Portfolio Optimization with Fuzzy Constraints

Thesis

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2012

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Introduction

Recently the importance of investment theory is increasing. Not only companies but also individual investors invest in stock, currency, land and property. It would be easy to decide where to invest money if we knew future returns a priori, however uncertainty from i.e. social conditions and behavior have a great influence on the future returns. The random ambiguous factors, the risk aversion of investors, the lack of information all take part of this. The problem is to reduce risk while making profit under such uncertain situations.

We call this problem a portfolio selection problem, which is concerned with selecting a combination of securities to best meet the investor's desire. The most influential theory for portfolio selection was proposed by Harry Markowitz. He considered returns of individual securities as random variables. The key principle of his mean-variance model was to use the expected return of a portfolio as the investment return, and the variance as the investment risk.

Since the introduction of Markowitz's mean-variance model, many efforts have been made to improve it, and of course made great achievements in portfolio selection theory. But the common assumption in them are that investors have enough historical data of securities and that the situation of asset markets in future can be correctly reflected by asset data in the past. They ignore, for example, the appearance of new stocks, or the above mentioned uncertain situations.

To deal with uncertainty, the focus of portfolio selection began to move towards fuzzy variables as returns instead of random variables. Fuzzy logic, first proposed by Lotfi Zadeh, allows expressing uncertain knowledge with subjective concepts, and uses a higher level of abstraction originating from our knowledge and experience. Fuzzy logic has been applied in many fields including control theory and artificial intelligence, and since behavior and human thinking take a great part in portfolio selection, it seems natural to frame fuzzy logic into portfolio selection problems.

In fuzzy portfolio selection there are alternative ways to measure a fuzzy event. Although possibility, necessity and credibility measures are all popular ways to deal with uncertainty, this thesis will only consider possibility and credibility as the two most applied.

Portfolio selection has many different approaches, taking the mean-variance model; mean-VaR; utility; considering liquidity; with transaction cost; single period model or rebalancing an existing portfolio, just to mention a few. Taking transaction costs into consideration is a very important part of finding an effective portfolio. We will discuss the original problem of mean-variance, and the one with transaction cost, both with fuzzy returns.

Chapter 2

Fuzzy sets

In this chapter we introduce some definitions and examples about fuzzy sets and numbers. In the second subsection we show a variety of measures for fuzzy sets to find the one we will use afterwards.

Fuzzy sets were introduced by Lotfi A. Zadeh (1965) as an extension of the classical set. In classical set theory an element either belongs or does not belong to the set. So a classical set can be represented by its characteristic function χ_A as a mapping from the elements of X to the set of $\{0, 1\}$. The value zero represent the non-membership and the value one represent the membership. What Zadeh introduced is the following: many sets have more than an either-or criterion for memberships. Take for example the set of tall people. It can be easily decided whether a 70 cm tall or a 230 cm tall person should be a member of this set. If we're faced with a 170 cm tall person then the same question can lead to different answers. (Our answer will be subjective in this case.)
Zadeh proposed a grade of membership, such that the transition from non-membership to membership is gradual rather than abrupt. The grade of membership for all its elements describes a fuzzy set. An item's grade of membership is normally a real number between 0 and 1, denoted by μ .
As we use an ordered pair $(x, \chi_A(x))$ to represent a classical set, we can use tuples to represent fuzzy sets too. The first element is from X, and the second element shows the grade of membership of element x in fuzzy set A for $x \in X$.

2.1 Fuzzy sets and logic

Definition 1 *[23] Let X be a nonempty set. A fuzzy set A in X is characterized by its membership function*

$$\mu_A : X \to [0, 1]$$

and $\mu_A(x)$ is interpreted as the degree of membership of element in a fuzzy set A for each $x \in X$.

It should be noted that the terms *membership function* and *fuzzy subset* are used interchangeably and $A(x)$ is often used instead of $\mu_A(x)$.

Example 1 *A fuzzy set of real numbers "close to 5" can be defined as*

$$A(x) = \exp(-\beta(t-5)^2)$$

where β is a positive real number.

The following definitions are used widely for example in [6], [3], [12], [17] or [7].

Definition 2 *An α-level set (or α-cut) of a fuzzy set A of X is a non-fuzzy set denoted by $[A]^\alpha$ and defined by*

$$[A]^\alpha = \{t \in X \mid A(t) \geq \alpha\}$$

Definition 3 *A fuzzy set is called convex if $[A]^\alpha$ is a convex subset for all $\alpha \in [0,1]$.*

Definition 4 *The support of a fuzzy set A is the set of elements with non-zero degree of membership.*

$$supp(A) = \{x \mid A(x) > 0\}$$

Definition 5 *The core of a fuzzy set A is the set of elements with 1 as degree of membership.*

$$core(A) = \{x \mid A(x) = 1\}$$

Definition 6 *The intersection of fuzzy sets A, B is*

$$\min\{A(x), B(x)\} = A(x) \wedge B(x), x \in \mathfrak{R}.$$

Definition 7 *The union of fuzzy sets A, B is*

$$\max\{A(x), B(x)\} = A(x) \vee B(x), x \in \mathfrak{R}.$$

2.2 Fuzzy numbers

Definition 8 *A fuzzy number A is a fuzzy set of the real line with a normal, convex and upper semi-continuous membership function of bounded support. The family of fuzzy numbers will be denoted by $\mathfrak{F}$*

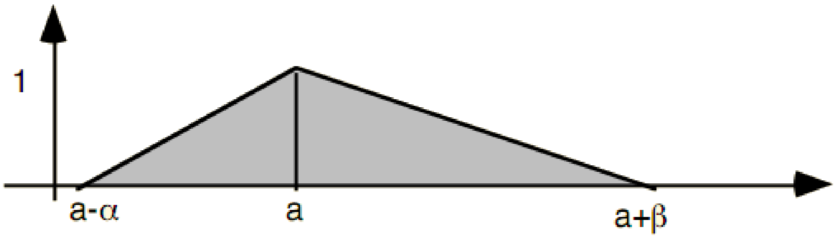

Figure 2.1: Triangular fuzzy number with center a

Definition 9 *A fuzzy set A is called a triangular fuzzy number with peak (or center) a, left width $\alpha > 0$ and right width $\beta > 0$ if its membership function has the following form*

$$A(t) = \begin{cases} 1 - \frac{a-t}{\alpha} & if a - \alpha \leqq t \leqq a \\ 1 - \frac{t-a}{\beta} & if a \leqq t \leqq a + \beta \\ 0 & otherwise \end{cases}$$

And we use the notation $A = (a, \alpha, \beta)$. The support of a triangular fuzzy number A is $(a - \alpha, a + \beta)$. A triangular fuzzy number with center a can be considered as: "x is approximately equal to a".

If $\alpha = \beta$ then we call the triangular fuzzy number symmetrical, and refer to it as (a, α). Let $A = (a, \alpha)$ and $B = (b, \beta)$ be two symmetrical triangular fuzzy numbers. Then

$$A + B = (a + b, \alpha + \beta), \lambda A = (\lambda a, |\lambda|\alpha)$$

Definition 10 *A fuzzy set A is called a trapezoidal fuzzy number, of it can be determined by four number $a < b < c < d$; and its membership function $\mu(x)$ is as follows*

$$\mu(x) = \begin{cases} 1 - \frac{x-a}{b-a} & if a \leq x \leq b \\ 1 & if b \leq x \leq c \\ 1 - \frac{x-d}{c-d} & if c \leq x \leq d \\ 0 & otherwise \end{cases}$$

and we use the notation $A = (a, b, c, d)$. The support of a trapezoidal fuzzy number is (a, d). A trapezoidal fuzzy number can be considered as: "x is approximately in the interval of $[b, c]$".

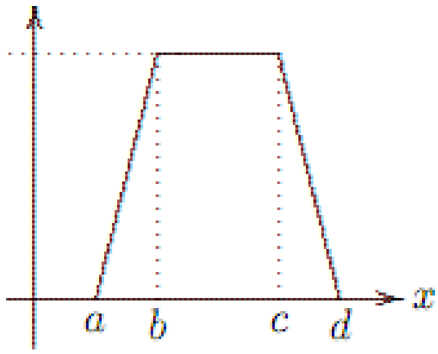

Figure 2.2: Trapezoidal fuzzy number

Definition 11 *Any fuzzy number $A \in \mathfrak{F}$ can be described as*

$$A(t) = \begin{cases} L\left(\frac{a-t}{\alpha}\right) & if t \in [a-\alpha, a] \\ 1 & if t \in [a, b] \\ R\left(\frac{t-b}{\beta}\right) & if t \in [b, b+\beta] \\ 0 & otherwise \end{cases}$$

where $[a, b]$ is the peak or core of A,

$$L : [0,1] \to [0,1], R : [0,1] \to [0,1]$$

are continuous and non-increasing shape functions with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$. We call this fuzzy interval of LR-type and refer to it as

$$A = (a, b, \alpha, \beta)_{LR}$$

The support of A is $(a-\alpha, b+\beta)$.

