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Metric dimension of distance-regular graphs and finite projective planes

BSc Thesis

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Introduction

The purpose of this thesis is to introduce the notion of metric dimension and summarise part of the results currently known about it especially those relating to distance-regular graphs and projective planes. Also, to present a result pertaining to the metric dimension of finite projective planes, first published in a 2012 article by Tamás Héger and Marcella Takáts [19] and refined further by the present author for the purpose of this thesis.

Metric dimension is a natural graph theoretic concept studied as far back as the 1950's and since then consequently by various authors. It proved to be a quite elusive topic however, as apart from a few general observations and theorems, the majority of the current results detail very special cases of graphs. This is due to the fact that the metric dimension has a strong connection to the structure of a graph and thus graphs with special constructions or generally of interesting properties prove to be more fruitful to investigate.

In this thesis we try to introduce as much of the general theory of the metric dimension as we can while slowly working our way towards the main result concerning projective planes. This involves a basic introduction where we also try to show how (and why) general results are hard to come by; then follows a brief overview of the metric dimension of distance-regular graphs which is a more ordered family providing more ordered results; while our final chapter deals with finite projective planes, a special family of distance-regular graphs where the stronger results concerning those do not yield satisfying results.

Chapters generally start with a detailed introduction containing definitions and proofs to familiarise the reader with the subject which are followed by an assortment of more powerful results to give a broader overview. The last chapter contains in most part the proof of our main theorem in which we precisely determine the metric dimension of finite projective planes of order $q \geqslant 13$.

Contents

Chapter 1

On metric dimension

1.1 Introduction to metric dimension

The metric dimension of a graph is not an unnatural property to consider and was studied as far back as 1953 by Blumenthal [10] while formally only introduced later by Slater [26] and independently by Harary and Melter [18] in 1975 and '76, respectively. The idea of "locating" vertices in a graph by their distances from a fixed set of other vertices can be interesting on its own, however, the topic was also investigated in connection with robot and sonar navigation, chemical drug discovery or strategies for the Mastermind game, among others. (A considerable bibliography can be found in [21].)

This chapter contains a detailed introduction together with a selection of more powerful results to found our general understanding of the topic before moving towards more specialised families of graphs to investigate.

1.1.1 Definitions

Let $\Gamma = (V, E)$ denote a graph on vertex set V and edge set E. Throughout this thesis, all graphs are assumed to be simple (that is, without loops and multiple edges), undirected and connected. The cardinalities of the vertex and edge sets will be denoted by $|V| = n$ and $|E| = m$. The distance between two vertices of Γ, $u, v \in V$ is the least number of edges in a path between u and v (that is, the length of a shortest path between them) and will be denoted by $d(u, v)$. (If the graph in question is not clear from context, we will use a suffix, e.g. $d_\Gamma(u, v)$.) The diameter of a graph is the longest distance between any two of its vertices and will be denoted by d.

A vertex w is said to *resolve* vertices u and v if $d(w,u) \neq d(w,v)$. This notation can be "interpreted" as u and v being distinguishable based only on their distance from w. A set of vertices $S \subset V$ is said to *resolve* a vertex v if for all vertices $u \neq v$ there is a vertex $w \in S$ that resolves u and v. Now it is straightforward to proceed with a definition of a set of vertices that is able to distinguish any pair of vertices in the graph in a similar manner.

Definition 1.1. *A set $S \subset V$ is called a* resolving set *of Γ if and only if it resolves all vertices of V, that is, for every pair of vertices $u, v \in V$, $u \neq v$, there is a vertex $w \in S$ such that $d(u,w) \neq d(v,w)$.*

For an even closer relation to this distinguishing quality we will also present an alternative, equivalent definition. For a vertex $v \in V$ and a set of vertices $S \subset V$, let us denote by $D(v \,|\, S)$ the list of distances $(d(v,s_1),\ldots,d(v,s_k))$ where $s_i \in S$, $i = 1,\ldots,k$, $|S| = k$, with an arbitrary but fixed ordering on the vertices of S.

Definition 1.2. *A set $S \subset V$ is called a* resolving set *of Γ if and only if for every pair of vertices $u, v \in V$, $D(u \,|\, S) = D(v \,|\, S) \Leftrightarrow u = v$. That is, the vertices of Γ are uniquely determined by the list of their distances from the vertices of S.*

It is straightforward to see that this alternative definition is equivalent to the original one. (Two lists of distances are distinct if and only if there is a position where they differ, that is, there is a vertex that resolves the vertices they represent.)

Definition 1.3. *The minimum cardinality of a resolving set in Γ is called the graph's* metric dimension *and will be denoted by $\mu(\Gamma)$. A resolving set of size $\mu(\Gamma)$ is called a* (metric) basis *of Γ.*

For yet another perspective on this concept see Chartrand, Eroh, Johnson, and Oellermann [15] where they give a characterisation of the problem of finding the metric dimension and the metric basis of a graph in terms of an integer programming problem.

Although the following sections will provide many examples, we take a moment here to show two graphs, the complete graph on four vertices and another with an additional fifth vertex in the middle, together with their resolving sets. (The vertices marked black form resolving sets in the graphs.)

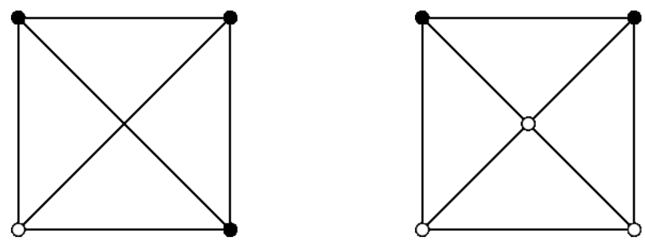

Figure 1.1: Two graphs and their resolving sets.

Note that if we leave out any vertex from the resolving set in the complete graph, the two missing vertices will both share the distance list $(1, 1)$, thus a set like that can not be a resolving set. Also, one vertex can not serve the purposes of a resolving set in either graph, which means that the sets in the figures are in fact metric bases.

The following sections are intended to introduce and bring closer the concept of metric dimension by presenting easy to understand but characteristic results and proofs. The properties detailed here will be used and deliberated upon in the following chapters as we gain more tools by investigating more specific families of graphs.

1.1.2 Bounds on the metric dimension

Note that for any set $W \subset V$, the vertices in W are trivially resolved by W as for any $w \in W$, the distance list $D(w \mid W)$ has a zero at $d(w, w)$, whereas for any other vertex $v \neq w$, $d(v, w)$ is positive. Therefore when testing whether a set of vertices W is a resolving set of Γ, we only need to be concerned with the vertices in $V \setminus W$. Consequently, as a set of vertices trivially resolves itself, V is a trivial resolving set for Γ.

Not much further consideration is needed to verify that $V \setminus \{v\}$ is also a resolving set with any vertex v missing. (For as long as there is only one vertex not in the set, it will be the only one with an all positive distance list). This gives a natural bound on the metric dimension of graphs, namely $1 \leqslant \mu(\Gamma) \leqslant n - 1$. However, if we also know the diameter of our graph, we can obtain slightly better bounds.

Proposition 1.1. *For a graph $\Gamma = (V, E)$ with diameter d and $|V| = n$, $\mu(\Gamma) \leqslant n - d$.*

Proof. Let u and v be two vertices at distance d and let $u = p_0, p_1, \dots, p_d = v$ be the vertices of a path of length d between them. We will show that $S = V \setminus \{p_1, p_2, \dots, p_d\}$ is a resolving set of Γ. For $i = 1, \dots, d$, $d(u, p_i) = i$ and it is the only vertex of the path that has this distance from u, thus u resolves any two vertices of the path and any vertices not on the path are resolved trivially. As S is a resolving set of size $n - d$, $\mu(\Gamma) \leqslant n - d$. $\square$

A lower bound can be obtained by considering the possible number of distance lists, that is, how many vertices can there be while still being distinguishable with regard to a specific resolving set. For writing the next proposition in a simpler form, for positive integers n and d, let $f(n, d)$ denote the least positive integer k for which $k + d^k \geqslant n$.

Proposition 1.2. *For a graph $\Gamma = (V, E)$ with diameter d and $|V| = n$, $f(n, d) \leqslant \mu(\Gamma)$.*

Proof. Let $\mu(\Gamma) = k$ and let B be a metric basis of Γ. The distance lists of the vertices of B are unique in the way that they have exactly one zero in them. Every other vertex in $V \setminus B$ has a distance list comprised of positive integers not exceeding d. As all lists need to be distinct, it follows that $n - k \leqslant d^k$ hence $f(n, d) \leqslant k = \mu(\Gamma)$. $\square$

It is worth noting that if we write the statement of this proposition as $n \leqslant \mu(\Gamma) + d^{\mu}(\Gamma)$, it asymptotically gives a $\log n$ lower bound on the metric dimension. (More precisely it gives the asymptotic bound for $d > 1$, while for complete graphs, where $d = 1$ and the logarithm is not defined, it gives a sharp bound.) We will see in the following chapter that in certain special cases, when we can give strong upper bounds too, this is enough to characterise the asymptotic behaviour of the metric dimension.

1.1.3 Extremal values

With very similar trains of thought we can characterise the graphs having extreme metric dimensions. We will denote by P_n the path on n vertices (that is, the path of length $n - 1$) and by K_n the complete graph on n vertices.

Proposition 1.3. $\mu(\Gamma) = 1$ *if and only if* $\Gamma = P_n$, *that is, it is a path on* $n \geqslant 1$ *vertices.*

Proof. In the proof of Proposition 1.1, we have already seen that an end vertex of a path resolves all its vertices.

Now let Γ be a graph on n vertices with $\mu(\Gamma) = 1$ and let r be the sole vertex of a basis of Γ. Now for every vertex v the distance list $D(v \mid \{r\}) = d(v, r)$, is a non-negative integer less than n. Since these representations of the vertices are distinct, there is a vertex u with $d(u, r) = n - 1$. Consequently, the diameter of Γ is $n-1$ which can only be true if Γ is a path on n vertices. □

Proposition 1.4. $\mu(\Gamma) = n - 1$ *if and only if* $\Gamma = K_n$*, that is, it is the complete graph on* $n \geqslant 2$ *vertices.*

Proof. Let Γ be the complete graph on $n \geqslant 2$ vertices and let B be a basis of Γ. For every vertex $v \notin B$, the distance list $D(v \mid B)$ is comprised of entirely ones and that implies that there can be only one such vertex. By Proposition 1.1 if Γ is not the complete graph then $\mu(\Gamma) \leqslant n - 2$.

(Note that Propositions 1.1 and 1.2, together, also give the very same result.) □

We will take a moment here to present an example that shows how a relatively simple construction can yield a wide range of metric dimensions; more precisely we will show graphs all on n vertices with $n - 1$ edges while their metric dimension ranges from 1 to $n - 2$. First consider a star graph on n vertices, a single vertex connected to $n - 1$ other vertices of degree 1. It is easy to see that this star has a metric dimension of $n-2$ as two "outer" vertices have the same distance list from any set of vertices that does not contain them. This means that at most one of these may be amiss from a resolving set, while the "central" vertex can also be omitted (as it is the only one whose distance is 1 from all other vertices) leaving us with a basis on $n - 2$ vertices.

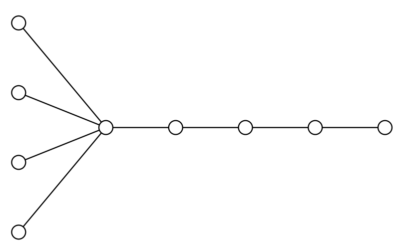

Figure 1.2: The graph B_4.

