{F}_i$ if and only if (i) $\varrho_D(X) \geq p(X)$ for each bi-set X where $p(X)$ denotes the number of $\mathcal{F}_i$'s containing X.*

Proof. Necessity can be seen easily. We prove sufficiency by induction on $\sum[|Z|-2: \ Z \in \mathcal{D}]$. If the sum is zero, then the dypergraph is actually a digraph and by Theorem 2.1.2 we are done. Suppose that there exists a dyperedge (Z,z) with $|Z| \geq 3$. Let u and v be two elements of $Z-z$. We call a bi-set X *tight* if $\varrho_D(X) = p(X) > 0$. If after replacing Z by $Z-u$ inequality (i) still holds then we are done by induction. In other case there is a tight bi-set X for which $u \notin X_O, \ Z-u \subseteq X_O, \ z \in X_I$. Similarly for v, we may assume that there is a tight bi-set Y for which $v \notin Y_O, \ Z-v \subseteq Y_O, \ z \in Y_I$.

But then the hyperedge Z shows that $d_D(X,Y) \geq 1$. We already saw that p is a positively intersecting supermodular bi-set function. By this, Lemma 2.3.1 and (i): $p(X)+p(Y) = \varrho_D(X) + \varrho_D(Y) = \varrho_D(X \cap Y) + \varrho_D(X \cup Y) + d_D(X,Y) \geq p(X \cap Y) + p(X \cup Y) + 1$, a contradiction. $\square$

We only mention here the corresponding version of Theorem [18]:

Theorem 2.3.3. *Suppose every dyperedge of the dypergraph $D=(V,\mathcal{D})$ has at least two elements. Let $R=\{r_1,...,r_k\}$ be a set of roots and S_i the set of nodes reachable from r_i. Then the dypergraph D includes disjoint dypergraphs $D_1=(S_1,\mathcal{D}_1),...,D_k=(S_k,\mathcal{D}_k)$ such that D_i is rooted-connected with root r_i and spans S_i if and only if $\varrho_D(X) \geq p(X)$ for each $X \subseteq V$ where $p(X)$ denotes the number of r_i's for which $r_i \notin X$ and $S_i \cap X \neq \emptyset$.*

The theorem can be proved easily by using the trimming method and the positively intersecting supermodular function p defined by the bi-set families as seen in 2.1.5.

Chapter 3

Bibranchings

Let $D = (V, A)$ be a directed graph and let V be partitioned into classes R and S. An $R - S$ *bibranching* is a set B of arcs such that in the graph (V, B), each node in S is reachable from R, and each node in R reaches S. The concept of bibranching can be considered as a generalization of branchings, and give rise to similar min-max relations and polyhedral characterizations.

3.1 Schrijvers's disjoint bibranchings theorem

In [25] Schrijver gave a min-max characterization of the number of $R - S$ bibranchings in a directed graph $D = (V, A)$, where $\{R, S\}$ is a partition of V. We call a set of arcs C to be an $R - S$ *bicut* if $C = \Delta^+(U)$ for some nonempty proper subset U of V satisfying $U \subseteq S$ or $S \subseteq U$. Schrijver showed the following:

Theorem 3.1.1 (Schrijver's disjoint bibranchings theorem)**.** *Let $D = (V, A)$ be a directed graph and let V be partitioned into sets R and S. Then the maximum number of disjoint $R-S$ bibranchings is equal to the minimumsize of an $R - S$ bicut.*

Let $D = (V + r, A)$ be a digraph and let $R = r$ and $S = V$. Then Theorem 3.1.1 implies Edmonds' weak theorem. Schrijver gave two proof of his result: one of them is based on an exchange property of branchings, the other uses Edmonds' disjoint branchings theorem and a coloring-type theorem of Schrijver [26] on supermodular functions. Our extension is based on the second approach.