In order to introduce fuzzy arithmetic first we have to mention an important concept from fuzzy set theory called extension principle.

Definition 12 *(Zadeh's extension principle) Let $\xi_1, \xi_2, \ldots, \xi_n$ are independent fuzzy variables with membership functions $\mu_1, \mu_2, \ldots, \mu_n$, and $f : \mathfrak{R}^n \to \mathfrak{R}$. Then the membership function μ of $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is derived from the membership functions $\mu_1, \mu_2, \ldots, \mu_n$ by*

$$\mu(x) = \sup_{x = f(x_1, x_2, \ldots, x_n)} \min_{1 \le i \le n} \mu_i(x_i)$$

for any $x \in \mathfrak{R}$. Here we set $\mu(x) = 0$ if there are no $x_1, x_2, \ldots, x_n$ such that $x = f(x_1, x_2, \ldots, x_n)$.

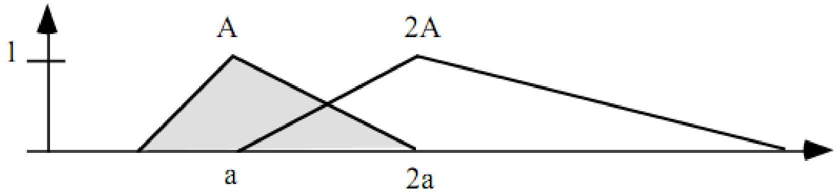

Figure 2.3: Membership function of $\lambda * \xi$

Example 2 *The sum of independent triangular fuzzy variables* $\xi = (a_1, a_2, a_3)$ *and* $\eta = (b_1, b_2, b_3)$ *is also a triangular fuzzy variable*

$$\mu + \eta = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

Example 3 *The product of a triangular fuzzy variable* $\xi = (a, b, c)$ *and a scalar number* $\lambda \in \Re$ *is also a triangular fuzzy number*

$$\lambda * \xi = \begin{cases} (\lambda a, \lambda b, \lambda c) & if \lambda \geq 0 \\ (\lambda c, \lambda b, \lambda a) & if \lambda < 0 \end{cases}$$

The result from substraction between two triangular fuzzy numbers is similar to the addition and also results a triangular fuzzy number. Multiplication or division of triangular fuzzy numbers do not result triangular fuzzy numbers.

Example 4 *Let* $\xi = (a_1, b_1, c_1)$ *and* $\eta = (a_2, b_2, c_2)$ *are triangular fuzzy numbers. Then the membership of their product is:*

$$\mu(x) = \begin{cases} \frac{-(a_1b_2 + a_2b_1 - 2a_1a_2) + \sqrt{(a_1b_2 - a_2b_1)^2 + 4(b_1 - c_1)(b_2 - a_2)x}}{2(b_1 - a_1)(b_2 - a_2)} & if a_1a_2 \leq x \leq b_1b_2 \\ \frac{-(c_1b_2 + c_2b_1 - 2c_1c_2) + \sqrt{(c_1b_2 - c_2b_1)^2 + 4(b_1 - c_1)(b_2 - c_2)x}}{2(b_1 - c_1)(b_2 - c_2)} & if b_1b_2 \leq x \leq c_1c_2 \\ 0 & otherwise. \end{cases}$$

Take for example $\xi = (2, 3, 5)$ *and* $\eta = (3, 5, 6)$, *then their product is*

$$\mu(x) = \begin{cases} \frac{1}{4}(-7 + \sqrt{1 + 8x}) & if 6 \leq x \leq 15 \\ \frac{1}{4}(17 - \sqrt{49 + 8x}) & if 15 \leq x \leq 30 \\ 0 & otherwise. \end{cases}$$

Theorem 2.2.1 *(Nguyen's theorem) [16] Let* $f : X \to X$ *be a continuous function and let* A *be a fuzzy number. Then*

$$[f(A)]^\alpha = f([A]^\alpha)$$

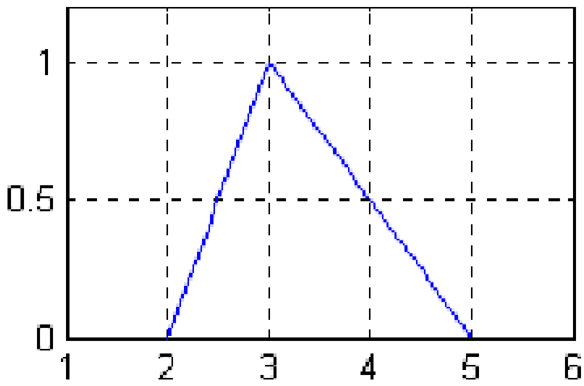

Figure 2.4: Membership function of ξ

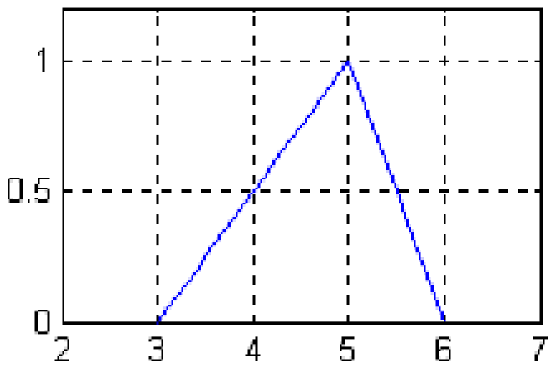

Figure 2.5: Membership function of η

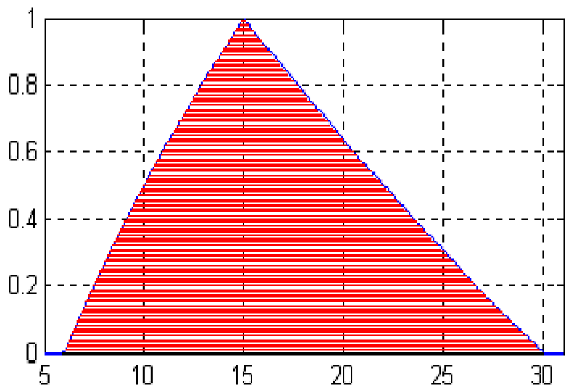

Figure 2.6: Membership function of $\xi * \eta$

where $f(A)$ is defined by the above mentioned extension principle, and

$$f([A]^\alpha) = \{f(x) \mid x \in [A]^\alpha\}.$$

If $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$ and f is monotone increasing then we get

$$[f(A)]^\alpha = f([A]^\alpha) = f([a_1(\alpha), a_2(\alpha)]) = [f(a_1(\alpha)), f(a_2(\alpha))].$$

Theorem 2.2.2 *(Nguyen's theorem 2.) [16] Let $f : X \times X \to X$ be a continuous function and let A and B be fuzzy numbers. Then*

$$[f(A, B)]^\alpha = f([A]^\alpha, [B]^\alpha)$$

where

$$f([A]^\alpha, [B]^\alpha) = \{f(x_1, x_2 \mid x_1 \in [A]^\alpha, x_2 \in [B]^\alpha)\}.$$

Example 5 *(fuzzy max) Let $f(x, y) = \max\{x, y\}$ and let $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$ and $[B]^\alpha = [b_1(\alpha), b_2(\alpha)]$ be two fuzzy numbers. Then*

$$[f(A, B)]^\alpha = f([A]^\alpha, [B]^\alpha) = \max\{[A]^\alpha, [B]^\alpha\} = [a_1(\alpha) \vee b_1(\alpha), a_2(\alpha) \vee b_2(\alpha)].$$

Example 6 *(fuzzy min) Let $f(x, y) = \min\{x, y\}$ and let $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$ and $[B]^\alpha = [b_1(\alpha), b_2(\alpha)]$ be two fuzzy numbers. Then*

$$[f(A, B)]^\alpha = f([A]^\alpha, [B]^\alpha) = \min\{[A]^\alpha, [B]^\alpha\} = [a_1(\alpha) \wedge b_1(\alpha), a_2(\alpha) \wedge b_2(\alpha)].$$

The fuzzy *min* and *max* operations are commutative, associative and distributive. The distributive property is the following: if A, B and C are fuzzy numbers, then

$$\max\{A, \min\{B, C\}\} = \min\{\max\{A, B\}, \max\{A, C\}\},$$

$$\min\{A, \max\{B, C\}\} = \max\{\min\{A, B\}, \min\{A, C\}\}.$$

Chapter 3

Measuring fuzzy numbers

In order to use fuzzy numbers later we first need to be able to measure them. Zadeh was the first who proposed a concept to measure fuzzy, this is called the possibility measure [23] [3]. In the first section we will discuss this measure. The possibility measure is widely used [24][19], although it has no self-duality property. Since this property is needed in theory and practice, Liu and Liu presented the concept of credibility measure [6]. So in the second section we will introduce the fundamentals of credibility measure, credibility space, credibility distribution, expected value and variance.