Now consider the following graph B_k on n vertices: have a vertex c that is the end of a path on $n-k$ vertices and attach k additional vertices of degree 1 to c. Now our previous consideration about stars can be adopted for the neighbours of c in the sense that no two of the degree 1 neighbours may be missing from a resolving set while c's neighbour on the path can only be left out if all the degree 1 neighbours or another vertex of the path is inside, meaning that at least k vertices must be in a resolving set of B_k. But (for example) the k neighbours of c having degree 1 do form a resolving set, as they naturally resolve themselves while the vertices of the path are resolved because there is exactly one at distance $1, 2, \ldots, n-k$ from any one of the vertices in the set. (Note that this shows that Proposition 1.1 is sharp as B_k has diameter $n-k$ and metric dimension $n-(n-k)$.)

This basic example shows how more than the number of vertices and edges, the underlying structure of a graph is the important part in determining its metric dimension. Some more families of graphs with easily characterisable metric dimensions follow, closing this section.

Graphs on n vertices having a metric dimension of $n-2$ have been fully characterised by Chartrand, Eroh, Johnson and Oellermann [15]. For graphs G and H, we denote by $G \cup H$ the disjoint union of the two graphs, and $G \oplus H$ denotes the graph obtained by joining every vertex of G with every vertex of H in $G \cup H$. $K_{s,t}$ will denote a complete bipartite graph, that is a graph with vertex set $V = S \cup T$ where $|S| = s$, $|T| = t$, edges go only between S and T, and every vertex in S is adjacent to every vertex in T. For a graph Γ, $\overline{\Gamma}$ will denote the complement of the graph, that is, a graph on the same vertex set with edges exactly between those vertices that are not adjacent in Γ. Using this notation, the characterisation is as follows.

Theorem 1.5. *[15] If G is a graph with $|V| = n \geq 4$, then $\mu(G) = n-2$ if and only if $G = K_{s,t}$ $(s, t \geq 1)$, $G = K_s \oplus \overline{K_t}$ $(s \geq 1, t \geq 2)$, or $G = K_s \oplus (K_1 \cup K_t)$ $(s, t \geq 1)$.*

On the other end of the spectrum, it is easy to see that for every $n \geq 3$, the metric dimension of a cycle of size n, denoted by C_n is 2. (Two neighbouring vertices always suffice as a basis.) However, graphs with metric dimension 2 can be much more complex than a cycle. The following propositions from Khuller, Raghavachari and Rosenfeld [24] give a few restrictions on how these graphs can look like.

Proposition 1.6. *[24] If $\mu(\Gamma) = 2$, then Γ cannot have K_5 or $K_{3,3}$ as a subgraph.*

Proposition 1.7. *[24] Whenever $\mu(\Gamma) = 2$, and $a, b \in V$ are a basis of Γ, the following holds.*

1. *The shortest path between a and b is unique.*

2. *The degrees of a and b are at most 3.*

3. *The degrees of the vertices on the shortest path between a and b are at most 5.*

1.2 Further results

In this section a variety of further results are presented on the metric dimension in general and also for specific graphs, while mostly omitting their proofs or even the precise statements themselves as they are predominantly beyond the scope and topic of this thesis. Their inclusion, however, is in the hope that they still provide the necessary perspective to further understand the fundamentals of the topic and confidently move along with our work.

1.2.1 General bounds

We have seen rudimentary results pertaining to the possible size of a metric basis in Propositions 1.1 and 1.2 which implicitly also provided restrictions on the size of a graph with given diameter and metric dimension. Much more detailed inspection of this question has been done by Hernando, Mora, Pelayo, Seara and Wood in their article [21] where they managed to derive a sharp upper bound using a more precise proof than we used for Proposition 1.2, deliberating the possible distance lists of the vertices of a graph.

Theorem 1.8. *[21] For all integers $d \geqslant 2$ and $\mu \geqslant 1$, the number of vertices a connected graph Γ with diameter d and metric dimension μ can have is*

$$n \leqslant \left(\left\lfloor \frac{2d}{3} \right\rfloor + 1 \right)^{\mu} + \mu \sum_{i=1}^{\lceil d/3 \rceil} (2i-1)^{\mu-1}.$$

For all such d and μ there exists a graph with diameter d, metric dimension μ and exactly this many vertices.

They also characterised graphs having minimum size with diameter d and metric dimension μ, that is, having a size of $\mu + d$ as was implicitly shown to be minimum by Proposition 1.1. They used the notion of twin vertices (vertices with identical neighbourhoods) to describe the twin graph (made up of the equivalence classes of the relation of being twins in the graph) which has to fit certain restrictions when the graph is of such size. As their final results are rather technical to write out and do not necessarily relate to the central topic of this thesis, for the sake of brevity, we will not detail them here. The curious reader should refer to [21].

A natural question besides that of bounding the size of a graph in terms of its metric dimension and diameter is whether there are graphs with unique bases or with bases of any size. The answer to this question proved to be positive as shown by Buckzkowski, Chartrand, Poisson and Zhang [13]. While the statement of Theorem 1.9 was implicitly shown already by our example of the B_k graphs, they also proved that for any metric dimension, there is a graph with a unique basis.

Theorem 1.9. *[13] For any pair of integers n, μ with $1 \leqslant \mu \leqslant n-1$ there exists a graph on n vertices with a metric dimension of μ.*

Theorem 1.10. *[13] For any integer $\mu \geqslant 2$ there exists a graph with metric dimension μ that has a unique metric basis.*

The last fundamental result we mention here will show however that, besides these naturally studied general cases, the metric dimension of an "arbitrary" graph is harder to talk about. In their article, Bollobás, Mitsche and Prałat [11] investigated the expected metric dimension of the random graph $G(n, p)$ which is a graph on n vertices where every edge has probability p of being present. Their results are rather technical thus we will only present a paraphrase of their explanation on the strange behaviour of the metric dimension of these graphs. They observed that (given some sensible restrictions on the expected degree of these random graphs) the metric dimension (in the sense of its asymptotic behaviour) shows a "zig-zag" pattern when viewed as a function of p. That is, the metric dimension is not monotone, but provably fluctuates as p (and consequently the number of edges in the graph) changes.

The intuitive explanation they give of their starting thought is as follows. Whenever the graph is dense enough to locally (that is, from any one vertex) "look" the same then, if we denote by $\Gamma_i(v)$ the set of vertices at distance i from a vertex v, the cardinality of these sets are roughly the same for all vertices. Thus if we find the two sets where $|\Gamma_i(v)|$ is the largest for a given vertex, then their relative size (the ratio between their cardinality) is also about constant across vertices. It turns out that this ratio is of crucial importance to the metric dimension as if these two largest sets are near equal in size than a typical vertex added to a possible resolving set distinguishes a lot of pairs of vertices and thus the metric dimension can be low. On the other hand if the smaller of the two sets is much smaller, then a typical vertex distinguishes only these fewer vertices from the rest and thus a metric basis has to be larger to cover all grounds. By formalizing these thoughts Bollobás et al. proved that this property is indeed non-monotonic in p and causes the metric dimension to change considerably too.

1.2.2 Metric dimension of specific graphs

The general problem of determining the metric dimension of a graph is NP-complete as shown (among others) by a reduction to 3-SAT by Khuller et al. [24]. In fact, Epstein et al. [17] mention that the problem is NP-complete even if restricted to planar graphs and then extend these results and show that these hard families of graphs include the bipartite, co-bipartite, and split graphs. Thus, considering the results presented in the previous sections too, talking about the metric dimension in general seems to be hard so it is advantageous to investigate graphs with special structural properties. These properties can either stem from their unique definitions or from the way they are constructed from other, simpler graphs. We will list a few examples to give a little insight into the state of the art on metric dimension.

Regular graphs are a natural choice for graphs with structural order, and in fact we will get back to them in the following chapter when we discuss distance-regular graphs. To give here an interesting result nonetheless, Bača et al. [3] investigated some special high valency, regular bipartite graphs and arrived at the following results.

Theorem 1.11. *[3] If $k \geqslant 3$, and Γ is a $k-1$ regular bipartite graph with k nodes in each bipartite half, then $\mu(\Gamma) = k-1$.*

Theorem 1.12. *[3] If $k \geqslant 5$, and H is a connected graph with $H = K_{k,k} - E(C_{2k})$ (that is, a complete bipartite graph with the edges of a Hamiltonian* *cycle deleted), then* $\mu(H) = \left\lfloor \frac{4k}{5} \right\rfloor$*.*

Next we will take a look at the results regarding the metric dimension of trees, already present in the papers of Harary and Melter [18] and Slater [26], as communicated by Chartrand et al. [15]. Trees form a naturally simple family of graphs to consider and while the results themselves are quite elementary, they are a bit different than what we saw so far and still worth writing out.

We will call a vertex of degree at least 3 a *major vertex* of Γ. A vertex u of degree 1 is a *terminal vertex of a major vertex* v if for every other major vertex w, $d(u,v) < d(u,w)$ holds. The *terminal degree*, of a major vertex v is the number of terminal vertices of v. A major vertex is an *exterior major vertex* if it has a positive terminal degree. Let $\sigma(\Gamma)$ and $ex(\Gamma)$ denote the sum of the terminal degrees of the major vertices of Γ and the number of exterior major vertices of Γ, respectively. With this notation the results are as follows.

Lemma 1.13. *[15] For every graph* Γ, $\mu(\Gamma) \geqslant \sigma(\Gamma) - ex(\Gamma)$*.*

Theorem 1.14. *[15] For every tree* T *that is not a path,* $\mu(T) = \sigma(T) - ex(T)$*.*

If T is a tree that is not a path, a metric basis of T can be constructed by having every terminal vertex except one for each exterior major vertex of T in it. (Note here that the example shown earlier with the graphs B_k is a special case of this theorem and base construction.)

These next few results were compiled by Imran, Bokhary, Ahmad and Semaničová-Feňovčiková [22] and concern simpler graphs described using the already introduced $\oplus$ operation (see the paragraph right before Theorem 1.5). The wheel, W_n, is defined as $W_n = K_1 \oplus C_n$ for $n \geqslant 3$, that is, a cycle of size n with a single added vertex adjacent to all the vertices in the cycle. Similarly a fan, f_n, is defined as $f_n = K_1 \oplus P_n$, while the Jahangir graph J_{2n} for $n \geqslant 2$ (also known as the gear graph) is obtained from the wheel W_{2n} by alternately deleting n "spokes".

Theorem 1.15. *[22] For wheels* W_n*, fans* f_n*, and Jahangir graphs* J_{2n}*, the following hold.*

1. *For* $n \geqslant 7$, $\mu(W_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor$*.*

2. *For* $n \geqslant 7$, $\mu(f_n) = \left\lfloor \frac{2n+2}{5} \right\rfloor$.

3. *For* $n \geqslant 4$, $\mu(J_{2n}) = \left\lfloor \frac{2n}{3} \right\rfloor$.

As mentioned before, studying new graphs constructed from structurally simpler ones is also a common approach. We will look at a few operations to do so and some of the so far discovered properties of the resulting graphs.

Poisson and Zhang [25] introduce a procedure to form new graphs from two smaller ones, constructing the so-called identification graph. Having two nontrivial graphs Γ_1 and Γ_2, and $v_1 \in \Gamma_1$, $v_2 \in \Gamma_2$, their *identification graph*, $\Gamma = \Gamma[\Gamma_1, \Gamma_2, v_1, v_2]$ is obtained by having the union of the two graphs but identifying v_1 and v_2 so that in the new graph $v_1 = v_2$. For the metric dimension of this identification graph, the following holds.