Schrijver proved the following theorem:

Theorem 3.1.2 (Supermodular colorings)**.** *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be intersecting families on the groundset S, let $g_1 : \mathcal{C}_1 \to \mathbb{Z}$ and $g_2 : \mathcal{C}_2 \to \mathbb{Z}$ be intersecting supermodular, and let $k \in \mathbb{Z}_+$ with $k \geq 1$. Then S can be partitioned into classes $L_1, ..., L_k$ such that*

$$g_i(U) \leq |\{j \in \{1, ..., k\} : \ L_j \cap U \neq \emptyset\}|$$

for each $i = 1, 2$ *and each* $U \in \mathcal{C}_i$ *if and only if*

$$g_i(U) \leq \min\{k, |U|\}$$

for each $i = 1, 2$ *and each* $U \in \mathcal{C}_i$*.*

This theorem also implies edge-coloring theorems for bipartite graphs, such as Kőnig's theorem. Let $G = (V_1, V_2; E)$ be a bipartite graph, and let $\mathcal{C}_i = \{\Delta^+(v) : v \in V_i\}$ for $i = 1, 2$. By choosing $g_i(\Delta^+(v)) = |\Delta^+(v)|$, Theorem 3.1.2 reduces to Kőnig's theorem.

3.2 Covering by bibranchings

First of all, we need a lemma which is an easy consequence of Theorem 2.1.2:

Lemma 3.2.1. *Let* $D = (V + r, A)$ *be a directed graph and* $\mathcal{F}_1, ..., \mathcal{F}_k$ *be intersecting families on the ground-set* V*. Assume that* $A_1, ..., A_k$ *is a subpartition of* $\Delta^+(r)$*. Then* $A_1, ..., A_k$ *can be extended to a partition of* A *such that* A_i *covers* $\mathcal{F}_i$ *if and only if* $p^{'}(U) \leq \varrho^{'}(U)$*, where*

$$p^{'}(U) = |\{i \in \{1, ..., k\} : \; U \in \mathcal{F}_i, \; A_i \textit{ does not cover } U\}|,$$

and $\varrho^{'}(U)$ *denotes the in-degree of* U *in* $A - \cup_i A_i$*.*

Proof. Let $\mathcal{F}_i^{'} = \{U \in \mathcal{F}_i : \; A_i \textit{ does not cover } U\}$. It is easy to see that $\mathcal{F}_1^{'}, ..., \mathcal{F}_k^{'}$ are intersecting families. The linking property also holds, since $A_1, ..., A_k$ only contain arcs from $\Delta^+(r)$, and there is no set $F \in \cup_i \mathcal{F}_i$ that contains r. So Szegő's theorem, on the families $\mathcal{F}_1^{'}, ..., \mathcal{F}_k^{'}$, implies the theorem. □

Earlier we observed that Edmonds' disjoint branchings theorem is based on the problem of covering intersecting families. This observation suggests the followings. Let $\mathcal{F}$ be a family on the ground-set S, and let $\mathcal{F}^{'}$ be a family on the ground-set R. A subset B of arcs *covers* $\mathcal{F} \cup \mathcal{F}^{'}$ if for each $X \in \mathcal{F}$ there is an arc $e \in B$ entering X, and for each $Y \in \mathcal{F}^{'}$ there is an arc $e \in B$ leaving Y. By mixing Szegő's theorem and supermodular colorings we get:

Theorem 3.2.1. *Let* $D = (V, A)$ *be a digraph and let* V *be partitioned into sets* R *and* S*. Let* $\mathcal{F}$ *and* $\mathcal{F}^{'}$ *be intersecting set-systems on the ground-sets* S *and* R*, respectively. Then* A *can be partitioned into sets* $A_1, ..., A_k$ *such that* A_i *covers* $\mathcal{F} \cup \mathcal{F}^{'}$ *if and only if*

$$\varrho(X) \geq k$$

for each $X \in \mathcal{F}$ *and*

$$\delta(Y) \geq k$$

for each $Y \in \mathcal{F}^{'}$*.*

Proof. Necessity being trivial, we prove sufficiency. Let $H = \Delta^+(R)$, and define the following collections of subsets of H:

$$\mathcal{C}_1 = \{\Delta^-_H(U) : U \in \mathcal{F}\},\ \mathcal{C}_2 = \{\Delta^+_H(U) : U \in \mathcal{F}^{'}\}.$$

From the intersecting property of $\mathcal{F}$ and $\mathcal{F}^{'}$ follows that $\mathcal{C}_1$ and $\mathcal{C}_2$ are intersecting families on H. Let $g_i : \mathcal{C}_i \to \mathbb{Z}$ for $i \in \{1, 2\}$ be defined as follows:

$$g_1(B) = \max\{k - \varrho_{A[S]}(U) : U \in \mathcal{F},\ B = \Delta^-_H(U)\}$$

for $B \in \mathcal{C}_1$ and

$$g_2(B) = \max\{k - \delta_{A[R]}(U) : U \in \mathcal{F}^{'},\ B = \Delta^+_H(U)\}$$

for $B \in \mathcal{C}_2$.