Example 7 *Probability vs Possibility [23] Consider the statement "Hans ate X eggs for breakfast", where $X \in U = \{1, 2, ..., 8\}$. We may associate a probability distribution p by observing Hans eating breakfast for 100 days,*

$$U = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$$

$$p = \begin{pmatrix} .1 & .8 & .1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A fuzzy set expressing the grade of ease with which Hans can eat X eggs may be the following so-called possibility distribution π,

$$U = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$$

$$\mu = \begin{pmatrix} 1 & 1 & 1 & 1 & .8 & .6 & .4 & .2 \end{pmatrix}$$

Where the possibility for $X = 3$ is 1*, the probability is only* 0.1*.*

3.1 Possibility measure

Definition 13 *Let $A, B \in \mathfrak{F}$ fuzzy numbers. The degree of possibility that "A is less than or equal to B" is true denoted by $Pos[A \leq B]$ and defined as:*

$$Pos[A \leq B] = \sup_{x \leq y} \min\{A(x), B(y)\} = \sup_{z \leq 0}(A - B)(z)$$

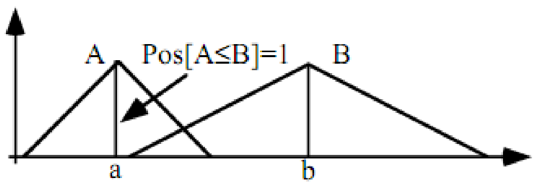

Figure 3.1: $Pos[A \leq B] = 1$

The "greater than" and "equal to" proposition can be described in a similar way. Let $A = (a, \alpha)$ and $B = (b, \beta)$ symmetric triangular fuzzy numbers. In this case

$$Pos[A \leq B] = \begin{cases} 1 & if a \leq b \\ 1 - \frac{a-b}{\alpha+\beta} & otherwise \\ 0 & if a \leq b + \alpha + \beta \end{cases}$$

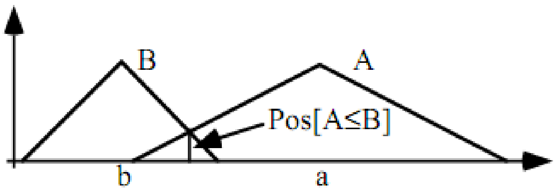

Figure 3.2: $Pos[A \leq B] < 1$

In 1986 Goetschel and Voxman introduced a method for ranking fuzzy numbers [7] : Let $A, B \in \mathfrak{F}$ be fuzzy numbers with $[A]^\gamma = (a_1(\gamma), a_2(\gamma))$ and $[B]^\gamma = (b_1(\gamma), b_2(\gamma))$, $\gamma \in [0, 1]$ than

$$A \leq B \Leftrightarrow \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma))d\gamma \leq \int_0^1 \gamma(b_1(\gamma) + b_2(\gamma))d\gamma$$

With this definition Goetschel and Voxman gave less importance to the lower level of fuzzy numbers.

Definition 14 *[2] Using the above mentioned ranking, the possibilistic mean value of a fuzzy number A can be defined as*

$$E(A) = \int_0^1 (a_1(\gamma) + a_2(\gamma))\gamma d\gamma$$

Which means that $E(A)$ is the level-weighted average of the arithmetic means of all γ-level sets.

Example 8 *Let $A = (a, \alpha, \beta)$ be a triangular fuzzy number. Then*

$$[A]^\gamma = [a - (1-\gamma)\alpha, a + (1-\gamma)\beta], \forall \gamma \in [0, 1]$$

$$E(A) = \int_0^1 \gamma[a - (1-\gamma)\alpha + a + (1-\gamma)\beta]d\gamma = a + \frac{\beta - \alpha}{6}$$

Definition 15 *[2] The possibilistic variance of a fuzzy number A is*

$$Var(A) = Cov(A, A) = \int_0^1 \sigma^2_{U_\gamma} 2\gamma d\gamma = \frac{1}{12}\int_0^1 (a_2(\gamma) - a_1(\gamma))^2 2\gamma d\gamma$$
$$= \frac{1}{6}\int_0^1 (a_2(\gamma) - a_1(\gamma))^2 \gamma d\gamma,$$

where U_γ is a uniform probability distribution on $[A]^\gamma$ and $\sigma^2_{U_\gamma}$ denotes the variance of U_γ

Example 9 *Let $A = (a, \alpha, \beta)$ be a triangular fuzzy number. Then its variance is as follows:*

$$Var(A) = \frac{1}{6}\int_0^1 \gamma(a + \beta(1-\gamma) - (a - \alpha(1-\gamma)))^2 d\gamma =$$
$$= \frac{(\alpha + \beta)^2}{72}.$$

Definition 16 *[2] Let A, B are fuzzy numbers. Then their covariance is defined as:*

$$Cov(A, B) = \frac{1}{2}\int_0^1 \gamma(a_2(\gamma) - a_1(\gamma))(b_2(\gamma) - b_1(\gamma))d\gamma$$

Example 10 *Let $A = (a_1, \alpha_1, \beta_1)$, $B = (a_2, \alpha_2, \beta_2)$ be triangular fuzzy numbers. Then their covariance is:*

$$Cov(A, B) = \frac{(\alpha_1 + \beta_1)(\alpha_2 + \beta_2)}{24}.$$

Example 11 *Let $A = (a, b, c, d)$ is a trapezoidal fuzzy number. Then the γ-level set, the expected value and the variance of A is as follows*

$$[A]^\gamma = [a + \gamma(b - a), d - \gamma(d - c)]$$
$$E(A) = \frac{1}{6}(a + d) + \frac{1}{3}(b + c)$$
$$V(A) = \frac{(a-d)^2}{4} + \frac{(c - d + a - b)^2}{8} + \frac{(d - c)(c - d + a - b)}{3}$$

Theorem 3.1.1 *[2] Let A, B are fuzzy numbers and let λ, $\mu \in \Re$. Then*

$$Var(\lambda A + \mu B) = \lambda^2 Var(A) + \mu^2 Var(B) + 2|\lambda\mu| Cov(A, B)$$

where the definition of the addition of fuzzy numbers and the multiplication of them by a scalar are defined by Zadeh.

So the possibilistic variance of linear combinations of fuzzy numbers are computed in a similar way as in probability theory.

Theorem 3.1.2 *Let $A_1, A_2, \ldots, A_n$ be fuzzy numbers and let $\lambda_0, \lambda_1, \lambda_2 \ldots \lambda_n$ are real numbers. Then*

$$E(\lambda_0 + \sum_{i=1}^{n} \lambda_i A_i) = \lambda_0 + \sum_{i=1}^{n} \lambda_i E(A_i),$$

$$Var(\lambda_0 + \sum_{i=1}^{n} \lambda_i A_i) = \sum_{i=1}^{n} \lambda_i^2 Var(A_i) + 2 \sum_{i<j=1}^{n} |\lambda_i \lambda_j| Cov(A_i, A_j).$$

3.2 Credibility measure

Let Θ be a nonempty set, and $\mathcal{P}$ the power set of Θ. Each element of $\mathcal{P}$ is called an event. In order to present credibility, we need to assign a number, $Cr\{A\}$ to each event A in Θ. In order to ensure that $Cr\{A\}$ has certain mathematical properties, we accept the following axioms:

Axiom 3.2.1 *(Normality)* $Cr\{\Theta\} = 1$

Axiom 3.2.2 *(Monotonicity) $Cr\{A\} \leq Cr\{B\}$ whenever $A \subseteq B$.?!?*

Axiom 3.2.3 *(Self-Duality) $Cr\{A\} + Cr\{A^C\} = 1$ for each event A.*

Axiom 3.2.4 *(Maximality) $Cr\{\bigcup_i A_i\} = \sup_i Cr\{A_i\}$ for any event A_i with $\sup_i Cr\{A_i\} < 0.5$.*

Definition 17 (Liu and Liu) *The set function Cr is called a credibility measure if it satisfies the normality, monotonicity, self-duality and maximality axioms.*

Theorem 3.2.5 *[13] Let Θ be a nonempty set, $\mathcal{P}$ the power set of Θ and Cr the credibility measure. Then $Cr\{\emptyset\} = 0$ and $0 \leq Cr\{A\} \leq 1$ for any $A \in \mathcal{P}$.*

Definition 18 *Let Θ be a nonempty set, $\mathcal{P}$ the power set of Θ and Cr the credibility measure. This triplet $\{\Theta, \mathcal{P}, Cr\}$ is called credibility space.*

Definition 19 *A fuzzy variable is defined as a (measurable) function from a credibility space $\{\Theta, \mathcal{P}, Cr\}$ to the set of real numbers.*

Example 12 *Take $\{\Theta, \mathcal{P}, Cr\}$ to be $\{\theta_1, \theta_2\}$ with $Cr\{\theta_1\} = Cr\{\theta_2\} = 0.5$. Then the function*

$$\xi(\theta) = \begin{cases} 0, & if \theta = \theta_1 \\ 1, & if \theta = \theta_2 \end{cases}$$

is a fuzzy variable.

Definition 20 *Let ξ be a fuzzy variable defined on the credibility space $\{\Theta, \mathcal{P}, Cr\}$. Then its membership function is derived from the credibility measure by*

$$\mu(x) = (2Cr\{\xi = x\}) \wedge 1, x \in \mathbb{R}.$$

Membership function represent the degree that the fuzzy variable ξ takes some prescribed value. The membership degree $\mu(x) = 0$ on if x is an impossible point, and $\mu(x) = 1$ if x is the most possible point that ξ takes.