Proposition 1.16. *[25] Let* Γ_1 *and* Γ_2*, be two nontrivial graphs and* $v_1 \in \Gamma_1$*,* $v_2 \in \Gamma_2$*,* $\Gamma = \Gamma[\Gamma_1, \Gamma_2, v_1, v_2]$ *is their identification graph. Then*

$$\mu(\Gamma) \geqslant \mu(\Gamma_1) + \mu(\Gamma_2) - 2.$$

Ishwadi, Baskoro, Salman and Simanjutak [23] go further and define the *amalgamation* operation for a finite collection of graphs $\{\Gamma_i\}$, each having a fixed vertex, v_i called a *terminal*. The amalgamation $Amal\{\Gamma_i, v_i\}$ is formed by having all the Γ_i's and identifying their terminals. They proceed to investigate amalgamations of cycles and arrive at the following result.

Theorem 1.17. *[23] If* $Amal\{C_i, v_i\}$ *is an amalgamation of cycles that consists of* t_1 *number of odd cycles and* t_2 *of even cycles, then*

$$\mu(Amal\{C_i, v_i\}) = \begin{cases} t_1, & \text{if } t_2 = 0, \\ t_1 + 2t_2 - 1 & \text{otherwise.} \end{cases}$$

Another common way to construct more complex graphs from simpler ones is via the Cartesian product. With graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$, their *Cartesian product* $\Gamma_1 \times \Gamma_2$ is a graph on vertex set $V_1 \times V_2$ with two vertices (u_1, u_2) and (v_1, v_2) adjacent if and only if $u_1 = v_1$ and $u_2 v_2 \in E_2$ or $u_2 = v_2$ and $u_1 v_1 \in E_1$. For a wide array of results concerning the metric dimension of Cartesian products of graphs, one should refer to the survey article of Cáceres and Puertas [20]. We present here a bound on the metric dimension from them.

Proposition 1.18. *[20] For graphs Γ_1 and Γ_2 the following holds.*

$$\max\{\mu(\Gamma_1),\, \mu(\Gamma_2)\} \leqslant \mu(\Gamma_1 \times \Gamma_2).$$

$$\mu(\Gamma_1 \times \Gamma_2) \leqslant \min\{\mu(\Gamma_1) + |\Gamma_2|,\, \mu(\Gamma_2) + |\Gamma_1|\} - 1.$$

The results concerning the wheels, fans, and Jahangir graphs from Imran et al. [22] were presented as counterpoints to other infinite families of graphs whose metric dimension does not grow as the graphs themselves do. They introduce three new families of graphs and prove that they have constant metric dimension; flower snarks (a special family of 3-regular graphs), graphs of convex polytopes and Harary graphs. For details on these results, refer to [22]. We will close this section with some simpler results they reprint from Hernando et al. [14] concerning Cartesian products that also have constant metric dimension.

Theorem 1.19. *[14] For integers $s, t \geqslant 3$, the following is true.*

$$\mu(P_s \times C_t) = \begin{cases} 2, & \text{if } t \text{ is odd;} \\ 3, & \text{if } t \text{ is even.} \end{cases}$$

$$\mu(C_s \times C_t) = \begin{cases} 3, & \text{if } s \text{ or } t \text{ is odd;} \\ 4, & \text{otherwise.} \end{cases}$$

Summary

This chapter was intended to serve as a general introduction to the concept of metric dimension. After defining the concept and investigating its characteristics in simpler graphs, we saw more powerful results that suggested that it would be fruitful to take a look at graphs with some special structural regularity; we did so presenting some assorted results. We will continue this line of thinking focusing our attention on a special family of graphs.

Distance-transitivity is the property of a graph that given two vertices u and v at distance i, for any other two vertices u' and v' at distance i there is an automorphism of the graph that carries u to u' and v to v'. We see that this property does not allow vertices to be much different from each other regarding their neighbourhood; an attribute we saw in the result of Bollobás et al. to be quite important when investigating the metric dimension.

Distance-transitive graphs are many and are widely studied while, in fact, the following chapter will deal with an even bigger family, a generalization of the distance-transitive concept: distance-regular graphs.

Chapter 2

On distance-regular graphs

2.1 Introduction to distance-regular graphs

Distance-regular graphs are a combinatorial generalisation of distance-transitive ones defined explicitly by constrictions on regularity and neighbourhood rather than implicitly via the automorphisms. It really is a generalisation, all distance-transitive graphs are distance-regular while the reverse is not true. It is a family of graphs that has been studied extensively and has plenty of literature, take for example the book of Brouwer, Cohen and Neumaier [12], that we will cite in the introduction frequently. Most results in this chapter however come from articles of Robert F. Bailey [4, 5, 6, 7, 8] who investigated the question of metric dimension and distance-regular graphs thoroughly.

The structural order of these graphs makes it easier to talk about their metric dimension and consequently more powerful results have been derived, as we will see. They are still fairly many however, and this makes the results we present here, after a brief introduction, a bit more general than most of the ones we have seen so far.

$\Gamma = (V, E)$ denotes a simple, connected graph as it did before. The set of vertices at distance $i = 0, 1, \ldots, d$ from a vertex $v \in V$ will be denoted by $\Gamma_i(v)$. Now we define the central concept of this chapter.

Definition 2.1. *A graph Γ with diameter d is called* distance-regular *if for all $i = 0, 1, \ldots, d$ and for all vertices $u, v \in V$ at distance $i = d(u, v)$, the number of neighbours of u at distances $i-1$, i, $i+1$ from v depend only on the distance i and not on the choices of u and v. More precisely, if there* *are constants* a_i, b_i, c_i *(called* intersection numbers *or* parameters*) such that among the neighbours of* u*, there are exactly* c_i *in* $\Gamma_{i-1}(v)$, a_i *in* $\Gamma_i(v)$ *and* b_i *in* $\Gamma_{i+1}(v)$*.*

It follows that if Γ is distance-regular, then it is regular with valency $k = b_0 = c_i + a_i + b_i$ for all $i = 0, 1, \ldots, d$. As the intersection numbers a_i can be expressed in terms of the others, it is standard to put those into the so called *intersection array*:

$$\left\{ \begin{array}{cccc} b_0 & b_1 & \ldots & b_{d-1} \\ c_1 & c_2 & \ldots & c_d \end{array} \right\}.$$

Note that $b_d = 0$ and $c_0 = 0$ are not included whereas $c_1 = 1$ is, which makes all numbers of the intersection array positive integers as shown by this next proposition which poses some restrictions on the intersection array.

Proposition 2.1. *[12] For a distance-regular graph with valency* k*, the following hold for the numbers of its intersection array.*

1. $0 = b_d < b_{d-1} \leqslant b_{d-2} \leqslant \ldots \leqslant b_1 < b_0 = k.$

2. $1 = c_1 \leqslant c_2 \leqslant \ldots \leqslant c_d \leqslant k.$

3. *If* $i + j \leqslant d$*, then* $c_i \leqslant b_j$*.*

Proof. For *1.* and *2.*, consider two vertices u, v at distance $i = 1, 2, \ldots, d$, and a vertex w adjacent to u and at distance $i - 1$ from v. Now the c_{i-1} neighbours of v at distance $i - 2$ from w are all at distance $i - 1$ from u, therefore $c_{i-1} \leqslant c_i$. Similarly, the b_i neighbours of v that are at distance $i+1$ from u are at distance i from w, hence $b_i \leqslant b_{i-1}$.

For *3.*, consider two vertices u and v at distance $i + j$ and a vertex w at distance i from u and j from v. Now the c_i neighbours of w at distance $i - 1$ from u are at distance $j + 1$ from v and thus $c_i \leqslant b_j$. □

Also, the number of vertices can be obtained using only the numbers in the intersection array. Moreover, it is also true that every vertex has a constant number of vertices at any given distance i, that is, there are numbers k_i such that $k_i = |\Gamma_i(v)|$ for every $v \in V$ with

$$k_0 = 1, \quad k_1 = k, \quad k_{i+1} = k_i b_i / c_{i+1} \qquad (i = 0, 1, \ldots, d - 1).$$

The values of k_0 and k_1 are straightforward to see, while the rest follows by counting the number of edges between $\Gamma_i(v)$ and $\Gamma_{i+1}(v)$. The number of vertices now follows as $n = k_0 + k_1 + \cdots + k_d$. (Note here that this property, the existence of these numbers k_i, is exactly what Bollobás et al. described as making it easier to talk about the metric dimension in their result that we detailed at the end of section 1.2.1.)

The distance-regular graphs of diameter 2 are considered special as they are precisely the connected strongly regular graphs. A graph is called *strongly regular* if it is regular and there are constants λ and μ such that any two adjacent vertices have λ common neighbours while any two non-adjacent vertices have μ common neighbours. Then if the graph has n vertices and its valency is k, then the strongly regular graph is said to have parameters (n, k, λ, μ). In terms of these parameters a strongly regular graph has the intersection array

$$\left\{ \begin{array}{cc} k & k-1-\lambda \\ 1 & \mu \end{array} \right\}.$$

Some easy examples of distance-regular graphs to consider are complete graphs and cycles. Complete graphs with $n > 1$ are distance-regular with the unique intersection array

$$\left\{ \begin{array}{c} n-1 \\ 1 \end{array} \right\}.$$

Cycles are the distance-regular graphs with valency 2 and have intersection arrays

$$\left\{ \begin{array}{cccc} 2 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \end{array} \right\} \text{ or } \left\{ \begin{array}{ccccc} 2 & 1 & \dots & 1 & 1 \\ 1 & 1 & \dots & 1 & 2 \end{array} \right\}$$

whenever n is odd or even, respectively.

The well-known Petersen graph is also distance-regular (strongly regular in fact as it has diameter 2) and it is uniquely determined by its intersection array

$$\left\{ \begin{array}{cc} 3 & 2 \\ 1 & 1 \end{array} \right\}.$$

2.2 Metric dimension of specific distance-regular graphs

To have a brief overview of general results we will take a look at some specific families of distance-regular graphs, whose metric dimension has been studied, before addressing the main question of this chapter in the following two sections. This section is intended to introduce the notion of distance-regular graphs a bit further and give a few examples.

First up is a quite straightforward result from Bailey concerning complete multipartite graphs which, in the special case when the parts are equal in size, are strongly regular and thus distance-regular graphs.

Proposition 2.2. *[7] Let* $\Gamma = K_{k_1,\dots,k_r}$ *be a complete multipartite graph with* r *parts of sizes* $k_1, \dots, k_r$*, with* $r \geqslant 2$*. Then we have* $\mu(\Gamma) = \sum_{i=1}^{r}(k_i - 1)$*.*

Another result worth mentioning here concerns a special part of the family of Hamming graphs, which are defined as Cartesian products (see the paragraph before Proposition 1.18). They were studied by Cáceres et al. [14].

Definition 2.2. *The* Hamming graph*,* $H(s,t)$*, is a Cartesian product of cliques,*

$$H(s,t) = \underbrace{K_t \times K_t \times \cdots \times K_t}_{s \text{ times}}.$$

The vertices of the Hamming graph $H(s,t)$ can be thought of as vectors of length s over $\{1,\dots,t\}$, with two vectors being adjacent if and only if they differ in exactly one coordinate (that is, their Hamming distance is 1). Two special cases will be especially important to us because of their distance-regularity.