The submodularity of the in-degree and out-degree functions implies that g_1 and g_2 are intersecting supermodular functions. If $g_1(B) = k - \varrho_{A[S]}(U)$ for some $U \in \mathcal{F}$ then

$$g_1(B) = k - \varrho_{A[S]}(U) \leq \varrho_A(U) - \varrho_{A[S]}(U) = \varrho_H(U) = |B|.$$

Similarly, if $g_2(B) = k - \delta_{A[R]}(U)$ for some $U \in \mathcal{F}^{'}$ then

$$g_2(B) = k - \delta_{A[R]}(U) \leq \delta_A(U) - \delta_{A[R]}(U) = \delta_H(U) = |B|.$$

Moreover, $g_i(B) \leq k$ for $i = 1, 2$ and $B \in \mathcal{C}_i$, so we can use Schrijver's supermodular colorings theorem. Then, by Theorem 3.1.2, H can be partitioned into subsets $H_1, ..., H_k$ such that

$$U \in \mathcal{F} \Rightarrow U \text{ is entered by at least } k - \varrho_{A[S]}(U) \text{ of the classes } H_i,$$

and if

$$U \in \mathcal{F}^{'} \Rightarrow U \text{ is left by at least } k - \delta_{A[R]}(U) \text{ of the classes } H_i.$$

Lemma 3.2.1 implies that $A[S]$ can be partitioned into subsets $B_1, ..., B_k$ such that B_i covers all $F \in \mathcal{F}$ not covered by H_i. Similarly, $A[R]$ can be partitioned into subsets $B^{'}_1, ..., B^{'}_k$ such that $B^{'}_i$ covers all $F \in \mathcal{F}^{'}$ not covered by H_i. Hence each $F \in \mathcal{F}$ is entered by at least one arc in $B_i \cup H_i$ and each $F \in \mathcal{F}^{'}$ is left by at least one arc in $B^{'}_i \cup H_i$. Then $A_i = B_i \cup H_i \cup B^{'}_i$ for $i \in \{1, ..., k\}$ is a partition of A into subsets such that A_i covers $\mathcal{F} \cup \mathcal{F}^{'}$, and we are done. □

3.3 Latin squares

In some ways Schrijver's disjoint bibranchings theorem can be considered as the extension of Edmonds' weak theorem to bibranchings. Hence it is a natural idea to study Edmonds' strong theorem, whether it can be extended in a similar way. Such an extension should answer the question, that given a digraph $D = (V, A)$, a partition $\{R, S\}$ of V, and subsets $A_1, ..., A_k$ of A then when can we extend these arc-sets to disjoint $R - S$ bibranchings in D. We will show that this problem is NP-hard, even in the special case of bipartite graphs.

3.3.1 Edge-colorings

Let $D = (V_1, V_2; A)$ be a complete bipartite graph with edges directed from V_1 to V_2 and $|V_1| = |V_2| = n$. Hence a $V_1 - V_2$ bibranching is a set B of arcs such that $\max\{\varrho_B(v), \delta_B(v)\} > 0$ for each $v \in V_1 \cup V_2$, which is equivalent with $d_B(v) > 0$ for each v in the underlying graph. Hence we can work with a bipartite graph $G = (V_1, V_2; E)$ and call a subset B of edges a $V_1 - V_2$ bibranching if $d_B(v) > 0$ holds for each node.

Assume now that for each i a subset $A_i \subseteq E$ is already given (where $A_i = \emptyset$ is also allowed), and we would like to extend these edge-sets to disjoint $V_1 - V_2$ bibranchings. Obviously, a sufficient condition is that each A_i must be a matching. So the problem can be reduced to the following: given a complete bipartite graph $K_{n,n}$ with a partial edge coloring with n colors, and we have to color the remaining edges as to get an n-edge coloring of $K_{n,n}$. In this form the problem is NP-hard. However, there are some special cases when we can still answer the question.