Theorem 3.2.6 (Credibility Inversion Theorem) *[13] Let ξ be a fuzzy variable with a membership function μ. Then for any set B of real numbers, we have*

$$Cr\{\xi \in B\} = \frac{1}{2}\left(\sup_{x\in B} \mu(x) + 1 - \sup_{x\in B^C} \mu(x)\right).$$

Proof: If $Cr\{\xi \in B\} \leq 0,5$, then by the Monotonicity axiom we have $Cr\{\xi = x\} \leq 0,5$ for each $x \in B$. It follows from the Maximality axiom that

$$Cr\{\xi \in B\} = \frac{1}{2}\left(\sup_{x\in B}(2Cr\{\xi = x\} \wedge 1)\right) = \frac{1}{2}\sup_{x\in B} \mu(x).$$

The self-duality of credibility measure implies that $Cr\{\xi \in B^c\} \geq 0,5$ and $\sup_{x\in B^c} Cr\{\xi = x\} \geq 0,5$, i.e.,

$$\sup_{x\in B^c} \mu(x) = \sup_{x\in B^c}(2Cr\{\xi = x\} \wedge 1) = 1.$$

It follows from the previous calculations that the statement holds.
If $Cr\{\xi \in B\} \geq 0,5$ then $Cr\{\xi \in B^c\} \leq 0,5$. It follows from the first case that

$$Cr\{\xi \in B\} = 1 - Cr\{\xi \in B^c\} = 1 - \frac{1}{2}\left(\sup_{x\in B^c} \mu(x) + 1 - \sup_{x\in B} \mu(x)\right) =$$

$$= \frac{1}{2}\left(\sup_{x\in B} \mu(x) + 1 - \sup_{x\in B^c} \mu(x)\right).$$

The theorem is proved. ∎

Example 13 *Let ξ be a fuzzy variable with a membership function μ. Then the following equations follow immediately from the Inversion Theorem:*

$$Cr\{\xi = x\} = \frac{1}{2}\left(\mu(x) + 1 - \sup_{y \neq x} \mu(y)\right), \forall x \in \mathbb{R};$$

$$Cr\{\xi \leq x\} = \frac{1}{2}\left(\sup_{y \leq x} \mu(y) + 1 - \sup_{y > x} \mu(y)\right), \forall x \in \mathbb{R},$$

$$Cr\{\xi \geq x\} = \frac{1}{2}\left(\sup_{y \geq x} \mu(y) + 1 - \sup_{y < x} \mu(y)\right), \forall x \in \mathbb{R}$$

Theorem 3.2.7 *(Sufficient and Necessary Condition for Membership Function) [13] A function $\mu : \mathbb{R} \to [0, 1]$ is a membership function if and only if* $\sup \mu(x) = 1$.

Proof: If μ is a membership function then there exists a fuzzy variable ξ whose membership function is just μ, and

$$\sup_{x \in \mathbb{R}} \mu(x) = \sup_{x \in \mathbb{R}} (2Cr\{\xi = x\}) \wedge 1.$$

If there is some point $x \in \mathbb{R}$ such that $Cr\{\xi = x\} \geq 0.5$, then $\sup \mu(x) = 1$. Otherwise we have $Cr\{\xi = x\} < 0.5$ for each $q \in \mathbb{R}$. It follows from the Maximality Axiom that

$$\sup_{x \in \mathbb{R}} \mu(x) = \sup_{x \in \mathbb{R}} (2Cr\{\xi = x\}) \wedge 1 = 2 \sup_{x \in \mathbb{R}} Cr\{\xi = x\} = 2(Cr\{\Theta\} \wedge 0.5) = 1.$$

Conversely, suppose that $\mu(x) = 1$. For each $x \in \mathbb{R}$, we define

$$Cr\{x\} = \frac{1}{2}\left(\mu(x) + 1 - \sup_{y \neq x} \mu(y)\right).$$

It is clear that

$$\sup_{x \in \mathbb{R}} Cr\{x\} \geq \frac{1}{2}(1 + 1 - 1) = 0.5.$$

For any $x^* \in \mathbb{R}$ with $Cr\{x^*\} \geq 0.5$, we have $\mu(x^*) = 1$ and

$$Cr\{x^*\} + \sup_{y \neq x^*} Cr\{y\} = \frac{1}{2}\left(\mu(x^*) + 1 - \sup_{y \neq x^*} \mu(y)\right) + \sup_{y \neq x^*} \frac{1}{2}\left(\mu(y) + 1 - \sup_{z \neq y} \mu(z)\right) =$$

$$1 - \frac{1}{2} \sup_{y \neq x^*} \mu(y) + \frac{1}{2} \sup_{y \neq x^*} \mu(y) = 1.$$

Thus $Cr\{x\}$ satisfies the credibility extension condition, and has a unique extension to credibility measure on $\mathfrak{P}(\mathbb{R})$ by using the credibility extension theorem. Now we define a fuzzy variable ξ as an identity function from the credibility space $(\mathbb{R}, \mathfrak{P}(\mathbb{R}), Cr)$ to $\mathbb{R}$. Then the membership function of the fuzzy variable ξ is

$$(2Cr\{\xi = x\}) \wedge 1 = \left(\mu(x) + 1 - \sup_{y \neq x} \mu(y)\right) \wedge 1 = \mu(x)$$

for each x. The theorem is proved. ∎

Theorem 3.2.8 *[13] A fuzzy variable ξ with a membership function μ is (a) non-negative if and only if $\mu(x) = 0 \forall x < 0$; (b) positive if and only if $\mu(x) = 0 \forall x \leq 0$; (c) simple if and only if μ takes nonzero values at a finite number of points; (d) discrete if and only if μ takes nonzero numbers at a countable number of points; (e) continuous if and only if μ is a continuous function.*

3.2.1 Credibility distribution and density function

Definition 21 *[12] The credibility distribution $\Phi : \mathbb{R} \to [0,1]$ of a fuzzy variable ξ is defined as*

$$\Phi(x) = Cr\{\theta \in \Theta \mid \xi(\theta) \leq x\}.$$

Which means that Φ is the credibility of a fuzzy variable ξ taking a value less than or equal to x. The credibility distribution is neither left nor right continuous.

Example 14 *The credibility distribution of a $\xi = (a, b, c)$ triangular fuzzy variable is*

$$\Phi(x) = \begin{cases} 0 & if x \leq a \\ \frac{x-a}{2(b-a)} & if a \leq x \leq b \\ \frac{x+c-2b}{2(c-b)} & if b \leq x \leq c \\ 1 & x \geq c \end{cases}$$

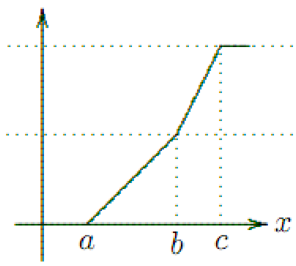

Figure 3.3: Credibility distribution of a triangular fuzzy variable

Theorem 3.2.9 *[13] Let ξ be a fuzzy variable with membership function μ. Then its credibility distribution is*

$$\Phi(x) = \frac{1}{2}\left(\sup_{y \leq x} \mu(y) + 1 - \sup_{y > x} \mu(y)\right), \forall x \in \mathbb{R}.$$

Proof: It follows from the Credibility Inversion Theorem. ■

Theorem 3.2.10 *[11] (Sufficient and Necessary Condition for Credibility Distribution) A function $\Phi : \mathbb{R} \to [0, 1]$ is a credibility distribution if and only if it is an increasing function with*

$$\lim_{x\to-\infty} \Phi(x) \le 0.5 \le \lim_{x\to\infty} \Phi(x),$$

$$\lim_{y\downarrow x} \phi(y) = \Phi(x) \text{ } if \text{ } \lim_{y\downarrow x} \Phi(y) > 0.5 \text{ } or \text{ } \Phi(x) \ge 0.5$$