First off are the square lattice graphs, $H(2,t)$, which can be thought of as having as vertices the points of a square grid with two vertices being adjacent if and only if they are in the same column or row (also known as the rook graph, for obvious reasons). These graphs are in fact strongly regular: it is easy to see they have diameter 2, every vertex has $2(t-1)$ neighbours ($t-1$ on its row and again on its column), any two adjacent vertices have $t-2$ common neighbours (the other points on the column or row that contains them both) and any two non-adjacent vertices have 2 common neighbours (the two other corners of the rectangle they span). That is, the square lattice graph $H(2,t)$ is distance-regular and its intersection array is

$$\left\{\begin{array}{cc} 2t-2 & t-1 \\ 1 & 2 \end{array}\right\}.$$

In their article, Cáceres et al. proved the following theorem concerning the metric dimension of Cartesian products of cliques which gives as a corollary an exact value for the square lattice graphs.

Theorem 2.3. *[14] For all $1 \leqslant s \leqslant t$,*

$$\mu(K_s \times K_t) = \begin{cases} \lfloor \frac{2}{3}(s+t-1) \rfloor & \text{if } s \leqslant t \leqslant 2s+1, \\ t-1 & \text{if } 2s+1 \leqslant t. \end{cases}$$

Corollary 2.4. *[14] For the square lattice graphs, with $t \geqslant 1$,*

$$\mu(H(2,t)) = \left\lfloor \frac{2}{3}(2t-1) \right\rfloor .$$

The other special case is of the hypercubes, $H(s,2)$. They have been studied in terms of the metric dimension by A. F. Beardon [9] with techniques from linear algebra. Interestingly, despite their regular structure, the exact values for the metric dimension of a hypercube of arbitrary size are not known. Beardon reduced the question of whether a set of vertices is a resolving set for the hypercube to whether a set of linear equations have a non-trivial solution.

More widely studied families are the Johnson and Kneser graphs. These few selected results are from Bailey et al. [4].

Definition 2.3. *The* Johnson graph*, $J(s,t)$ and the* Kneser graph*, $K(s,t)$ both have as vertex sets the t-subsets of the s-set $[s] = \{1,\ldots,s\}$, with edges between subsets in $J(s,t)$ if and only if their intersection has a size of $t-1$, and in $K(s,t)$ if and only if the subsets are disjoint.*

It is straightforward to see that the Kneser graph is connected only if $s > 2t$ (if $s < 2t$, there are no edges, if $s = 2t$, the graph is a perfect matching). Also the Johnson graphs $J(s,t)$ and $J(s,s-t)$ are isomorphic, so in the following results we only need to consider the case $s > 2t$ for Kneser and $s \geqslant 2t$ for Johnson graphs.

Johnson graphs are always distance-regular, while the Kneser graphs only in the special cases of $K(s, 2)$ and $K(2t + 1, t)$ (also known as the Odd graph). Despite this, it is beneficial to investigate the metric dimension of both because of their connection shown by the following Lemma.

Lemma 2.5. *[4] Whenever $s > 2t$, any resolving set R of the Kneser graph $K(s, t)$ is a resolving set for the Johnson graph $J(s, t)$. Thus $\mu(J(s, t)) \leqslant \mu(K(s, t))$.*

Exact values for the metric dimensions of these graphs have been determined only in the special cases of $J(s, 2)$ and $K(s, 2)$ by Bailey and Cameron in a previous article [5]. Their result is as follows.

Theorem 2.6. *[5] Whenever $s \geqslant 6$, for the metric dimension of the Johnson graph $J(s, 2)$ and the Kneser graph $K(s, 2)$, where $s \equiv i \pmod 3$ (for $i \in \{0, 1, 2\}$), we have*

$$\mu(J(s, 2)) = \mu(K(s, 2)) = \frac{2}{3}(s - i) + i.$$

Bailey et al. go on to provide various bounds on the metric dimension of Johnson and Kneser graphs stemming from constructions of resolving sets using symmetric designs, projective and affine planes and more. For details, one should refer to their article [4].

A survey-like article [7], collecting numerous precise values for metric dimension of small distance regular graphs, was also written by Bailey. It presents results of computer calculations for all distance regular graphs on up to 34 vertices, for distance-regular graphs of valency 3 and 4 on up to 189 vertices and also low-valency distance-transitive graphs for up to valency 13, and up to 100 vertices.

2.3 Primitive and imprimitive graphs

2.3.1 Definitions

Primitivity of a distance-regular graph is an important attribute to consider as we will see shortly in light of the results of Babai, introduced in the next section. For our definitions, let the *distance-i graph* of Γ, denoted by Γ_i, be a graph with vertex set V where two vertices u and v are adjacent if and only if $d_\Gamma(u, v) = i$.

Definition 2.4. *A (distance-regular) graph* Γ *with diameter* d *is called* primitive *if all its distance-*i *graphs,* Γ_i $i = 1, 2, \ldots, d$*, are connected. It is* imprimitive *otherwise.*

Easy examples of imprimitive graphs are the bipartite graphs as their distance-2 graph consists of two disconnected parts naturally arising from the bipartition. Another family of imprimitive graphs, important in the distance-regular case, is that of the antipodal graphs. A graph of diameter d is called *antipodal* if its distance-d graph consists of a disjoint union of cliques. These cliques will be referred to as *antipodal classes* and if they all have size t then Γ is said to be t*-antipodal*. An easy example of a t-antipodal graph is the complete multipartite graph $K_{t,\ldots,t}$.

It is possible for a distance-regular graph to be both bipartite and antipodal, the hypercubes providing a straightforward example. As we have seen, the hypercube $H(n, 2)$ can be seen as having as vertices the vectors $\{0, 1\}^n$ and thus a bipartition naturally arises by having the vertices whose coordinates add up to an even number in one part and the rest in the other. Adjacent vertices differ in exactly one coordinate and the difference is exactly 1, thus the parity of the sum necessarily changes between neighbours. For seeing the antipodality, we only need to observe that for any vertex there is exactly one other whose every coordinate is different and thus the distance-d graph (which is the distance-n graph in fact) is composed of K_2 graphs, disjoint edges.

In fact, a theorem of D. H. Smith (who proved it for distance-transitive graphs only in [27]) shows that for distance-regular graphs other than cycles these aforementioned two are the only possibilities for being imprimitive.

Theorem 2.7. *[12] An imprimitive distance-regular graph of valency* $k \geqslant 3$ *is bipartite or antipodal (or both).*

When Γ is connected and bipartite, the distance-2 graph Γ_2 has two connected components; the graphs induced by these components are called the *halved graphs* of Γ and will generally be referred to as Γ^+ and Γ^-. If Γ is antipodal, its *folded graph*, denoted by $\widetilde{\Gamma}$, is defined as having the antipodal classes as vertices with two classes being adjacent in $\widetilde{\Gamma}$ if and only if they contain vertices adjacent in Γ.

As seen in the book of Brouwer et al. [12, §4.2/A,B], these operations of halving and folding do not change the distance-regularity of a graph but may be used to reduce imprimitive graphs to primitive ones. In particular an imprimitive distance-regular graph with valency $k \geqslant 3$ may be reduced to a primitive one by halving at most once and folding at most once. Also, the intersection arrays of the halved and folded graphs may be obtained from that of Γ.

An extended version of Smith's theorem, categorising all distance-regular graphs into classes based on the primitivity and structure of their halved and folded graphs, has been published by Alfuraidan and Hall [1]. They regarded primitive graphs as a generic case for distance-regular graphs and then divided the imprimitive ones into eleven classes based on how they reduce to this generic case with the operations of halving and folding. This classification provided a base for studying the metric dimension of distance regular graphs in connection with their primitivity by Bailey. His work serves as an outline for the next section.

2.3.2 Metric dimension of primitive and imprimitive distance-regular graphs

The main goal of Bailey in his aforementioned article [8] was to characterise the asymptotic behaviour of the metric dimension of distance-regular graphs. As we also mentioned, the generic case was of the primitive distance-regular graphs motivated by a strong result derived from works of László Babai [2] on the order of permutation groups. These results, reframed for the case of the metric dimension by Bailey, are as follows.

Theorem 2.8. *[8] Let Γ be a primitive, distance-regular graph on n vertices, with valency $k \geqslant 3$ and diameter $d \geqslant 2$. Then*

1. $\mu(\Gamma) < 4\sqrt{n}\log n$;
2. *if $d = 2$ (that is, the graph is strongly regular), then*
   (a) $\mu(\Gamma) < 2\sqrt{n}\log n$ *and*
   (b) $\mu(\Gamma) < \dfrac{2n^2}{k(n-k)}\log n < \dfrac{4n}{k}\log n$ *(where $k \leqslant n/2$);*
3. *if $K_\Gamma = \max\{k_1, \ldots, k_d\}$ is the maximum size of a set of vertices at a given distance from any vertex of Γ, we have*

$$\mu(\Gamma) < 2d\frac{n}{n-K_\Gamma}\log n.$$

This is a powerful result, incomparable to our previously presented trivial bounds. An example from Bailey shows that there are families where this theorem is enough to characterise the asymptotic behaviour of their metric dimension. If q is a prime power such that $q \equiv 1 \pmod 4$, then the *Paley graph* P_q has vertex set $\mathbb{F}_q$, the finite field on q elements, and two of them are adjacent if and only if their difference is a square in $\mathbb{F}_q$. P_q is strongly regular with valency $\frac{1}{2}(q-1)$ and thus *(b)* in our theorem gives $\mu(P_q) \leqslant 8\frac{q}{q-1}\log q$ which together with the trivial bound given in Proposition 1.2 gives an asymptotic magnitude of $\log q$ for the metric dimension of the Paley graphs.

The already mentioned square lattice graphs show however that the $\sqrt{n}$ factor cannot be eliminated in general. The square lattice graph $H(2,t)$ is by definition $K_t \times K_t$ and thus has t^2 vertices and is strongly regular which, using *(a)* of the theorem, gives $\mu(H(2,t)) \leqslant 4t\log t$. We know from our previously presented Proposition 2.4 however that the exact value is $\left\lfloor \frac{2}{3}(2t-1) \right\rfloor$.

Establishing this foundation with the primitive case is essential, but for imprimitive graphs we need further considerations. As we have seen that imprimitive distance-regular graphs may be reduced to primitive ones by the operations of halving and folding, it is desirable to obtain relationships between the resolving sets and metric dimension of imprimitive graphs and their halved or folded graphs.

Both of these next results are form Bailey [8]. The first concerns bipartite graphs and does not even assume that the graph is distance-regular.

Theorem 2.9. *[8] Let $\Gamma = (V,E)$ be a connected bipartite graph with bipartition $V = V^+ \cup V^-$ and let $\Gamma^+ = (V^+, E^+)$ and $\Gamma^- = (V^-, E^-)$ be its halved graphs. Then $\mu(\Gamma) \leqslant \mu(\Gamma^+) + \mu(\Gamma^-)$.*

Proof. Let $R^+ \subset V^+$ and $R^- \subset V^-$ be resolving sets for Γ^+ and Γ^-, respectively. We will show that $R = R^+ \cup R^-$ is a resolving set for Γ.

With $u \in V^+$ and $v \in V^-$ we have for any $w \in V$ that one of $d_\Gamma(u,w)$ and $d_\Gamma(v,w)$ is odd while the other is even (as one of u and v is in the same bipartite half as w while the other is not), and thus any $w \in R$ resolves u, v chosen such.

So we only need to consider the case where u and v are both in the same bipartite half. If for example $u, v \in V^+$, then there exists $w \in R^+$ such that $d_{\Gamma^+}(u,w) \neq d_{\Gamma^+}(v,w)$ and thus

$$d_\Gamma(u,w) = 2 \cdot d_{\Gamma^+}(u,w) \neq 2 \cdot d_{\Gamma^+}(v,w) = d_{\Gamma^+}(v,w),$$

i.e. w resolves u and v (in Γ). The case $u, v \in V^-$ is similar.