3.3.2 Evans' conjecture

The example above showes that extending Edmonds' strong theorem to bibranchings is hopeless. The previous problem can be reformulated in terms of *latin squares*. A *partial latin square of side* n is an $n \times n$ matrix in which each cell is empty or filled with one of $\{1, ..., n\}$, and no number occurs twice in any row or column. It is a *(complete) latin square* if all cells are filled. In 1960 Evans conjectured that a partial latin square of order n with at most $n - 1$ cells occupied can be completed [4]. The conjecture was independently confirmed by Häggkvist [15] for $n \geq 1111$, by Smetaniuk [27] for all n, and by Andersen and Hilton [1] for all n. In fact Andersen and Hilton proved the stronger statement that n cells can be preassigned except in certain cases which can be specified:

Theorem 3.3.1 (Andersen and Hilton, 1983)**.** *A partial edge coloring φ of at most n edges of $K_{n,n}$ can be extended to an n-edge coloring of $K_{n,n}$ except the following two cases:*

(a) *For some uncolored edge xy there are n colored edges of different coors, each one incident with x or y.*

(b) *For some node x and some color i, the color i is not incident to x, but it is incident to all vertices y for which xy is uncolored.*

It follows from the foregoing that we can answer the proposed question about bipartite graphs when $\sum_{i=1}^{n} |A_i| \leq n$.

Chapter 4

Directed cuts and matroid intersection

Let $D = (V, A)$ be a directed graph. We call a subset C of arcs a *directed cut* if $C = \Delta^-(U)$ for some nonempty proper subset U of V such that $\Delta^+(U) = \emptyset$. A *dijoin* is a set of arcs intersecting each directed cut. It is easy to see that contracting the arcs of a dijoin makes the graph strongly connected.

Lucchesi and Younger showed [21] that the minimum size of a dijoin is equal to the maximum number of disjoint directed cuts. This theorem is a central result of the theory of dijoins and has been originally conjectured by Robertson and by Younger. Lucchesi, Karzanov, and Frank showed that a minimum-sized dijoin and a maximum packing of directed cuts can be found in polynomial time. Later Frank gave a strongly polynomial-time algorithm for finding a minimum-size dijoin [7]. In [10] Frank and Tardos reduced the problem of finding a minimum-length directed cut k-cover to weighted matroid intersection and so a strongly polynomial-time algorithm were at hand. But the reduction of the problem to matroid intersection did not give back the theorem itself, because the meaning of the dual solution was not read out.

In the followings we show how the meaning of the dual solution can be read out and so give a new proof for the theorem using matroid intersection.

4.1 Finding dijoin algorithmically

Let $D = (V, A)$ be a directed graph. The problem of finding a dijoin can be solved with the matroid intersection algorithm as follows [11]. We put two new nodes on each arc $e = uv \in A$: let e_v be the *head-node* and e_u be the *tail-node* of the arc. The set of new nodes will be denoted by S. Let $\mathcal{P} = \{Z \subseteq V : \ \Delta^+(Z) = \emptyset\}$ and for all $Z \in \mathcal{P}$ let

$$F(Z) = \{e_v : \ e = uv \in A, \ v \in Z\} \cup \{e_u : \ e = uv \in A, \ u \in Z\}.$$

Then the family $\mathcal{F} = \{F(Z) : Z \in \mathcal{P}\}$ is crossing. We define the function $p : \mathcal{F} \to \mathbb{Z}$ as follows:

$$p(\emptyset) = 0,\ p(S) = |A|$$

and

$$p(X) = i(Z) + 1$$

if $X = F(Z)$ for some $Z \in \mathcal{P}$ such that $Z \neq \emptyset, S$. The p we get is a crossing-supermodular function on a crossing family. Hence the family

$$\mathcal{B} = \{B \subseteq S :\ |B| = p(S),\ |B \cap X| \geq p(X)\ \forall X \in \mathcal{F}\}$$

is the collection of the bases of a matroid $\mathcal{M}_1$. We also know that $\mathcal{B} \neq \emptyset$ since it contains the set of the head-nodes. Let $\mathcal{M}_2$ be the partition-matroid on the ground-set S in which $F \in \mathcal{M}_2$ is independent if and only if $F \cap \{e_v, e_u\} \leq 1$ for each $e = uv \in A$.