Proof: It is obvious that the credibility distribution Φ is an increasing function. The first inequalities follow from the credibility asymptotic theorem immediately. Assume that x is a point at which $\lim_{y\downarrow z} \Phi(y) > 0.5$. That is,

$$\lim_{y\downarrow z} Cr\{\xi \le y\} > 0.5.$$

Since $\{\xi \le y\} \downarrow \{\xi \le x\}$ as $y \downarrow x$, it follows from the credibility semicontinuity law that

$$\Phi(y) = Cr\{\xi \le y\} \downarrow Cr\{\xi \le x\} = \Phi(x)$$

as $y \downarrow x$. When x is a point at which $\Phi(x) \ge 0.5$, if $\lim_{y\downarrow x} \Phi(y) \ne \Phi(x)$, then we have

$$\lim_{y\downarrow x} \Phi(y) > \Phi(x) \ge 0.5.$$

For this case, we have proved that $\lim_{y\downarrow x} \Phi(y) = \Phi(x)$. Thus both inequalities are proved. Conversely, if $\Phi : \Re \to [q, 1]$ is an increasing function satisfying the inequalities, then

$$\mu(x) = \begin{cases} 2\Phi(x), & if \Phi(x) < 0.5 \\ 1, & if \lim_{y\uparrow x} \Phi(y) < 0.5 \le \Phi(x) \\ 2 - 2\Phi(x), & if 0.5 \le \lim_{y\uparrow x} \Phi(y) \end{cases}$$

takes values in $[0, 1]$ and $\sup \mu(x) = 1$. It follows from the Sufficient and Necessary Condition for Membership Function Theorem that there is a fuzzy variable ξ whose membership function is just μ. Let us verify that Φ is the credibility distribution of ξ, i.e., $Cr\{\xi \le x\} = \Phi(x)$ for each x. The argument breaks down into two cases. (i) If $\Phi(x) < 0.5$, then we have $\sup_{y>x} \mu(y) = 1$, and $\mu(y) = 2\Phi(y)$ for each y with $y \le x$. Thus

$$Cr\{\xi \le x\} = \frac{1}{2}\left(\sup_{y\le x} \mu(y) + 1 - \sup_{y>x} \mu(y)\right) = \sup_{y\le x} \Phi(y) = \Phi(x).$$

(ii) If $\Phi(x) \ge 0.5$, the we have $\sup_{y\le x} \mu(y) = 1$ and $\Phi(y) \ge \Phi(x) \ge 0.5$ for each y with $y > x$. Thus $\mu(y) = 2 - 2\Phi(y)$ and

$$Cr\{\xi \le x\} = \frac{1}{2}\left(\sup_{y\le x} \mu(y) + 1 - \sup_{y>x} \mu(y)\right) = \frac{1}{2}\left(1 + 1 - \sup_{y>x} 2 - 2\Phi(y)\right) =$$

$$= \inf_{y>x} \Phi(y) = \lim_{y \downarrow x} \Phi(y) = \Phi(x).$$

The theorem is proved. ■

Definition 22 (Liu) *The credibility density function $\phi : \Re \to [0, +\infty)$ of a fuzzy variable ξ is a function such that*

$$\Phi(x) = \int_{-\infty}^{x} \phi(y)dy, \forall x \in \Re,$$

$$\int_{-\infty}^{+\infty} \phi(y)dy = 1$$

where Φ is the credibility distribution of the fuzzy variable ξ.

Example 15 *The credibility density function of a triangular fuzzy variable (a, b, c) is*

$$\phi(x) = \begin{cases} \frac{1}{2(b-a)} & , if a \leq x \leq b \\ \frac{1}{2(c-b)} & , if b \leq x \leq c \\ 0 & , otherwise. \end{cases}$$

Example 16 *The credibility density function of a trapezoidal fuzzy variable (a, b, c, d) is*

$$\phi(x) = \begin{cases} \frac{1}{2(b-a)} & , if a \leq x \leq b \\ \frac{1}{d-c} & , if c \leq x \leq d \\ 0 & , otherwise. \end{cases}$$

3.2.2 Credibility expected value

For fuzzy numbers there are many ways to define the expected value. The most accepted definition is given by Liu and Liu. This definition is not only applicable to continuous fuzzy variables but also discrete ones.

Definition 23 (Liu and Liu) *[1] Let ξ be a fuzzy variable. Then the expected value of ξ is defined by*

$$E[\xi] = \int_{0}^{+\infty} Cr\{\xi \geq r\}dr - \int_{-\infty}^{0} Cr\{\xi \leq r\}dr$$

provided that at least one of the two integrals is finite.

Theorem 3.2.11 *[1] Let ξ be a fuzzy variable with a continuous membership function μ. If its expected value exists, and there is a point x_0 such that $\mu(x)$ is increasing in $(-\infty, x_0)$ and decreasing in (x_0, ∞) then its expected value can be calculated as the following:*

$$E[\xi] = x_0 + \frac{1}{2} \int_{x_0}^{+\infty} \mu(x)dx - \frac{1}{2} \int_{-\infty}^{x_0} \mu(x)dx.$$

Proof: If $x_0 \geq 0$, then

$$Cr\{\xi \geq r\} = \begin{cases} \frac{1}{2}[1+1-\mu(x)] & ,if 0 \leq r \leq x_0, \\ \frac{1}{2}\mu(x) & ,if r > x_0. \end{cases}$$

and $Cr\{\xi \leq r\} = \frac{1}{2}\mu(x)$, so we have

$$E[\xi] = \int_0^{x_0} [1 - \frac{1}{2}\mu(x)]dx + \int_{x_0}^{\infty} \frac{1}{2}\mu(x)dx - \int_{-\infty}^{0} \frac{1}{2}\mu(x)dx =$$

$$x_0 + \frac{1}{2}\int_{x_0}^{\infty} \mu(x)dx - \frac{1}{2}\int_{-\infty}^{x_0} \mu(x)dx.$$

The case of $x_0 < 0$ is similar. ∎

Example 17 *Let $\xi = (a, b, c)$ be a triangular fuzzy variable. Then its expected value is*

$$E[\xi] = b + \frac{1}{2}\int_b^c \frac{x-c}{b-c}dx - \frac{1}{2}\int_a^b \frac{x-a}{b-a}dx =$$

$$= b + \frac{c-b}{4} + \frac{a-b}{4} = \frac{a+2b+c}{4}.$$

Example 18 *Let $\xi = (a, b, c, d)$ be a trapezoidal fuzzy variable. Then its expected value is $E[\xi] = \frac{a+b+c+d}{4}$.*

Theorem 3.2.12 (Liu and Liu) *[22] Let ξ and η be independent fuzzy variables with finite expected values. Then for any numbers a and b, we have*

$$E[\xi + b] = E[\xi] + b$$

$$E[a\xi] = aE[\xi]$$

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta].$$

Theorem 3.2.13 (Liu) *[11] Let ξ be a fuzzy variable whose credibility density function ϕ exists. If*

$$\lim_{x\to\infty} \phi(x) = 0, \lim_{x\to\infty} \phi(x) = 1, and$$

the Lebesgue integral

$$\int_{-\infty}^{+\infty} x\phi(x)dx$$

is finite, then we have

$$E[\xi] = \int_{-\infty}^{+\infty} x\phi(x)dx.$$

Definition 24 (Liu and Liu)*[1] Let ξ be a fuzzy variable whose expected value is finite. Then the variance of ξ is defined by:*

$$Var[\xi] = E[(\xi - E[\xi])^2].$$

Example 19 *Let $\xi = (a, b, c)$ be a triangular fuzzy number. Then its variance is:*

$$Var(\xi) = \int_0^{+\infty} Cr\{(\xi - E(\xi))^2 \geq x\}dx.$$

We will use our previous knowledge about the expected value of a triangular fuzzy number:

$$E(\xi) = \frac{a + 2b + c}{4}$$

to get

$$Cr\{(\xi - E(\xi))^2 \geq x\} = Cr\{\xi \geq \sqrt{x} + E(\xi)\} =$$
$$= Cr\left\{\xi \geq \sqrt{x} + \frac{a + 2b + c}{4}\right\} = Cr\{\xi \geq k\} =$$

$$= \frac{1}{2}\left(\sup_{y \geq k} \mu(y) + 1 - \sup_{y<k} \mu(y)\right) = \begin{cases} 2 & if k \leq a \\ 1 - \frac{1}{2}\frac{k-a}{b-a} & if a < k \leq b \\ \frac{1}{2}\frac{k-c}{b-c} & if b < k \leq c \\ 0 & if c < k, \end{cases}$$

where $k = \sqrt{x} + \frac{a+2b+c}{4}$. Defining α and β, so that $\alpha = b - a$, $\beta = c - a$, we can continue the previous line as:

$$= \begin{cases} 1 & if \sqrt{x} \leq \frac{-3\alpha-\beta}{4} \\ 1 - \frac{3\alpha+\beta}{8\alpha} - \frac{\sqrt{x}}{2\alpha} & if \frac{-3\alpha-\beta}{4} < \sqrt{x} \leq \frac{\alpha-\beta}{4} \\ \frac{3\beta+\alpha}{8\beta} - \frac{\sqrt{x}}{2\beta} & if \frac{\alpha-\beta}{4} < \sqrt{x} \leq \frac{\alpha+3\beta}{4} \\ 0 & if \frac{\alpha+3\beta}{4} < \sqrt{x} \end{cases}$$

If we have a symmetric triangular fuzzy number ($\alpha = \beta$), then the first two cases are 0 as well as the last one. So what we have is:

$$Var(\xi) = \int_0^{+\infty} Cr\{(\xi - E(\xi)^2) \geq x\}dx = \frac{1}{2(b-c)} \int_0^{\left(\frac{\alpha+3\beta}{4}\right)^2} \frac{3\beta + \alpha}{8\beta} - \frac{\sqrt{x}}{2\beta}dx =$$

$$= \frac{1}{c-a} \int_0^{\left(\frac{c-a}{2}\right)^2} \sqrt{x} - \frac{c-a}{2}dx = \frac{(c-a)^2}{24}.$$

The non-symmetric case is similar:

$$Var[\xi] = \begin{cases} \frac{33\alpha^3+11\alpha\beta^2+21\alpha^2\beta-\beta^3}{348\alpha} & , \alpha > \beta \\ \frac{(c-a)^2}{24} & , \alpha = \beta \\ \frac{33\beta^3+11\beta\alpha^2+21\beta^2\alpha-\alpha^3}{348\beta} & , \alpha < \beta \end{cases}$$

Theorem 3.2.14 *[13] If ξ is a fuzzy variable whose variance exists, a and b are real numbers, then $Var[a\xi + b] = a^2Var[\xi]$.*

Proof: From the definition of variance we get

$$V[a\xi + b] = E[(a\xi + b - aE[\xi] - b)^2] = a^2 E[(\xi - E[\xi])^2] = a^2 V[\xi].$$

■

Chapter 4

Portfolio optimization

The name Markowitz sounds familiar to those working on portfolio selection. His well-known and widely used mean-variance model used probability theory to chose between portfolios. In this section we'll give a short introduction on this model and turn it into a possibility/credibility model to fit our needs.