Hence R is a resolving set for Γ of size $\mu(\Gamma^+) + \mu(\Gamma^-)$ and thus $\mu(\Gamma) \leqslant \mu(\Gamma^+) + \mu(\Gamma^-)$. □

A simple application of this theorem is for the complete bipartite graph $K_{n,n}$, whose halved graphs are the complete graphs K_n and the bound is thus $\mu(K_{n,n}) \leqslant 2(n-1)$. We already know (for example from Proposition 2.2) that this is the exact value and the bound holds with equality.

It need not always do however, as shown by an example of Bailey which also shows that even in the case of distance-regular graphs, the halved graphs need not be isomorphic to each other and thus we cannot even assume that $\mu(\Gamma^+)$ and $\mu(\Gamma^-)$ are equal. Indeed, the halved graphs of the incidence graph of the generalised quadrangle $GQ(3,3)$ have metric dimension 7 and 8 while the incidence graph has metric dimension 10 (for details on these, see [7]).

Bailey's second theorem concerns metric bases in distance-regular antipodal graphs. Suppose that Γ is t-antipodal with V partitioned into s antipodal classes, $W_1, \dots, W_s$. We say that a partition of the vertices of Γ is a *t-antipodal partition* if each of its classes have exactly one vertex from each W_i, $i = 1, \dots, s$. The following theorem states that in a distance-regular antipodal graph a resolving set can be obtained by taking the "inverse image" of a resolving set of the folded graph with regard to an antipodal partition (and possibly adding a few vertices).

Theorem 2.10. *[8] Let $\Gamma = (V, E)$ be an $(r+1)$-antipodal, distance-regular graph with diameter d and let $V = V^0 \cup V^1 \cup \ldots \cup V^r$ be an $(r+1)$-antipodal partition of V. Let $\widetilde{R}$ be a resolving set for the folded graph $\widetilde{\Gamma} = (\widetilde{V}, \widetilde{E})$ whose diameter is $\widetilde{d}$. For $v \in \widetilde{V}$ let $\{v^0, v^1, \dots, v^r\}$ be its inverse image in V with $v^i \in V^i$ for each $i \in \{0, 1, \dots, r\}$. Let $R = \{w^1, \dots, w^r : w \in \widetilde{R}\}$.*

1. *If $u^i, v^j \in V$ with $u, v \in \widetilde{V}$, $i, j \in \{0, 1, \dots, r\}$ and $u \neq v$, then there exists a $w^\ell \in R$ with $w \in \widetilde{R}$ and $\ell \in \{1, \dots, r\}$ such that $d_\Gamma(u^i, w^\ell) \neq d_\Gamma(v^j, w^\ell)$ (i.e. w^ℓ resolves u^i and v^j in Γ).*

2. *If $d = 2\widetilde{d} + 1$ is odd, or for every $u \in \widetilde{V}$ there exists a $w \in \widetilde{R}$ such that $d_{\widetilde{\Gamma}}(u, w) < \widetilde{d}$, then R is a resolving set for Γ. In particular, $\mu(\Gamma) \leqslant r \cdot \mu(\widetilde{\Gamma})$.*

3. *If* $d = 2\widetilde{d}$ *is even and there exists* $u \in \widetilde{V}$ *such that* $d_{\widetilde{\Gamma}}(u, w) = \widetilde{d}$ *for all* $w \in R$*, then* $R^* = R \cup \{u^1, \dots, u^r\}$ *is a resolving set for* Γ*. In particular,* $\mu(\Gamma) \leqslant r \cdot (\mu(\widetilde{\Gamma}) + 1)$*.*

Complete multipartite graphs are an example where the bound is met with equality while Bailey gives other examples again where the bound is not met (the 3-antipodal Conway-Smith graph has metric dimension 6 while its folded graph, the Kneser graph $K(7, 2)$, has 4) and even where the antipodal graph has smaller metric dimension than its folded graph (in the case of the Tutte 8-cage and its folded graph the Foster graph; for details see [7]).

In the aforementioned classification theorem of Alfuraidan and Hall, eight of the eleven imprimitive classes are really problematic (the rest are the easy and extremal cases of cycles, complete and complete multipartite graphs). Four out of these eight can be satisfactorily handled using the reductions in Theorems 2.9 and 2.10 and Bailey did derive the applying theorems in [8] using the results of Babai.

Problems arise when, after applying the reductions, we cannot use Theorem 2.8 to sharpen our bounds. These are cases when halving or folding results in a complete graph leaving us with only the trivial linear upper estimate. As the bounds of these previous theorems are not always achieved, this lack of further information can be quite problematic. These cases however are not many (the four remaining of the already mentioned eight classes in the theorem of Alfuraidan and Hall) and in some of them Bailey even obtained partial results (for some 2-antipodal graphs and Taylor graphs).

Summary

As we have seen, these last results on distance-regular graphs give fairly stronger bounds than the introductory ones in the beginning of the first chapter, while also being able to apply to a much wider family of graphs than the specialised results at the end of the first chapter; thus Bailey's article [8] proves an invaluable addition to the ongoing investigation of metric dimension. In the following chapter we will discuss in detail one of the cases which he was not able to handle.

It is easy to see that a connected, bipartite, distance-regular graph with diameter 3 has complete graphs as its halved graphs and, as stated, can not be handled just by Theorem 2.9. The classification theorem of Alfuraidan and Hall states that this is indeed one of the possible cases of imprimitive distance-regular graphs, manifesting as the incidence graph of some symmetric design.

The remaining part of this thesis concerns a special case of this class of graphs, namely the incidence graphs of projective planes, and aims to determine precisely the metric dimension of these. The following chapter, after a brief introduction on the topic of projective planes, contains the proof of our main result, a theorem giving the exact metric dimension of all projective planes of order $q \geqslant 13$.

Chapter 3

On projective planes

3.1 Introduction to projective planes

Projective planes are geometric structures that extend our usual idea of a plane where the concept of parallel lines makes it so that lines can be in two different relations to each other: either they intersect or they are parallel. A projective plane is constructed in a way that two lines always intersect and two points always have a connecting line. In the finite case that we will concern ourselves with, they are in fact a special case of symmetric designs. A *symmetric design* with parameters (v, k, λ) is a set of v points together with a family of v k-subsets of points called blocks, such that any two blocks intersect in exactly λ points and any two points are contained in exactly λ blocks together.

This symmetry and order is the reason that their incidence graphs prove very comfortable to work with regarding questions of metric dimension. In the following section we introduce the basic notions and combinatorial properties of finite projective planes and then move on to investigate the metric dimension of their incidence graphs which culminates in the proof of our main result.

3.1.1 Definitions

Definition 3.1. *A* projective plane Π *is a non-empty set* $\mathcal{P}$ *of* points *together with a non-empty set* $\mathcal{L}$ *of* lines *and an* incidence *relation between these points and lines satisfying the following axioms:*

A1 *Given two lines, there is exactly one point incident with both of them.*

A2 *Given two points, there is exactly one line incident with both of them.*

A3 *There exist four points in general position, that is, there is no line incident with three of them.*

As it is the canonical example, we too will present here the *Fano plane*, which is the smallest possible projective plane. It has seven points and lines (note that the circle at the center is also a “line”). The truth of axioms A1 and A2 is quite easy to check by hand, and the three corners and the middle point gives an example of four points in general position. Also each line is incident with three points and each point is incident with three lines which, as we will see a bit later, is characteristic of finite projective planes.

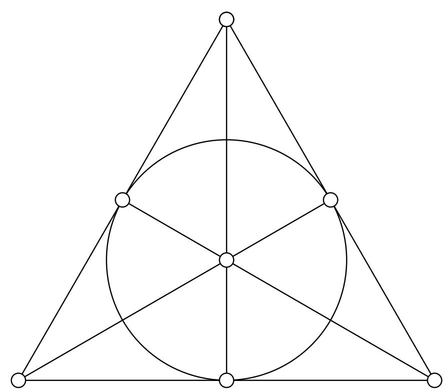

Figure 3.1: The Fano plane.

As our work focuses on the metric dimension which is a graph theoretic concept, we will regard the projective planes we work with as bipartite graphs, more precisely we will identify a projective plane with its incidence graph. The *incidence graph* of a projective plane Π is a bipartite graph with vertex set $\mathcal{P} \cup \mathcal{L}$ where vertices are adjacent if and only if the points and lines they represent are incident by the relation defined above. We will still use the notation regarding points and lines usually used in geometry as the meaning will always be clear in this context too. This manner of discussion should not cause any difficulties, in fact it is of the ones that reflect best the dual nature of projective planes.

Before delving deeper into the topic, we will introduce some notation. For points of the projective plane $P, Q \in \mathcal{P}$, naturally PQ will denote the unique line joining them. For a point P, let $[P]$ denote the set of lines incident with P, while similarly for a line ℓ, let $[\ell]$ denote the set of points incident with ℓ. This distinction is necessary as we do not regard lines as the subset of points they are incident with. One could also think of this $[\,.\,]$ operation as one that assigns its open neighbourhood to a vertex of the incidence graph. For a subset of vertices S in the incidence graph we will have $\mathcal{P}_s = S \cap \mathcal{P}$ denote the points among them and $\mathcal{L}_s = S \cap \mathcal{L}$ the lines. If one such subset S is fixed we will call its elements *inner* points and lines, while the rest *outer* ones. We say that a line is an i-secant of a set of points if it intersects it in i points. For convenience, 0-secants will be called skew lines and 1-secants tangents. Similarly, we say that a point is i-covered by a set of lines if it is incident with i of them.

The *dual* of a statement about geometries is one where the role of points and lines are reversed. As we can see, axioms A1 and A2 are duals of each other. We will show that, in the case of projective planes, this is not only a syntactic operation but carries some meaning too.

Proposition 3.1. *In a projective plane, the dual of axiom A3 is also true.*

Proof. We have to see that in a projective plane there exist four lines so that no three intersect in the same point. Let us have P_1, P_2, P_3, P_4 as the four points in general position guaranteed by axiom A3. The four lines P_1P_2, P_2P_3, P_3P_4, P_4P_1 are distinct and any one of the points is on exactly two lines. Now if we suppose that there exists a point P such that three of the lines intersect in P (which is different from $P_1, \ldots, P_4$ as they each are on two lines) we get a contradiction with A1 as there will be two lines intersecting in P as well as in one of $P_1, \ldots, P_4$, two distinct points. □

No we see that for any statement about points and lines proved using the axioms, one could have the dual statement and prove it in a very similar way using the duals of the axioms. This "interchangeable" nature of points and lines in statements about them will be referred to as *duality* in the projective plane.

We note that the usual notion of duality in projective planes is slightly different than this. It is the observation that for any true statement about points and lines in a projective plane, the dual statement is true in the plane's dual (that is, in the projective plane we get if we swap the points and lines "along" the incidence relation or equivalently, relabel the bipartition of the incidence graph). Our notion of duality is stronger because it is able to tell us something about the original plane itself but also a bit restricted in the sense that it can only be applied to statements proved using the axioms alone. This does not inconvenience us, as our results rely only on the combinatorial properties of the planes provided by the axioms.

3.1.2 Basic properties

We will now look at some basic properties of finite projective planes both to provide a foundation for our main results and to familiarise the reader with the basic ideas and methods of proofs in this subject. Every projective plane we consider from now on is supposed to be finite.

Lemma 3.2. *In a finite projective plane Π, for any point $P \in \mathcal{P}$ and any line $\ell \in \mathcal{L}$ not incident with P, the number of lines intersecting P is exactly the number of points incident with ℓ.*

Proof. Assume that the line ℓ has k points on it. Each one of these points is on exactly one line that is also incident with P (by axiom A2), so P has at least k lines (as these connecting lines are each distinct). But also every line through P intersects ℓ in some distinct point (axiom A1) so there is no more than k lines thorough P. That is, P has exactly k lines through it. □

Now we can introduce the notion of order and show the most prominent combinatorial properties relating to it. At first it would seem like the order is not properly defined, but we will see in the proposition directly following the description that in fact it is.