Clearly, if $C \subseteq A$ is a cut cover then the head-nodes of C and the tail-nodes of $A - C$ form a common base of the two matroids, and conversely, if B is a common base of $\mathcal{M}_1$ and $\mathcal{M}_2$ then the arcs with head-nodes in B form a cut cover. If we put weights 1 on the head-nodes and weights 0 on the tail-nodes, then the weighted matroid intersection algorithm finds a dijoin with minimal weight.

4.2 Gröflin-Hoffman theorem

The bases of a matroid can be defined by intersecting submodular functions, but in applications we often meet supermodular functions. However, a proper intersecting supermodular function defines the generators of a matroid as follows (see in [11]).

Let $p : 2^S \to \mathbb{Z} \cup \{-\infty\}$ be an intersecting supermodular function such that $p(\emptyset) = 0$ and $p(X) \leq |X|$ for each $X \subseteq S$. Let

$$\mathcal{G}^p = \{Z \subseteq S :\ |Z \cap X| \geq p(X)\ \forall X \subseteq S\}.$$

By $p(S) \leq |S|$ we get $\mathcal{G}^p \neq \emptyset$ as $S \in \mathcal{G}^p$. The following theorem holds:

Theorem 4.2.1. *$\mathcal{G}^p$ forms the generator-system of a matroid $\mathcal{M}^p$. The co-rank of the matroid is*

$$\max\Big\{\sum_i p(X_i) :\ \{X_1, ..., X_k\} \textit{ is a subpartition of } S\Big\}.$$

The co-rank function of the matroid is also determined by p:

Theorem 4.2.2. *The co-rank function of the matroid $\mathcal{M}^p$ is*

$$t^p(Z) = \max\Big\{\sum_i p(X_i) - |\cup_i X_i - Z| :\ \{X_1, ..., X_k\} \textit{ is a subpartition of } S\Big\}.$$

Gröflin and Hoffman gave the following interesting application of the weighted matroid intersection theorem in [16]. This theorem will be the base of our proof:

Theorem 4.2.3 (Gröflin and Hoffman, 1981)**.** *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be matroids on the ground-set S having a common base. Then for each $R \subseteq S$:*

$$\min\big\{|R \cap B| :\ B \textit{ is a common base}\big\} = \max\Big\{\sum_{i=1}^{t}[k - r_{12}(S - R_i)]\Big\},$$

where the maximum is taken over all partitions $\{R_1, ..., R_t\}$ of R, $r_{12}(T)$ denotes the cardinality of the maximal common independent set lying in T, and k is the rank of the matroids.

By Edmonds' matroid intersection theorem

$$r_{12}(T) = \min_{X \subseteq T}\{r_1(X) + r_2(T - X)\}.$$

So the previous equality can be reformulated as follows:

$$\min\big\{|R \cap B| :\ B \textit{ is a common base}\big\} = \max\Big\{\sum_{i=1}^{t}[k - (r_1(X_i) + r_2(Y_i))]\Big\},$$

where $\{R_1, ..., R_t\}$ is a partition of R, and $\{R_i, X_i, Y_i\}$ is a partition of S for each $i \in \{1, ..., t\}$.

4.3 Lucchesi-Younger theorem

Now we turn to the proof of the Lucchesi-Younger theorem:

Theorem 4.3.1 (Lucchesi-Younger, 1978)**.** *Let $D = (V, A)$ be a digraph. The minimum size of a dijoin is equal to the maximum number of disjoint directed cuts.*

Proof. We use the matroids described in Section 4.1, so we put two new nodes e_u and e_v on each $e = uv \in A$ and denote the set of new nodes by S. Let $\mathcal{P} = \{Z \subseteq V :\ \Delta^+(Z) = \emptyset\}$ and for all $Z \in \mathcal{P}$ let

$$F(Z) = \{e_v :\ e = uv \in A,\ v \in Z\} \cup \{e_u :\ e = uv \in A,\ u \in Z\}.$$

We define $\mathcal{M}_1$ as a partition-matroid on S in which $F \subseteq S$ is independent if and only if $F \cap \{e_v, e_u\} \leq 1$ for each $e = uv \in A$. The function p will be defined in a slightly different way than previously. Let $p(\emptyset) = 0$ and $p(X) = i(Z) + \sigma(Z)$ if $X = F(Z)$ for some $\emptyset \neq Z \in \mathcal{P}$, where $\sigma(Z)$ denotes the number of components in $D - Z$. Hence $p(S) = |A|$. Furthermore, p is an intersecting supermodular function on an intersecting set-system. We extend p to 2^S as follows: if $p(X)$ is not yet defined for some $X \subseteq S$, let $p(X) = -\infty$. Then p is intersecting supermodular on 2^S. By Theorem 4.2.1, p determines the generator-set of a matroid $\mathcal{M}_2$.