4.1 Markowitz model

The fundamental goal of portfolio theory is to optimally allocate your investments between different assets. Mean variance optimization is a quantitative tool which will allow you to make this allocation by considering the trade-off between risk and return. When making investment decision, the investor would always strike a balance between maximizing the return and minimizing the risk. At the original Markowitz model the performance of individual securities were considered as random variables. The return of the portfolio was quantified as the mean and the return was quantified as the variance.

In the case of maximizing the return at a given specific level of risk (called the single period Markowitz mean variance optimization), the standard formulation of Markowitz model is as follows [15]:

$$\begin{cases} \max E[x_1\xi_1 + x_2\xi_2 + \ldots + x_n\xi_n] \\ \text{subject to:} \\ Var[x_1\xi_1 + x_2\xi_2 + \ldots + x_n\xi_n] \leq \gamma \\ x_1 + x_2 + \ldots + x_n = 1 \\ x_i \geq 0, i = 1, 2, \ldots, n. \end{cases}$$

, where E denotes the expected value operator, Var denotes the variance, x_i are the investment proportions, in securities i, ξ_i represents the risk of the i-th security $(i = 1, \ldots, n)$ and γ is the maximum risk level the investor can tolerate.

In this scenario our portfolio will not include short-selling. If short-selling is allowed, we have to delete the $x_i \geq 0.i = 1, 2, \ldots, n$ constraints. Short-selling means that the investor can sell shares without owning them. This can be done by borrowing shares from the broker and only reasonable when the prices fall.

Let C be the variance-covariance matrix of the returns as follows:

$$\begin{pmatrix} \sigma_{11} & \ldots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \ldots & \sigma_{nn} \end{pmatrix}$$

The previous model can be converted into the following, using $x = (x_i, x_2, \ldots, x_n)$, $e = (E[\xi_1], E[\xi_2], \ldots, E[\xi_n])$, and C for the variance-covariance matrix of random vector of the returns $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$:

$$\begin{cases} \max xe^t \\ \text{subject to:} \\ xCx^t \leq \gamma \\ \sum_{i=1}^{n} x_i = 1 \\ x_i \geq 0, i = 1, 2, \ldots, n. \end{cases}$$

The rates of returns are not necessarily random variables in real life. If we know historical data for returns of the considered assets, we can calculate the next period's return as the mean of the previous returns of one asset.

To get the returns of an asset if only the stock-market prices are given we can use the following formula: $\xi_i = 100 \frac{P_i - P_{i-1}}{P_i}$, where P_i is the price of the asset in the ith period.

The first approach to solve this constrained maximization problem is to eliminate (i.e.) x_n with the help of the $\sum_{i=1}^{n} x_i = 1$, and get a problem which includes only $n - 1$ unknown parameters.

However, contrary to its theoretical reputation, the Markowitz's mean-variance model is not used extensively to construct large-scale portfolios. One of the most important reasons for this is the computational difficulty associated with solving a large-scale quadratic programming problem with a dense covariance matrix.

As a basic problem we can consider the Markowitz model's simpler version where we are dealing only with the maximization of the return or the minimization of the risk. If we also include short-selling, we will get the following problems:

$$\begin{cases} \max xe^t \\ \text{subject to: } \sum_{i=1}^n x_i = 1 \end{cases}$$

or

$$\begin{cases} \min xCx^t \\ \text{subject to: } \sum_{i=1}^n x_i = 1 \end{cases}$$

After eliminating one x_i (for example x_n) we get a "simple" n-asset portfolio optimization problem with $n-1$ unknown parameters. To eliminate x_n, let us define an n-vector $\alpha = \begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -1 \end{pmatrix}$, which is an $(n-1)xn$ matrix. Using $y_i = x_i (i = 1, \dots, n-1)$, $x = \alpha + y\beta$. And we get the following:

$$\min(\alpha + y\beta)C(\alpha + y\beta)^t$$

, or for the maximization problem:

$$\max(\alpha + y\beta)e^t$$

One way to make the computation easier is to use absolute deviation risk function instead of the original. This new mean absolute deviation portfolio optimization model maintain the favorable properties of the Markowitz model, but removes most of the difficulties of solving it, since it can be reduced to a linear programming problem.

4.2 Markowitz mean-variance model with fuzzy returns

In order to use the mean-variance model, it is necessary to estimate the expected return vector and a covariance matrix. In the original mean-variance model uncertainty of the return is equated with randomness, but it turned out that fuzzy number is a more powerful tool to describe an uncertain environment. In important cases, it might be easier to estimate the possibility or credibility distributions of rates of return on risky assets than the corresponding probability distributions. Based on these facts, we discuss the portfolio selection problem under the assumption that the returns of assets are fuzzy numbers.

Let's consider a financial market with n risky assets and a risk-less asset. Let r_0 be the interest rate of the risk-less asset. Analogue to the Markowitz mean variance model the possibilistic/credibilistic mean value is the measure of the investment return and possibilistic/credibilistic variance is the measure of the investment risk.

4.2.1 Involving fuzzyness into the model

As mentioned before, the rate of return of an asset is calculated from the historical data of the asset with the mean or the median of the previous periods. More information can be added to our model if we consider including not only the mean or the variance but a minimum and maximum return of an asset in the selected period. This can be done with fuzzy numbers. Let us take triangular fuzzy numbers as returns with the mean ($\overline{\xi}$) as peak (a) and $\overline{\xi} - \xi_{\min}$ as α, $\xi_{\max} - \overline{\xi}$ as β.

We will use the following theorems here [19]:

$$E(\lambda_0 + \sum_{i=1}^{n} \lambda_i A_i) = \lambda_0 + \sum_{i=1}^{n} \lambda_i E(A_i),$$

$$Var(\lambda_0 + \sum_{i=1}^{n} \lambda_i A_i) = \sum_{i=1}^{n} \lambda_i^2 Var(A_i) + 2 \sum_{i<j=1}^{n} |\lambda_i \lambda_j| Cov(A_i, A_j),$$

And the possibilistic mean value, variance and covariance of a triangular fuzzy number $A_i = (a_i, \alpha_i, \beta_i)$:

$$E(A) = a + \frac{\beta - \alpha}{6}$$

$$Var(A) = \frac{(\alpha + \beta)^2}{72}$$

$$Cov(A_1, A_2) = \frac{(\alpha_1 + \beta_1)(\alpha_2 + \beta_2)}{24}$$

where $A_1, A_2, \ldots, A_n$ are fuzzy numbers and $\lambda_0, \lambda_1, \lambda_2 \ldots \lambda_n$ are real numbers.

We get the following optimization problem:

$$\max \sum_{i=1}^{n} x_i \left(a_i + \frac{\beta_i - \alpha_i}{6} \right)$$

$$s.t. \sum_{i=1}^{n} x_i^2 \frac{(\alpha_i + \beta_i)^2}{72} + 2 \sum_{i \neq j=1}^{n} \frac{(\alpha_i + \beta_i)(\alpha_j + \beta_j)|x_i||x_j|}{24} \leq \gamma$$

, which is with the previously defined a_i, α_i and β_i can be transformed into

$$\max \sum_{i=1}^{n} \frac{\xi_{i,\max} + \xi_{i,\min} + 4\overline{\xi}}{6}$$

$$s.t. \sum_{i=1}^{n} x_i^2 \frac{(\xi_{i,\max} - \xi_{i,\min})^2}{72} + 2 \sum_{i \neq j=1}^{n} \frac{(\xi_{i,\max} - \xi_{i,\min})(\xi_{j,\max} - \xi_{j,\min})|x_i||x_j|}{24} \leq \gamma$$

4.2.2 Possibilistic mean-variance with one risk-free asset

Some say that there are no risk-free asset in real life because all assets carry some degree of risk. However some assets' (treasuries from the U.S. or from stable Western governments) level of risk is so small that they can be technically considered risk-free or risk-less.