Definition 3.2. *A finite projective plane is said to have* order q *if there is a line that has exactly $q + 1$ points incident with it.*

Proposition 3.3. *For a finite projective plane Π of order q, the following properties hold:*

1. *Every point is incident with $q + 1$ lines.*
2. *Every line is incident with $q + 1$ points.*
3. *There are $q^2 + q + 1$ points.*
4. *There are $q^2 + q + 1$ lines.*

Proof. For *1.*, consider again four points $P_1, \ldots, P_4$ in general position. Let k be the number of lines through P_1. By Lemma 3.2, the lines P_2P_3, P_3P_4 and P_4P_2 all have k points on them. Now for any point P in Π, at least one of these three lines does not contain P and thus P has k lines through it again by Lemma 3.2. That is, every point of Π has k lines through it and as Π has order q and there is a line with $q+1$ points on it, $k = q+1$. For proving *2.* also, we only need to note that for every line there is a point not incident with it and thus, by Lemma 3.2, every line also has $q+1$ points on it.

For *3.*, consider any point $P \in \mathcal{P}$. Any other point R of Π "can be seen" from P in the sense that there is a line through R that also contains P (by axiom A2). We have $q+1$ lines through P and each of those have q distinct points on them besides P. This gives a sum of $q(q+1)+1 = q^2+q+1$ points in Π. The dual argument proves the same for lines. □

A projective plane of order q will be denoted by Π_q. We note here that for any $q = p^k$ with a prime p and positive integer k, one could construct a projective plane of order q using the finite field on p^k elements. In fact it is conjectured that only projective planes with an order of a prime power exist as, although there are constructions other than those based on finite fields, there are none, so far, that provide a plane with an order that is not a power of a prime. In our present thesis however, we only need to consider the combinatorial properties of the planes provided by the axioms by which they are defined and thus we refrain from describing any of these special constructions. Our results apply to any projective plane.

The following formulae known as the *standard equations*, besides giving further insight into how projective planes work, will be used in the proof of our main result.

Lemma 3.4. *Let Π_q be a projective plane of order q and let us have a subset of lines $X \subset \mathcal{L}$. For $0 \leqslant i \leqslant q+1$ denote by n_i the number of i-secants to X, that is, $n_i = |\{\ell \in \mathcal{L} : |\ell \cap X| = i\}|$. Then the following equations hold:*

$$\sum_{i=0}^{q+1} n_i = |\mathcal{L}| = q^2+q+1,$$

$$\sum_{i=0}^{q+1} i n_i = (q+1)|X|,$$

$$\sum_{i=0}^{q+1} i(i-1)n_i = |X|(|X|-1).$$

Proof. Seeing the truth of the first equation is straightforward. As each line intersects X in some number of points, each line adds one to n_i for some i. If we add up all of these we naturally get the total number of lines.

For the second equation, let us count in two ways the number of point-line pairs where a line intersects X, that is, determine the cardinality of $S := \{(P, \ell) : P \in X, \ell \in \mathcal{L}, P \in [\ell]\}$. Counting from the angle of lines, each i-secant adds i to the cardinality of S, that is, in_i for every value of i possible, the left hand side. Counting from the points of X, each point has $q+1$ lines that intersect it and thus give $q+1$ to $|S|$. That is true for all $|X|$ points of X, giving the right hand side of the equation.

For the third equation, in a similar manner count $D := \{(P, Q, \ell) : P, Q \in X, P \neq Q, \ell \in \mathcal{L}, P, Q \in [\ell]\}$, that is, the number of triplets with two points in X and a line incident with both. For each line ℓ intersecting X in i points, each of these points can be paired with another one of the $i-1$ remaining on ℓ to form an element of D; this gives the left hand side of the equation. For each point in X we can choose another one of the remaining $|X|-1$ in X to (together with the line joining them) attain a unique element of D, which gives the right hand side. Note that in both cases we counted triplets with the same two points twice, with the order of P and Q accounted for. □

Finally, some remarks about the properties of a projective plane as an incidence graph. By Proposition 3.3, the incidence graph is $(q+1)$-regular and the vertex classes in the bipartition have equal size. It is easy to see that the diameter of the graph is 3, as vertices in the same class always share a neighbour (points their connecting line by axiom A2 and lines their intersection point by A1) while a point and line if not incident (and thus at distance 1) can be "reached" through the intersection of the line in question and any line through the point. A short calculation then proves that the graph of a projective plane of order q is indeed distance-regular as stated in the previous chapter and has an intersection array of

$$\left\{ \begin{matrix} q+1 & q & q \\ 1 & 1 & q+1 \end{matrix} \right\}.$$

Now that we have established the basic properties of projective planes in general and also those of their incidence graphs, we will begin the discussion of the metric dimension of them.

3.2 Resolving sets in projective planes

As we have stated in the introduction, the main result of this thesis is a refinement of the earlier result in a 2012 article by Héger and Takáts [19]. First we present here without considerable alterations their preliminary results concerning resolving sets in projective planes as they provide the foundation for the following proof.

It is worth noting that any vertex s of the incidence graph (be it a point or a line) trivially resolves a pair of vertices if they correspond to one point and one line, as one of them will be in the same bipartite half as s and thus will have an even distance to it, while the other will not.

Lemma 3.5. *[19] Let $S = \mathcal{L}_s \cup \mathcal{P}_s$ be a set of vertices in the incidence graph of a finite projective plane. Then any line ℓ intersecting $\mathcal{P}_s$ in at least two points (that is, $|[\ell] \cap \mathcal{P}_s| \geqslant 2$) is resolved by S. Dually, if a point P is covered by at least two lines of $\mathcal{L}_s$, then P is resolved by S.*

Proof. Let ℓ be a line with $\{P, Q\} \subset [\ell] \cap \mathcal{P}_s$, $P \neq Q$. Then any line e different from ℓ may contain at most one of P or Q, hence e.g. $P \notin [e]$ and $d(P, \ell) = 1 \neq 3 = d(P, e)$. By duality, the same holds for points as well. □

Proposition 3.6. *[19] $S = \mathcal{P}_s \cup \mathcal{L}_s$ is a resolving set in a finite projective plane if and only if the following properties hold:*

P1 *There is at most one outer line skew to $\mathcal{P}_s$.*

P1' *There is at most one outer point not covered by $\mathcal{L}_s$.*

P2 *Through every inner point, there is at most one outer line tangent to $\mathcal{P}_s$.*

P2' *On every inner line, there is at most one outer point that is 1-covered by $\mathcal{L}_s$.*

Proof. To prove that a set S with these properties is a resolving set, by duality and Lemma 3.5, it is enough to see that S resolves lines not in $\mathcal{L}_s$ that are skew or tangent to $\mathcal{P}_s$.

Property P1 guarantees that there is at most one outer skew line, that is, the only line whose distance list contains only numbers at least 2. Thus all skew lines are resolved.

Now take a tangent line $\ell \notin \mathcal{L}_s$. If there were another line e with the same distance list as ℓ's, hence $e \notin \mathcal{L}_s$, then both e and ℓ would be outer tangents to $\mathcal{P}_s$ through the point $[\ell] \cap \mathcal{P}_s$, which is not possible by property P2.

It is straightforward to see that all resolving sets have to have these properties (again by duality and Lemma 3.5 we refer only to skew and tangent lines) as all outer skew lines have the same distance list, and so do any two outer tangents that go through a given point of $\mathcal{P}_s$. □

Proposition 3.7. *[19] The metric dimension of a projective plane of order $q \geqslant 3$ is at most $4q - 4$.*

Proof. We construct a set of $2q - 2$ points and $2q - 2$ lines that satisfy the properties outlined in Proposition 3.6.

Let P, Q and R be three arbitrary points in general position. Let

$$\mathcal{P}_s = [PQ] \cup [PR] \setminus \{P, Q, R\},$$

$$\mathcal{L}_s = [P] \cup [R] \setminus \{PQ, PR, RQ\}.$$

The following figure shows a schematic representation of this construction. The black dots and continuous lines form the proposed resolving set.

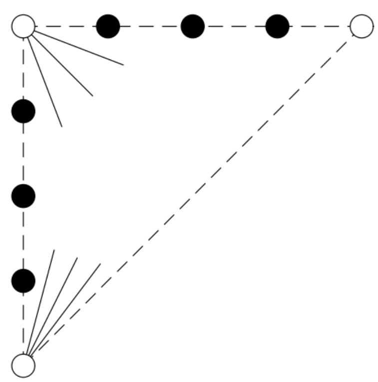

Figure 3.2: A resolving set of size $4q - 4$.

Now for checking the properties of Proposition 3.6:

- P1: The only outer line skew to $\mathcal{P}_s$ is RQ.
- P1': The only outer point not covered by $\mathcal{L}_s$ is Q.

- P2: As $q \geqslant 3$, $|[PQ] \cap \mathcal{P}_s| = |[PR] \cap \mathcal{P}_s| = q-1 \geqslant 2$, thus PQ and PR cannot be tangents to $\mathcal{P}_s$ themselves. Thus through a point $A \in [PQ] \setminus \{P, Q\}$ the only tangent is AR (which is in $\mathcal{L}_s$) and through a point $A \in [PR] \setminus \{P, R\}$ the only tangent is AQ.

- P2': As $q \geqslant 3$, $|[P] \cap \mathcal{L}_s| = |[R] \cap \mathcal{L}_s| = q-1 \geqslant 2$. Thus the only point on a line $l \in [R] \setminus \{RP, RQ\}$ that is 1-covered by $\mathcal{L}_s$ is $[l] \cap [PQ]$ (which is in $\mathcal{P}_s$) and the only 1-covered point on a line $l \in [P] \setminus \{PR, PQ\}$ is $[l] \cap RQ$.

Hence S is a resolving set of size $4q-4$. □

In their original article, Héger and Takáts [19] used Proposition 3.7 as a starting point and went on to prove that for projective planes of order $q \geqslant 23$, the size of their resolving sets also has to be at least $4q-4$, thus concluding that any metric basis of those planes has a size of $4q-4$.

In our present thesis however, we are going to employ a proof by contradiction for a stronger result. We will use a starting assumption of $|\mathcal{P}_s| + |\mathcal{L}_s| \leqslant 4q-5$ instead of $4q-4$ and then use the same ideas with the updated and modified proofs of the original article to arrive at the same conclusion as they did but for planes of order $q \geqslant 13$; that is, that resolving sets are also of size at least $4q-4$, thus arriving at a contradiction which nevertheless provides us with the consequence that $|\mathcal{P}_s| + |\mathcal{L}_s| \geqslant 4q-4$. This, together with Proposition 3.7, will prove the following theorem.

Theorem 3.8. *The metric dimension of a finite projective plane of order $q \geqslant 13$ is $4q-4$.*

The original article [19] also provided an exhaustive characterisation of the metric bases of this size in finite projective planes of order $q \geqslant 23$. As the tools they used for this came from the information gathered during the proof of their theorem, this line of continuation of the results is problematic in our case. The proof by contradiction we employ gives the same result on the size of the metric bases as theirs but lacks the additional information they gained about the structure of the resolving sets of size $4q-4$. Because of this (and of the thesis' limited length) we will not attempt this characterisation of the remaining planes.

3.3 The proof of Theorem 3.8

Before we present the propositions that form the body of the proof, we briefly outline the process we will go through and also note the major additions and changes to that of the original article.