From now R denotes the set of the head-nodes in S. It is easy to see that applying the Gröflin-Hoffman theorem to $\mathcal{M}_1$ and $\mathcal{M}_2$ the minimum on the left-hand side is exactly the minimum size of a dijoin. Our aim is to show that the maximum on the right-hand side is at most the maximum number of disjoint directed cuts which would prove the Theorem since the direction min $\geq$ max is clear.

Our first observation is the following:

Claim 4.3.1. *For each $e = uv \in A$ such that $e_v \notin R_i$ we can assume that both $e_u, e_v \in X_i$ or both $e_u, e_v \notin X_i$.*

Proof. Suppose that

$$\sum_{i=1}^{t}[k - (r_1(X_i) + r_2(Y_i))] = \sum_{i=1}^{t}[|A| - r_1(X_i) - r_2(Y_i)]$$

attains the maximum on the sets $R_1, ..., R_t;\ X_1, ..., X_t;\ Y_1, ..., Y_t$, where $\{R_1, ..., R_t\}$ is a partition of R and $\{R_i, X_i, Y_i\}$ is a partition of S for each i.

Assume that $e_v \in X_i$ but $e_u \notin X_i$ for some $e = uv \in A$. Giving e_u to X_i would not increase $r_1(X_i)$ while $r_2(Y_i)$ could only decrease. Hence the sum would surely not decrease. The other case when $e_u \in X_i$ but $e_v \notin X_i$ can be handled similarly. □

By Claim 4.3.1, we only have to study the cases when the X_i's have this important property. With the help of this observation the maximum on the right hand side can be expressed as follows:

$$\max\{\sum_{i=1}^{t}[|A| - r_1(X_i) - r_2(Y_i)]\} =$$

$$= \max\{\sum_{i=1}^{t}[|A| - \frac{|X_i|}{2} - (|A| - t_2(S - Y_i)]\} =$$

$$= \max\{\sum_{i=1}^{t} t_2(R_i \cup X_i) - \frac{|X_i|}{2}\}.$$

We exchange the co-rank function to the formula given by Theorem 4.2.2 and we get:

$$\max\{\sum_{i=1}^{t}[[\sum_{j=1}^{k_i} p(Z_j^i) - |(\cup_j Z_j^i) - (R_i \cup X_i)|\ ] - \frac{|X_i|}{2}]\},$$

where the maximum is taken over all partitions $\{R_1, ..., R_t\}$ of R, $X_i \subseteq S - R_i$ with the structure described in Claim 4.3.1 and all subpartitions $\{Z_1^i, ..., Z_{k_i}^i\}$ of S.

From now we scrutinize this expression. We say that a directed cut $C \subseteq A$ *enters* a set $X \subseteq S$ if $e_v \in X$ for each $e \in C$.

Claim 4.3.2. *If R_i is fixed then the maximum of the sum*

$$[[\sum_{j=1}^{k_i} p(Z_j^i) - |(\cup_j Z_j^i) - (R_i \cup X_i)|\ ] - \frac{|X_i|}{2}]$$

is at most the maximum number of disjoint directed cuts entering R_i.

Proof. Assume that X_i and $\{Z_j^i\}_{j=1}^{k_i}$ attains the maximum. Obviously, $p(Z_j^i) \neq -\infty$ since in other case the sum is $-\infty$. Hence, by the definition of p, $Z_j^i = F(H_{Z_j^i})$ for some $H_{Z_j^i} \subseteq V$ such that $\delta(H_{Z_j^i}) = 0$, which also means that $p(Z_j^i) = i(H_{Z_j^i}) + \sigma(H_{Z_j^i})$.

Now we show that there is no edge $e = uv \in I(H_{Z_j^i})$ such that $e_u, e_v \notin R_i \cup X_i$. In other case we could strictly increase the sum by giving e_u and e_v to X_i , a contradiction. The directed cut $\Delta^-(H_{Z_j^i})$ can be partitioned into $\sigma(H_{Z_j^i})$ disjoint directed cuts. Let $\mathcal{C}_j^i$ denote the set of cuts from these not entering R_i.