Let's take first the possibilistic approach of the above mentioned mean-variance model. The general model with n risky and one risk-less asset is the following:

$$\min Var[\xi^t x + \xi_0(1 - F^t x)]$$

$$s.t. E[\xi^t x + \xi_0(1 - F^t x)] \geq \mu,$$

$$x \in \mathbf{H}$$

where $x = (x_1, x_2, \ldots, x_n)$, $r = (r_1, r_2, \ldots, r_n)$, $F = (1, 1, \ldots, 1)$ $\mathbf{H}$ is a convex set representing the additional constraints on the choice of x. The return rate of the jth asset is ξ_j, and the proportion of total investment to this asset is x_j. We use the prime (^{t}) to denote matrix transposition.

The the possibilistic mean value of the return of the portfolio is given by:

$$E[\xi^t x + \xi_0(1 - F^t x)] = E(\sum_{i=1}^{n} \xi_i x_i) + \xi_0(1 - \sum_{i=1}^{n} x_i) =$$

$$= \sum_{i=1}^{n} E(\xi_i) x_i + \xi_0(1 - \sum_{i=1}^{n} x_i) =$$

$$= \sum_{i=1}^{n} (a_i + \frac{\beta_i - \alpha_i}{6}) x_i + \xi_0(1 - \sum_{i=1}^{n} x_i).$$

And the corresponding possibilistic variance of the return is:

$$Var[\xi^t x + \xi_0(1 - F^t x)] = Var[\sum_{i=1}^{n} \xi_i x_i] = \frac{1}{72}\left[\sum_{i=1}^{n} (\alpha_i + \beta_i)^2 x_i^2\right] + \frac{1}{12}\sum_{i \neq j = 1}^{n} (\alpha_i + \beta_i)(\alpha_j + \beta_j)|x_i||x_j|$$

$$= \frac{1}{72}\left[\sum_{i=1}^{n} (\alpha_i + \beta_i)|x_i|\right]^2.$$

So the possibilistic mean-variance model can be described by:

$$\min \frac{1}{72}\left[\sum_{i=1}^{n} (\alpha_i + \beta_i)^2 x_i^2\right] + \frac{1}{12}\sum_{i \neq j = 1}^{n} (\alpha_i + \beta_i)(\alpha_j + \beta_j)|x_i||x_j|$$

$$s.t. \sum_{i=1}^{n} (a_i + \frac{\beta_i - \alpha_i}{6}) x_i + \xi_0(1 - \sum_{i=1}^{n} x_i) \geq \mu,$$

$$x \in \mathbf{H},$$

where $\mathbf{H}$ is a convex set representing the additional constraints of x. The previous model is equal to the following:

$$\min \frac{1}{72}\left[\sum_{i=1}^{n}(\alpha_i+\beta_i)^2 x_i^2\right] + \frac{1}{12}\sum_{i\neq j=1}^{n}(\alpha_i+\beta_i)(\alpha_j+\beta_j)|x_i||x_j|$$

$$s.t. \sum_{i=1}^{n}(a_i - \xi_0 + \frac{\beta_i-\alpha_i}{6})x_i \geq \mu - \xi_0,$$

$$x \in \mathbf{H},$$

where $\mathbf{H}$ is a convex set representing the additional constraints of x.

This model contains only $3n$ unknown parameters as opposed to the probabilistic mean-variance model, which contains $(n^2+3n+2)/2$ unknown parameters. We could further decrease the number of unknown parameters by using symmetric triangular fuzzy numbers $r_i = (a_i, \alpha_i)$ $(\alpha_i = \beta_i)$.

4.2.3 Credibilistic mean-variance with one risk-free asset

The credibilistic model doesn't differ much from the possibilistic one. Since the credibilistic variance of triangular fuzzy numbers had three cases, we will consider only the symmetric one, where the left-width equals the right-width, which means $\xi_i = (a_i, \alpha_i)$, where a_i is the peak and α_i is the left and right width.

The expected value and variance of a symmetric triangular fuzzy number is

$$E[\xi_i] = a_i$$

$$Var[\xi_i] = \frac{\alpha_i^2}{6}$$

Since adding two (symmetric) triangular fuzzy numbers also results a triangular fuzzy number with peak $a_1 + a_2$ and left/right width $\alpha_1 + \alpha_2$, we have:

$$E[\sum_{i=1}^{n}\xi_i] = \sum_{i=1}^{n} a_i$$

and

$$Var[\sum_{i=1}^{n}\xi_i] = \frac{\sum_{i=1}^{n}\alpha_i}{6}.$$

With other features of the credibilistic expected value, the return of the portfolio x is

$$E[\xi^t x + \xi_0(1 - F^t x)] = E[\sum_{i=1}^{n}\xi_i x_i] + \xi_0(1 - \sum_{i=1}^{n} x_i) =$$

$$\sum_{i=1}^{n} E[\xi_i]x_i + \xi_0(1 - \sum_{i=1}^{n} x_i) = \sum_{i=1}^{n} a_i x_i + \xi_0(1 - \sum_{i=1}^{n} x_i),$$

and the corresponding credibilistic variance is:

$$Var[\xi^t x + \xi_0(1 - F^t x)] = Var[\sum_{i=1}^{n} \xi_i x_i] = \frac{(\sum_{i=1}^{n} x_i \alpha_i)^2}{6}.$$

Thus the credibilistic mean-variance model of the portfolio x is given by

$$\min \frac{(\sum_{i=1}^{n} x_i \alpha_1)^2}{6}$$

$$s.t. \sum i = 1^n a_i x_i + \xi_0(1 - \sum_{i=1}^{n} x_i) \geq \mu$$

$$x \in \mathbf{H},$$

where $\mathbf{H}$ is a convex set representing the additional constraints of x.

This problem equals with

$$\min \sum_{i=1}^{n} x_i \alpha_i$$

$$s.t. \sum_{i=1}^{n} (a_i - \xi_0) x_i \geq \mu - \xi_0$$

$$x \in \mathbf{H},$$

where $\mathbf{H}$ is a convex set representing the additional constraints of x.

One way to improve this model is to use fuzzy numbers that suit better the behavior of 'risk'. We can use trapezoidal or even LR-type fuzzy number for this case.

Chapter 5

Mean-variance model with transaction costs

Transaction cost is one of the main concerns for portfolio managers, since it has a significant effect on investment strategy. Ignoring them would result inefficient portfolios. So first let's have a brief overview of transaction costs.

Investors has to pay a certain amount of money after every investment they make whether he/she purchases (invest) or sells (disinvest) assets. The transaction cost associated with the amount of investment of an asset is a non-decreasing concave function which means that the transaction cost is relatively high when the amount of fund is small. This can change after a certain point, so beyond a certain amount of transaction this function turns into a convex function. This happens when the amount of investment into an asset is large and there's not enough supply on the market, so the price will increase which results a convex function.

Let's have n securities, and x_i be the investment proportion in securities i. The i-th security return is a fuzzy number ξ_i (it can be defined as $\xi_i = (p'_i + d_i - p_i)/p_i, i = 1, \ldots, n$ respectively, where p'_i is the estimated closing prices of the securities i in the next year, p_i is the closing prices of the securities i at present and d_i the estimated dividends of the securities i during the coming year). Let α be the maximum risk level the investor can tolerate, and c_i be the transaction cost associated with the investment into the i-th security. The investor wants to maximize the cost effected return at a given specific risk level, so his/her problem can be formulated the following way:

$$\max E[\sum_{i=1}^{n} \xi_i x_i] - \sum_{i=1}^{n} c_i(x_i)$$

$$s.t. Var[\sum_{i=1}^{n} \xi_i x_i] \leq \alpha$$

$$\sum_{i=1}^{n} x_i = 1$$
$$x_i \geq 0, i = 1, \ldots, n$$

5.1 Hybrid intelligent algorithm

Genetic algorithm is considered successful in providing good solutions to many complex optimization problems, and widely used for solving fuzzy optimization problems as well. In solving this problem first a fuzzy simulation is applied to compute the expected value and variance of a fuzzy variable then fuzzy simulation and genetic algorithm are integrated to produce a hybrid intelligent algorithm. With these we will be able to handle all forms of fuzzy membership functions.

5.1.1 Fuzzy simulation

Fuzzy simulation for credibility: [20] [10] Let ξ_i be fuzzy variables with membership functions $\mu_i, i = 1, \ldots, n$. Let $\mathbf{x} = (x_1, \ldots, x_n)$, $\xi = (\xi_1, \ldots, \xi_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$. Let μ denote the membership function of ξ.

For our model we will need to compute the expected values and variances. Since $Var[\mathbf{x}\xi] = E[(\mathbf{x}\xi - E[\mathbf{x}\xi])^2]$, therefore computing $E[f(\mathbf{x}\xi)]$ is enough for us. We know that $E[\xi] = \int_0^\infty Cr\{\xi \geq r\}dr - \int_{-\infty}^0 Cr\{\xi \leq r\}dr$.