After establishing some general bounds on the number of points and lines that make up a resolving set S, we will see that any line is either a quite short or quite long secant of $\mathcal{P}_s$. In fact, we prove that there must be two at least $(q-5)$-secants to $\mathcal{P}_s$. This proof contains an additional idea for separating cases that was not present in the original and makes the proof much stronger. Then we go on to prove that these long secants are in fact much longer and can only have a total of three points that are not in $\mathcal{P}_s$. This proof is the one that deviates most from the original as the cases needed to be eliminated are more numerous and complex. We derive sharper bounds and even utilise a short computer calculation to proceed. A small part of the original proof will be omitted here as our new assumptions prove strong enough to proceed to the conclusion directly by simply counting the lines we know to be in $\mathcal{L}_s$ by then. Now the detailed proof follows.

General assumption: From now on, throughout the proof we assume that S is a resolving set of a finite projective plane of order q, with $|\mathcal{P}_s| + |\mathcal{L}_s| \leqslant 4q - 5$.

Proposition 3.9. *$2q - 5 \leqslant |\mathcal{P}_s| \leqslant 2q$, $2q - 5 \leqslant |\mathcal{L}_s| \leqslant 2q$.*

Proof. Let t denote the number of tangents in $\mathcal{L} \setminus \mathcal{L}_s$ (that is, the number of outer tangents). By property P2, $t \leqslant |\mathcal{P}_s|$ and by property P1 we know that there may be at most one skew line that is not in $\mathcal{L}_s$.

Now we double count the pairs $\{(P, \ell) : P \in \mathcal{P}_s,\ P \in [\ell],\ |[\ell] \cap \mathcal{P}_s| \geqslant 2\}$. Every line through a point in $\mathcal{P}_s$ which is not a tangent has to be an at least 2-secant while every line not in $\mathcal{L}_s$ (as we know nothing of those), which is not a tangent or the possibly one skew line, has to intersect $\mathcal{P}_s$ in at least two points. This gives us

$$|\mathcal{P}_s|(q+1) - t \geqslant 2(q^2 + q + 1 - 1 - t - |\mathcal{L}_s|),$$

which, when rearranged, gives

$$q|\mathcal{P}_s| \geqslant 2(q^2 + q - |\mathcal{L}_s|) - t - |\mathcal{P}_s| \geqslant 2(q^2 + q - (|\mathcal{L}_s| + |\mathcal{P}_s|)),$$

$$q|\mathcal{P}_s| \geqslant 2(q^2 - 3q + 5).$$

Thus $|\mathcal{P}_s| \geqslant 2q - 6 + 10/q$, and as it is an integer, $|\mathcal{P}_s| \geqslant 2q-5$. Dually, $|\mathcal{L}_s| \geqslant 2q-5$ also holds. From $|\mathcal{P}_s| + |\mathcal{L}_s| \leqslant 4q-5$, the upper bounds follow. □

Assumption: From now on, by duality, we assume $|\mathcal{P}_s| \leqslant |\mathcal{L}_s|$. Thus, as $|\mathcal{P}_s| + |\mathcal{L}_s| \leqslant 4q-5$, $|\mathcal{P}_s| \leqslant 2q-3$ follows.

Proposition 3.10. *Let $q \geqslant 13$. Then any line intersects $\mathcal{P}_s$ in either at most 4 or at least $q-5$ points.*

Proof. Suppose that for a line ℓ, $|[\ell] \cap \mathcal{P}_s| = x$, $2 \leqslant x \leqslant q$. For a point $P \in [\ell] \setminus \mathcal{P}_s$, let $s(P)$ and $t(P)$ denote the number of skew or tangent lines to $\mathcal{P}_s$ through P, respectively. Denote by s the number of skew lines and by t the total number of tangents intersecting ℓ outside $\mathcal{P}_s$. (Note that here we count not only the outer lines, but all of them.)

Counting the points of $\mathcal{P}_s$ on ℓ and the other lines through P we get

$$2q - 3 \geqslant |\mathcal{P}_s| \geqslant x + t(P) + 2(q - t(P) - s(P))$$

and thus

$$x \leqslant 2s(P) + t(P) - 3.$$

Now adding up the inequalities for all $P \in [\ell] \setminus \mathcal{P}_s$ we obtain

$$(q+1-x)x \leqslant 2s + t - 3(q+1-x).$$

Proposition 3.6 yields that $s \leqslant |\mathcal{L}_s| + 1$ and $s + t \leqslant 1 + (|\mathcal{P}_s| - x) + |\mathcal{L}_s|$, whence $2s + t \leqslant 2|\mathcal{L}_s| + |\mathcal{P}_s| - x + 2$. Using Proposition 3.9 we get $(|\mathcal{L}_s| + |\mathcal{P}_s|) + |\mathcal{L}_s| \leqslant (4q-5) + 2q = 6q - 5$. Combined with the previous inequality we get

$$qx + x - x^2 \leqslant 6q - 5 - x + 2 - 3q - 3 + 3x,$$
$$0 \leqslant x^2 - (q-1)x + 3q - 6.$$

Substituting $x = 5$, we get $0 \leqslant 25 - 5q + 5 + 3q - 6 = 24 - 2q$, that is, the right hand side is negative when $q \geqslant 13$, therefore, as x is an integer, we conclude that $x \leqslant 4$. We also know that the roots of the quadratic expression on the right hand side add up to $q-1$ (Viète's formulas) and thus our result is the same when $x = q-6$, giving us $x \geqslant q-5$ when $q \geqslant 13$. □

Proposition 3.11. *Let $q \geqslant 13$. Then there exist two lines intersecting $\mathcal{P}_s$ in at least $q-5$ points.*

Proof. By Proposition 3.10, every line is either a $\leqslant 4$ or a $\geqslant q-5$-secant. Let ℓ be a longest secant of $\mathcal{P}_s$ and suppose to the contrary that every other line intersects $\mathcal{P}_s$ in at most 4 points. Let $x = |[\ell] \cap \mathcal{P}_s| \geqslant 2$; note that $x \leqslant 4$ is also possible. Let n_i denote the number of i-secants to $\mathcal{P}_s$ different from ℓ. To be convenient, let $n_0 = s$, $n_1 = t$ and $b = |\mathcal{P}_s|$. (Note that s and t here represent the total number of skew and tangent lines.) Then the standard equations, with a slight rearrangement, yield

$$\sum_{i=2}^{4} n_i = q^2 + q + 1 - s - t - 1 = q^2 + q - (s+t),$$

$$\sum_{i=2}^{4} i n_i = (q+1)b - t - x,$$

$$\sum_{i=2}^{4} i(i-1)n_i = b(b-1) - x(x-1).$$

Thus

$$0 \leqslant \sum_{i=2}^{4} (i-2)(4-i)n_i =$$

$$= -\sum_{i=2}^{4} i(i-1)n_i + 5\sum_{i=2}^{4} i n_i - 8\sum_{i=2}^{4} n_i =$$

$$= -b^2 + (5q+6)b + x(x-6) + 3(s+t) + 5s - 8(q^2+q).$$

Now we have several subcases based on the supposed size of x:

Case 1: $x = q+1$

In this case we have all points of ℓ in $\mathcal{P}_s$ and thus there cannot be skew lines (as every line intersects ℓ in some point and that point is necessarily in $\mathcal{P}_s$), there can only be $q+1$ outer tangents (at most one through each point of ℓ) and also the lines in $\mathcal{L}_s$ cannot be guaranteed not to be tangents each. This all gives us $s = 0$ and $t \leqslant q + 1 + |\mathcal{L}_s| \leqslant 3q + 1$, thus

$$0 \leqslant -b^2 + (5q+6)b + q^2 - 4q - 5 + 9q + 3 - 8q^2 - 8q =$$

$$-b^2 + (5q+6)b - 7q^2 - 3q - 2.$$

By duality, we assumed $b = |\mathcal{P}_s| \leqslant 2q - 3$. Substituting $b = 2q - 3$ (which is on the left hand side of the above parabola if considered as a function of b) the above inequality yields

$$0 \leqslant -4q^2 + 12q - 9 + 10q^2 - 3q - 18 - 7q^2 - 3q - 2 =$$

$$= -q^2 + 6q - 29,$$

which is always negative, a contradiction.

Case 2: $x = q$

Here $s \leqslant q$ as all skew lines must pass through the outer point on ℓ (q such lines overall), while $s + t \leqslant 2q + |\mathcal{L}_s| \leqslant 4q$ as the lines through the outer point of ℓ can be either tangents or skew lines but give q lines still, also there can be at most one outer tangent through every inner point of ℓ and the lines of $\mathcal{L}_s$ can also be tangents. Thus

$$0 \leqslant -b^2 + (5q + 6)b + q^2 - 6q + 12q + 5q - 8q^2 - 8q =$$

$$= -b^2 + (5q + 6)b - 7q^2 + 3q$$

Substituting $b = 2q - 3$ we get

$$0 \leqslant 6q^2 + 9q - 27 - 7q^2 + 3q = -q^2 + 12q - 27,$$

which is negative whenever $q \geqslant 13$, a contradiction.

Case 3: $x \leqslant q - 1$

Now as a general estimation, we can have one outer tangent through each point of $\mathcal{P}_s$, an outer skew line overall and the lines of $\mathcal{L}_s$ still cannot be guaranteed to not be skew or tangent which gives $s + t \leqslant |\mathcal{P}_s| + |\mathcal{L}_s| + 1 \leqslant 4q - 4$ and $s \leqslant |\mathcal{L}_s| + 1 \leqslant 2q + 1$, thus

$$0 \leqslant -b^2 + (5q + 6)b + q^2 - 8q + 7 + 12q - 12 + 10q + 5 - 8q^2 - 8q.$$

Substituting $b = 2q - 3$, we get $-q^2 + 15q - 27$ which is negative whenever $q \geqslant 13$, a contradiction. $\square$

Now we see that there exist two distinct lines, e and f such that $|[e] \cap \mathcal{P}_s \setminus ([e] \cap [f])| = q - l$ and $|[f] \cap \mathcal{P}_s \setminus ([e] \cap [f])| = q - k$ with $k \leqslant l \leqslant 6$.

Let $[e] \cap [f] = P$ and denote the set of points of $\mathcal{P}_s$ outside of $[e] \cup [f]$ by Z. Further denote by E_{in} the set of $q - l$ points of $\mathcal{P}_s$ in $[e] \setminus \{P\}$ and by E_{out} the set of l outer points in $[e] \setminus \{P\}$. Note that P may be an inner or outer point but in either case it is not in any of the two sets defined here. We define F_{in} and F_{out} similarly for the sets of inner and outer points of $[f] \setminus \{P\}$.

Proposition 3.12. *Suppose $q \geqslant 13$. Then $k + l \leqslant 3$.*

Proof. Suppose to the contrary that $k+l \geqslant 4$. We will estimate the number of lines in $\mathcal{L}_s$ by counting potentially skew or tangent lines (as by Proposition 3.6, there can not be too many of those outside $\mathcal{L}_s$). These "problematic" lines are the following: the $q-1$ lines in $[P] \setminus \{e, f\}$ as, if they don't intersect Z, they are either tangent or skew depending on P being in $\mathcal{P}_s$ or not, respectively; the lq lines connecting the points of E_{out} with those of $[f] \setminus \{P\}$, as the lines between E_{out} and F_{in} are tangents if they don't intersect Z and the lines between E_{out} and F_{out} can be skew or tangent (or $\geqslant$ 2-secants) depending on how many points they intersect Z in. The kq lines between F_{out} and $[e] \setminus \{P\}$ will also be counted here with a similar reasoning but among these two bunches we listed twice the kl lines connecting E_{out} with F_{out}. Altogether we have

$$q - 1 + kq + lq - kl$$

of these problematic lines.