Let $e_1, ..., e_m$ be an order of the arcs such that $\{e = uv \in A: \ e_u, e_v \in X_i\} = \{e_1, ..., e_{\frac{|X_i|}{2}}\}$ and $\{e = uv \in A: \ e_v \in R_i\} = \{e_{m-|R_i|+1}, ..., e_m\}$. We assign the node $e_u \in \cup_j(Z_j^i) - R_i$ to each edge $uv = e \in I(Z_j^i)$. Moreover, we assign the node $e_v \in \cup_j(Z_j^i) - R_i$ to each $C \in \mathcal{C}_j^i$ where e is the first arc in the order for which $e \in C$ (we do these for all j). The main observation of the proof is the following: as $\{Z_j^i\}_{j=1}^{k_i}$ is a subpartition, this assignment is an injection which means that we use each node from S at most once. Moreover, for each $e = uv$ with $e_u, e_v \in X_i$ at most one of e_u and e_v is used. But the existence of this injection means that

$$[[\sum_{j=1}^{k_i} p(Z_j^i) - |(\cup_j Z_j^i) - (R_i \cup X_i)| \,] - \frac{|X_i|}{2}] =$$

$$= [[\sum_{j=1}^{k_i} [i(H_{Z_j^i}) + \sigma(H_{Z_j^i})] - |(\cup_j Z_j^i) - (R_i \cup X_i)| \,] - \frac{|X_i|}{2}] \leq$$

$$\leq \sum_{j=1}^{k_i} [\sigma(H_{Z_j^i}) - |\mathcal{C}_j^i|].$$

The right-hand side of this inequality is clearly lower or equal to the maximum number of disjoint directed cuts entering R_i, which proves the Claim. □

Claim 4.3.2 means that we can take the partition $R_1, ..., R_k$ of R anyhow, the maximum on the right-hand side of the Gröflin-Hoffman theorem will be never larger than the maximum number of disjoint directed cuts such that each of them enters an R_i. Hence the maximum is clearly at most the maximum number of disjoint directed cuts, and we are done. □

Conclusions

We close this work with some questions more or less related to this area. Some of them are well known open problems, others came up in connection with this work and just show new horizons that maybe could be interesting. We would like to examine these questions in the near future.

The theory of packing and covering seemed to be closed for a long time. The Japanese arborescences theorem infused life into this area and raised several questions. The extension of Szegő's theorem is a beautiful application of bi-sets and also proves that this structure hides great potentialities. As it plays an important role in node-connectivity augmentations, an aim is to find other problems where this technique can be used successfully. Such a question offers itself from the generalization of Szegő's theorem. This extension shows that covering bi-set families satisfying the intersecting- and linking properties is equal to covering a special positively intersecting supermodular bi-set function. What is the necessary and sufficient condition of covering an arbitrary supermodular bi-set function? If the graph is undirected, when can it be oriented as to cover a fixed supermodular bi-set function?

Another question arises when we would like to reformulate the Japanese theorem to undirected graphs. Let $G = (V, E)$ be an undirected graph and let $R_1, ..., R_k$ be subsets of V. When can we find disjoint trees $F_1, ..., F_k$ such that F_i spans R_i? However, this question can be answered using matroids: let $\mathcal{M}_i$ be the matroid on the ground set E within a set $E^{'}$ of edges is independent if and only if $E^{'}$ is a forest and $E^{'} \subseteq I(R_i)$. Then the requested arborescences exist if and only if the sum of these matroids has rank $\sum_{i=1}^{k}[|R_i| - 1]$. Hence we can answer the question but the solution is really complicated. It would be interesting to find a simpler characterization.

Kriesell proposed the following conjecture in [9]. Let $G = (V, E)$ be a graph and $T \subseteq V$. Is it true that if G is $2k$-edge-connected in T, then G contains k edge-disjoint Steiner trees for T? If G is $3k$-edge-connected and $V - T$ is stable then the conjecture holds. What can we say in other cases?

Chapter 4 deals with dijoins. The Lucchesi-Younger theorem also has an algorithmic proof from which the structure of disjoint directed cuts can be read out. It seems to us that this can also be done from our approach, since the sum we get by combining the Gröflin-Hoffman theorem with Edmonds' matroid intersection tells a lot about this structure.