The algorithm is the following:

1. Set $E = 0$.
2. Randomly generate $u_{1j}, u_{2j}, \ldots, u_{nj}$ from the ϵ-level sets of $\xi_1, \xi_2, \ldots, \xi_n$ fuzzy variables. Let's have $\mathbf{u_j} = (u_{1j}, u_{2j}, \ldots, u_{nj}), j = 1, \ldots, N$, where ϵ is a sufficiently small positive number and N is a sufficiently large number.
3. Set $a = f(\mathbf{xu_1}) \wedge f(\mathbf{xu_2}) \wedge \ldots \wedge f(\mathbf{xu_N})$ and
   $b = f(\mathbf{xu_1}) \vee f(\mathbf{xu_2}) \vee \ldots \vee f(\mathbf{xu_N})$.
4. Randomly generate r from $[a, b]$.
5. If $r \geq 0$, then $E := E + Cr\{f(\mathbf{x}\xi) \geq r\}$.
6. If $r < 0$, then $E := E - Cr\{f(\mathbf{x}\xi) \leq r\}$.
7. Repeat steps from the fourth to the sixth for N times.
8. $E[\mathbf{x}\xi] = a \vee 0 + b \wedge 0 + E * (b - a)/N$.

In order to compute $E[\mathbf{x}\xi]$ by the definition of the credibilistic expected value we need $Cr\{\mathbf{x}\xi \geq r\}, r \geq 0$ and $Cr\{\mathbf{x}\xi < r\}, r < 0$. Let's denote $\tilde{\mu}$ as the membership function of $\mathbf{x}\xi$

$Cr\{\mathbf{x}\xi \geq r\} = \frac{1}{2}\left(\sup_{y\geq r}\tilde{\mu}(y) + 1 - \sup_{y<r}\tilde{\mu}(y)\right)$. We will randomly generate u_{ij} $(j = 1, \ldots, N)$ from the ϵ-level set of ξ_i, where N is a sufficiently large number. Using Zadeh's extension principle we will calculate results from the following formula:

$$C = \frac{1}{2}\Big(\max_{1\leq j\leq N}\{\min_{1\leq i\leq n}\{\mu_i(u_{ij})\} \mid \mathbf{x}\mathbf{u_j} \geq r\} + 1 -$$

$$\max_{1\leq j\leq N}\{\min_{1\leq i\leq n}\{\mu_i(u_{ij})\} \mid \mathbf{x}\mathbf{u_j} < r\}\Big) :=$$

$$= \frac{1}{2}(N + 1 - K)$$

1. $j := 1, N =: 0, K := 0$.
2. Randomly generate u_{ij} from the ϵ-level set of ξ_i, $i = 1, \ldots, n$, where ϵ is a sufficiently small number. $\mathbf{u_j} = (u_{1j}, u_{2j}, \ldots, u_{nj})$
3. Let's take $m_j := \min_{1\leq i\leq n}\{\mu_i(u_{ij})\}$.
4. If $\mathbf{x}\mathbf{u_j} \geq r$ then update N, if not, update K.
5. $j := j + 1$, and go back to the second step, if $j \leq N$.
6. Return $\frac{1}{2}(N + 1 - K)$.

Fuzzy simulation for possibility: We need the expected value and the variance to optimize our portfolio, but since in possibility theory they are both obtained by the following formulas:

$$E[A] = \int_0^1 (a_1(\gamma) + a_2(\gamma))\gamma d\gamma$$

$$Var(A) = \frac{1}{6}\int_0^1 (a_2(\gamma) - a_1(\gamma))^2\gamma d\gamma,$$

we can use these in our genetic algorithm.

5.1.2 Genetic algorithm

Representation Structure: [20] [10] In order to encode a solution into a chromosome we need a mapping. This mapping between a solution $\mathbf{x} = (x_1, \ldots, x_n)$ and a chromosome $V = (v_1, \ldots, v_n)$ can be the following:

$$x_i = \frac{v_i}{v_1 + v_2 + \ldots v_n}, i = 1, \ldots, n$$

Initialization: Let the number of chromosomes in the population called *pop_size*, and initialize *pop_size* chromosomes randomly. We generate random point $V = (v_1, \ldots, v_n)$ in a $[0,1]^n$ hypercube and check its feasibility: $Var[\sum_{i=1}^{n} v_i x_i] \leq \alpha$. If it's feasible we'll keep it, if not we will regenerate another point from the hypercube until a feasible is obtained. Repeat this process until the initial feasible *pop_size* chromosomes $V_1, \ldots, V_{pop\_size}$ is generated.

Feasibility means that a chromosome $V = (v_1, v_2, \ldots, v_n)$ has to return 1 for $Var[\sum_{i=1}^{n} \xi_i v_i] \leq \alpha$.

Evaluation function: Evaluation function $Eval(V)$ is to assign a probability of reproduction to each chromosome. Chromosomes with higher fitness will have more chance to produce offspring by using roulette wheel selection.

The most popular evaluation function is the rank-based evaluation function, which means that among our chromosomes $V_1, V_2, \ldots, V_n$ with the smaller ordinal number is better. This evaluation function is defined as $Eval(V_i) = a(1-a)^{i-1}$ for each $i = 1, \ldots, n$, where $a \in (0,1)$ is a given parameter.

Selection Process: The selection process is based on spinning the roulette wheel *pop_size* times. In each round one chromosome is selected via the fitness-proportional selection. (The reason of the roulette-association is because the a proportion of the original wheel is assigned to each of the possible selection based on their fitness value.)

Before spinning the wheel we assume use an order relationship among our chromosomes using the their objective value. If two chromosomes have the same objective value then we're indifferent between them and we can rearrange them randomly. Now our *pop_size* chromosomes are arranged from good to bad: $V_1, \ldots, V_{pop\_size}$ (V_1 is the best, V_n is the worst).

We first calculate the cumulative probability for each chromosome V_i:

$$p_0 = 0, p_i = \sum_{j=1}^{i} Eval(V_j), i = 1, \ldots, n,$$

and divide all p_i-s by p_{pop_size}, so that $p_{pop\_size} = 1$. *Note*, that $i = 1$ means the best individual and $i = n$ is the worst.

Generate a random number $r \in (0,1]$, and select a chromosome V_i such that $p_{i-1} < r \leq p_i$. Repeat the last two steps *pop_size* times, so then we have *pop_size* copies of chromosomes. *Note*, that the 'best' chromosome has more chance to represent itself in our selection.

Crossover Operation: Primary operations exist for modifying structures in genetic algorithms. One of them is the crossover operation, in which two chromosomes (later called parents) are combined to from two new solutions. The parents are chosen by the following method:

Let's take a probability parameter $\mathbf{P_c}$ for showing the expected amount of chromosomes going through the crossover operation ($\mathbf{P_c} * pop\_size$), and generate a random r from $(0, 1)$ *pop_size* times. V_i will be selected as a parent if $r < \mathbf{P_c}$ at the i-th selection of r. After this we have the parents: $V_1', V_2', \ldots$, and we simply make couples with pairing them: $(V_1', V_2'), (V_3', V_4'), \ldots$.

The crossover will look like this: Let's take a couple, e.g. (V_1', V_2'), and generate a random number s from $(0, 1)$. Then the two children of the couple will be X, Y, such that:

$$X = sV_1' + (1 - s)V_2', Y = (1 - s)V_1' + sV_2'.$$

If the feasible set is convex, then both children will be in the feasible set. If not, we can still check feasibility of a chromosome. If both children are feasible, we replace the parents with them. If not, we keep the feasible one (if it exist), and redo the process of crossover by generating a new s until we obtain two feasible children, or a given number of cycles is finished.

Mutation This is another way to modify structures in genetic algorithm. Mutation can result new chromosomes, preventing the population from stagnating.

Let's have a probability parameter $\mathbf{P_m}$ for showing the expected number of chromosomes going through the process of mutation. This number is $\mathbf{P_m} * pop\_size$.

We generate a random q from (0,1) *pop_size* times, and select V_i as a parent of a mutation if the actual $q < \mathbf{P_m}$.

For each selected parent $V = (v_1, v_2, \ldots, v_n)$ the mutation will be the following:

Let M be a large positive number. We randomly choose a mutation direction $\mathbf{d} \in \Re^n$. If $V + M * \mathbf{d}$ is feasible, we replace the child with its parent (the child will be part of the next population). If it's not feasible then we set M as a random number between 0 and M, until it's feasible. If we can not find a feasible mutation of a parent in a predetermined number of iterations, then we set $M = 0$.

Hybrid Intelligent Algorithm The hybrid intelligent algorithm will terminate after a given number of cycles. The steps will be the following:

1. Fuzzy simulation and initializing *pop_size* chromosomes.

2. Update the chromosomes by crossover and mutation, in which fuzzy simulation will be used.

3. Calculate the objective values for the chromosomes ($E[\sum_{i=1}^n \xi_i x_i] - \sum_{i=1}^n c_i(x_i)$)

4. Compute the fitness of each chromosome with the objective values.

5. Spinning the roulette wheel for selecting chromosomes.

6. Repeat steps from the second to the fifth for a given number of cycles.

7. Encode best chromosome to get the solution of the portfolio selection.

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