Some of these lines however cannot be assured to be in $\mathcal{L}_s$ for some reason or other. Some of the lines can be blocked by the points of Z: $|Z|$ of the $q-1$ lines incident with P, and also $k|Z|$ and $l|Z|$ lines among the kq and lq lines mentioned above. Furthermore each point in E_{in} and F_{in} can have at most one outer line through them that is tangent to $\mathcal{P}_s$ (by property P2 in Proposition 3.6), those are an additional $q-k$ and $q-l$ lines possibly not in $\mathcal{L}_s$, as they cannot be guaranteed to coincide with the lines intersecting Z excluded before. And lastly there can be exactly 1 outer skew line (as of property P1). We will write our estimations separately for two cases.

Case 1: $P \notin \mathcal{P}_s$. In this case, the lines through P which are not blocked by Z are skew thus already accounted for by the above considerations. We can also estimate the size of Z using $(q-k) + (q-l) + |Z| = |\mathcal{P}_s| \leqslant 2q - 3$, which gives $|Z| \leqslant k + l - 3$. Combining all the above we get

$$|\mathcal{L}_s| \geqslant q - 1 + kq + lq - kl - |Z| - k|Z| - l|Z| - (q-k) - (q-l) - 1 \geqslant$$

$$\geqslant q(k + l - 1) - (k + l + 1)|Z| + (k + l) - kl - 2 \geqslant$$

$$\geqslant q(k+l-1)-(k^2+l^2-3(k+l)+3kl-1).$$

Substituting $|\mathcal{L}_s| \leqslant 2q$ from Proposition 3.9, we get an estimate on q.

$$q \leqslant \frac{k^2+l^2-3(k+l)+3kl-1}{k+l-3} =: f_1(k,l).$$

Case 2: $P \in \mathcal{P}_s$. In this case the lines through P not blocked by Z are tangents and as such there can be an additional line not in $\mathcal{L}_s$, that is, an outer tangent through P. The estimate on $|Z|$ however is lower because $(q-k)+(q-l)+|Z|+1 = |\mathcal{P}_s| \leqslant 2q-3$ which gives $|Z| \leqslant k+l-4$. Thus in this case we get

$$|\mathcal{L}_s| \geqslant q(k+l-1)-(k+l+1)|Z|+(k+l)-kl-3 \geqslant \tag{3.1}$$

$$\geqslant q(k+l-1)-(k^2+l^2-4(k+l)+3kl-1),$$

and the resulting estimation on q is

$$q \leqslant \frac{k^2+l^2-4(k+l)+3kl-1}{k+l-3} =: f_2(k,l).$$

Substituting the possible values of k and l with $(k+l) \in [4,12]$ when $P \in \mathcal{P}_s$ and $(k+l) \in [4,10]$ when $P \notin \mathcal{P}_s$ (as whenever k or l equals 6, P must be in $\mathcal{P}_s$ because if it would not then e or f would contain fewer than $q-5$ points of $\mathcal{P}_s$ which is impossible by Proposition 3.11), we see that whenever $q \geqslant 13$ most of the cases are impossible.

$k+l$	k	l	$f_1(k,l)$	$k+l$	k	l	$f_1(k,l)$
4	0	4	3	6	3	3	8.67
4	1	3	6	7	2	5	9.25
4	2	2	7	7	3	4	9.75
5	0	5	4.5	8	3	5	10.8
5	1	4	6.5	8	4	4	11
5	2	3	7.5	9	4	5	12.17
6	1	5	7.33	10	5	5	**13.43**
6	2	4	8.33				

Table 3.1: Values of f_1, the $P \notin \mathcal{P}_s$ case.

$k+l$	k	l	$f_2(k,l)$	$k+l$	k	l	$f_2(k,l)$
4	0	4	-1	7	2	5	7.5
4	1	3	2	7	3	4	8
4	2	2	3	8	2	6	8.6
5	0	5	2	8	3	5	9.2
5	1	4	4	8	4	4	9.4
5	2	3	5	9	3	6	10.33
6	0	6	3.67	9	4	5	10.67
6	1	5	5.33	10	4	6	11.86
6	2	4	6.33	10	5	5	12
6	3	3	6.67	11	5	6	**13.25**
7	1	6	6.5	12	6	6	**14.56**

Table 3.2: Values of f_2, the $P \in \mathcal{P}_s$ case.

Some of them however are needed to be excluded by further considerations. Rearranging our previous formula (3.1) (which being the lower of the two bounds on $|\mathcal{L}_s|$ is true in both the case $P \in \mathcal{P}_s$ and $P \notin \mathcal{P}_s$), we get a lower estimate of $|Z|$, that is,

$$2q \geqslant q(k+l-1) - (k+l+1)|Z| + (k+l) - kl - 3 \Leftrightarrow$$

$$|Z| \geqslant \frac{q(k+l-3) + (k+l) - kl - 3}{k+l+1}.$$

Using our estimates we can now enumerate all the remaining cases needed to be excluded (E_{val} and R will be explained a bit later):

P	k	l	q	$\|Z\|$	L_{max}	E_{val}	R
$\in \mathcal{P}_s$	6	6	14	8	26	31	8
$\in \mathcal{P}_s$	6	6	13	8	24	25	13
$\in \mathcal{P}_s$	6	6	13	7	25	32	7
$\in \mathcal{P}_s$	5	6	13	7	24	28,5	4,5
$\notin \mathcal{P}_s$	5	5	13	7	24	25	3

Table 3.3: The remaining cases.

L_{max} indicates the maximum number of lines that can be in $\mathcal{L}_s$ without getting into contradiction with our assumption of $|\mathcal{P}_s| + |\mathcal{L}_s| \leqslant 4q - 5$. It is calculated using our knowledge of how many points are in $\mathcal{P}_s$ and the bound on $|\mathcal{L}_s| + |\mathcal{P}_s|$, as follows. In the $P \in \mathcal{P}_s$ case we have

$$|\mathcal{P}_s| = (q-k) + (q-l) + |Z| + 1 \Rightarrow$$

$$|\mathcal{L}_s| \leqslant (4q-5) - (2q+1-(k+l)+|Z|) = 2q-6+(k+l)-|Z|.$$

The $P \notin \mathcal{P}_s$ case is almost identical, without the ± 1 term introduced by P being in $\mathcal{P}_s$, and thus we get

$$|\mathcal{L}_s| \leqslant 2q-5+(k+l)-|Z|.$$

Now to ensure the existence of more than this many lines in $\mathcal{L}_s$ (and thus, by contradiction exclude the remaining cases), we will improve our existing estimate on $|\mathcal{L}_s|$. Let's have

$$r := |\{\text{lines between } E_{out} \text{ and } F_{out} \text{ intersecting } Z\}|.$$

It can be easily seen that in our previous considerations these lines have been subtracted twice when we counted the lines possibly blocked by Z, therefore we can add r and our estimation in (3.1) still remains true. For simplicity's sake we will only use estimations for the case $P \in \mathcal{P}_s$, since these are lower (and thus still true in the $P \notin \mathcal{P}_s$ case) but will still give the desired results. That gives

$$|\mathcal{L}_s| \geqslant q(k+l-1)-(k+l+1)|Z|+(k+l)-kl-3+r =: B_1(q,k,l,|Z|,r).$$

We will also use a generally rougher estimate, which is still better when r is low, using only the lines through P and the many skew lines between E_{out} and F_{out} (that is, every line there that does not intersect Z is guaranteed to be skew). Therefore our second estimation is

$$|\mathcal{L}_s| \geqslant q-2-|Z|+kl-1-r =: B_2(q,k,l,|Z|,r).$$

Since both of these expressions change linearly in r it is enough to check whether whenever one equals the other, they are still strictly greater than the aforementioned L_{max} (that is, their maximum is always greater than L_{max}).

The last two rows of Table 3.3 show the results of this check; E_{val} is the value at which the two estimates meet, more precisely, for the fixed values of q, k, l, and $|Z|$ of a certain problematic case,

$$E_{val}(q,k,l,|Z|) = \min_r(\max(B_1(q,k,l,|Z|,r),\ B_2(q,k,l,|Z|,r))),$$

while R is the corresponding r value. We see that in all cases $L_{max} < E_{val}$, that is, there must be more lines in $\mathcal{L}_s$ than is possible, and with that we arrive at a contradiction and are able to exclude them too.

With these last considerations we have comprehensively shown that whenever $q \geqslant 13$, $k+l \geqslant 4$ is not possible, therefore proving Proposition 3.12. □

Proposition 3.13. *Whenever $q \geqslant 13$, $|\mathcal{P}_s| = 2q-3$ and $|\mathcal{L}_s| \geqslant 2q-1$.*

Proof. With $k+l \leqslant 3$ we see at least $(q-k)+(q-l) \geqslant 2q-3$ points in $\mathcal{P}_s$ just on e and f. Together with our assumption of $|\mathcal{P}_s| \leqslant 2q-3$ we have $|\mathcal{P}_s| = 2q-3$, $|Z| = 0$ and $P \notin \mathcal{P}_s$ (and $k+l < 3$ is not possible).

First we will see that $l = 3$ is not possible. Suppose for a contradiction that $l = 3$ and thus $k = 0$. We see at least $q-2$ lines in $\mathcal{L}_s$ through P and further $2q$ among the $3q$ lines between E_{out} and F_{in} (as each point in F_{in} can have at most one outer tangent through it). That gives $|\mathcal{L}_s| \geqslant 3q-2$ while $|\mathcal{L}_s| \leqslant 2q$ is also true, a contradiction. (Note that we could not have used for example our estimate in (3.1) to arrive at this contradiction as they prove too weak in this particular case. The reason is that the $q-l$ lines we subtracted there as possible outer tangents between E_{in} and F_{out} can not be tangents here as there are no outer points among $[f] \setminus \{P\}$.)

This means that $l = 2$ and $k = 1$. Let the outer points on $e \setminus \{P\}$ be denoted by R_1 and R_2 and the outer point on $f \setminus \{P\}$ by Q.

Now we will count lines that are sure to be in $\mathcal{L}_s$. The above considerations ensure that there are no points in $\mathcal{P}_s$ outside of $[e] \cup [f]$, and thus the $q-1$ lines in $[P] \setminus \{e,f\}$ are skew, which together with QR_1 and QR_2, gives $q+1$ skew lines in total. As property P1 in Proposition 3.6 says there can only be one outer skew line, q of these have to be in $\mathcal{L}_s$. Then for every point $F \in [f] \setminus \{P,Q\}$ at least one of the lines FR_1, FR_2 has to be in $\mathcal{L}_s$ by property P2 as both are tangents to $\mathcal{P}_s$. This gives further $q-1$ lines at least. Together this gives at least $q+(q-1) = 2q-1$ lines in $\mathcal{L}_s$. □

Conclusion

Proposition 3.13 gives the overall result of $|\mathcal{P}_s| + |\mathcal{L}_s| \geqslant 4q-4$ for any resolving set, whenever $q \geqslant 13$ with which we arrive at a contradiction with our starting assumption of $|\mathcal{P}_s| + |\mathcal{L}_s| \leqslant 4q - 5$. This means that when $q \geqslant 13$, $|\mathcal{P}_s| + |\mathcal{L}_s| \geqslant 4q - 4$ is true instead which together with the result of Proposition 3.7 gives our final conclusion: whenever $q \geqslant 13$, the metric dimension of the incidence graph of the finite projective plane of order q is $4q - 4$. This concludes the proof of Theorem 3.8. □

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