Bibliography

[1] L.D. Andersen, A.J.W. Hilton, *Thank Evans!*, Proc. London Math. Soc. 47 (1983), 507–522.

[2] E. Brehm, *3-Orientations and Schnyder 3-Tree-Decompositions, Construction and Order Structure*, Diploma Thesis, FB Mathematik und Informatik, Freie Universität Berlin (2000)

[3] J. Edmonds, *Edge-disjoint branchings*, Combinatorial Algorithms (Courant Computer Science Symposium 9, Monterey, California), Algorithmics Press (1973), 91–96.

[4] T. Evans, *Embedding incomplete latin rectangles*, Amer. Math. Monthly, 67 (1960), 958–961.

[5] I. Fáry, *On straight line representing of planar graphs*, Acta. Sci. Math. (Szeged), 11 (1948), 229–233.

[6] A. Frank, *Kernel systems of directed graphs*, Acta Scientiarum Mathematicarum **41** (1979), 63–76.

[7] A. Frank, *How to make a digraph strongly connected*, Combinatorica 1 (1981), 145–153.

[8] A. Frank, T. Jordán, *Minimal edge-coverings of pairs of sets*, J. Combinatorial Theory, Ser. B. Vol. 65., No. 1 (September, 1995), 73–110.

[9] A. Frank, T. Király, M. Kriesell, *On decomposing a hypergraph into k connected sub-hypergraphs*, in Submodularity, (guest editor S. Fujishige) Discrete Applied Mathematics, Vol. 131, Issue 2. (September, 2003), 373–383.

[10] A. Frank, É. Tardos, *Matroids from crossing families*, Finite and Infinite Sets Vol.I.

[11] A. Frank, *Matroidelmélet, Lcture notes*

[12] A. Frank, *Gráfelmélet, Lecture notes*

[13] A. Frank, *Kombinatorikus optimalizálás, Lecture notes*

[14] H. de Fraysseix, J. Pach, R. Pollack, *Small sets supporting Fáry embeddings of planar graphs*, Proc. 20th Ann. ACM Symp. on Theory of Computing (1988), 426-433.

[15] R. Häggkvist, *A solution to the Evans conjecture for Latin squares of large size*, Colloqu. Math. Soc. Janos Bolyai 18 (1978), 405–513.

[16] H. Gröflin, A.J. Hoffman, *On matroid intersections*, Combinatorica 1 (1981), 43–47.

[17] S.L. Hakimi, *On the degrees of the vertices of a directed graph*, Journal of the Franklin Institute 279 (1965), 290–308.

[18] N. Kamiyama, N. Katoh, A. Takizawa, *Arc-disjoint In-trees in Directed Graph*, SODA 2008

[19] N. Kamiyama, N. Katoh, *Covering Directed Graphs by In-trees*, (2008)

[20] L. Lovász, *On two minimax theorems in graph*, Journal of Combinatorial Theory, Series B **21** (1976), 96–103.

[21] C.L. Lucchesi, D.H. Younger, *A minimax theorem for directed graphs*, The Journal of the London Mathematical Society (2) 17 (1978), 369–374.

[22] P. Rosenstiehl, R.E. Tarjan, *Rectilinear Planar Layouts and Bipolar Orientations of Planar Graphs*, Disc. Comp. Geom. **1** (1986), 343–353.

[23] L. Szegő, *Note on covering intersecting set-systems by digraphs*, Discrete Mathematics **234** (2001), 187–189.

[24] W. Schnyder, *Embedding Planar Graphs on the Grid*, Proceedings of the first annual ACM-SIAM symposium on Discrete algorithms (1990), 138–148.

[25] A. Schrijver, *Combinatorial Optimization Vol.B*, Springer

[26] A. Schrijver, *Supermodular colourings*, Matroid Theory (Proceedings Colloquium on Matroid Theory, SZeged, 1982), North-Holland, Amsterdam (1985), 327–343.

[27] B. Smetaniuk, *A new construction for Latin sqares I. Proof of the Evans conjecture*, Ars Combin. 11 (1981), 155-172.

[28] K. Vidyasankar, *Covering the edge set of a directed graph with trees*, Discrete Mathematics **24** (1978), 79–85.