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Wind Energy Producers - A Model with Cooperative Games

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Chapter 1

Introduction

The study of the theory of games gained popularity after the publication of the classical book “Theory of games and economic behaviour“ by von Neumann and Morgenstern in 1944. The basic framework of cooperative game was also introduced in this book (that is called the characteristic function).

In game theory, a coalitional game (or cooperative game) there is a competition between groups of players (coalitions), where the purpose is not to overtake the other players like in non-cooperative games, but to receive a payoff as much as possible by associating with each other.

In the cooperative game theory, there are two key questions according to a game:

- The **selection** problem: which kind of coalitions are going to form
- The **sharing** problem: after forming the coalitions, how to distribute the payoff for the members of coalition fairly.

In this thesis we are going to focus on the sharing problem: we assume that players manage to form the great coalition (the coalition containing all participants), and then build different interpretation of fairness for the solution concepts.

In Chapter 2, we are going to introduce the basic terms of cooperative game theory, and we present three different examples. We define superadditive games, where it is possible for the grand coalition to form.

After this, in Chapter 3, we expose the the concept of fair payoffs. We define the imputation, s payoff which is fair for all individual players. Then we introduce the core, which is good for not only the individuals, but for smaller groups of players too. Related to this, we define two groups of games (convex and balanced games) which can be characterized by the non-emptiness of the core. After this, we show two alternative concept of fair payoffs called the Shapley-value and the nucleolus, and mention some of their important properties.

In Chapter 4, we present the game-theoretic model for wind energy producers. We prove that the the game is superadditive and balanced, and then we discuss an example detailed, visualizing the core, the Shapley-value and the nucleolus with the help of the programs Matlab and Polymake.

Finally, in Chapter 5 we improve the model by introducing new variables and following a more complex market structure.

Chapter 2

Basics of cooperative game theory

2.0.1 Definition. *Let $n \geq 2$ denote the number of players in the game, and let N denote the set of players, $N = \{1, 2, ..., n\}$. A **coalition** S is defined to be a subset of N, $S \subseteq N$, and the set of all coalitions is denoted by 2^N. The empty set as a coalition is the **empty coalition**, and the set N is called the **grand coalition**.*

In case of two players, there are four coalitions: $\{\emptyset, \{1\}, \{2\}, \{12\} = N\}$ If there are three players, then the coalitions are $\{\emptyset, \{1\}, \{2\}, \{3\}, \{12\}, \{13\}, \{23\}, \{123\} = N\}$. We are going to work mostly with these two cases.

In the following, we use a utility-based approach and we assume that "everything has a price ": each agent has a utility function that is expressed in currency units. We call this the characteristic function.

2.0.2 Definition. *The **characteristic function** of a game is a real-valued function defined on all coalitions, $v\colon 2^N \to \mathbb{R}$, where $v(\emptyset) = 0$.*

2.0.3 Definition. *The **coalitional form** of a game is given by the pair (N, v).*

2.0.4 Remark. Cooperative or coalitional games are also called TU-games, but in this thesis we are going to use only the first two names. Although, the references may contain the last name too.

Now we are going to introduce three examples to show some everyday problems in which cooperative games can be used. The first two examples are presented in detail in [2] and [3]. The third example is probably the most ancient game-theoretic problem, which was covered in mystery for a long time, and finally in 1985, Robert J. Aumann and Michael Maschler managed to analyze the bankruptcy problem mathematically.

2.0.5 Example. (glove market, Shapley, 1959) Let N consists of two types of players, $N = L \cup R$, where $L \cap R = \emptyset$. Each player of L owns a left-hand glove, and each player of R owns a right-hand glove. One complete pair of glove is worth 1, gloves without pairs are worth nothing. This way, if a coalition consists of k members of L and j members of R, they have $\min\{j, k\}$ pairs of gloves, $v(S) = \min\{|S \cap L|, |S \cap R|\}$.

2.0.6 Example. (horse traders) There are three traders in the market. Trader A has a horse to sell, that he is unwilling to sell for less than 200 dollars. B and C would buy the horse for not more than 280, respectively 300 dollars. So we have: $v(A) = 200$, $v(B) = 280$ and $v(C) = 300$. If A and B can agree in a price p, then together they receive: $v(AB) = p + (280 + 280 - p) = 560$, which is independent from p. Similar: $v(AC) = p + (300 + 300 - p) = 600$. B and C can only "trade" money between them, so $v(BC) = 280 + 300 = 580$. All together they can receive not more than: $v(ABC) = 600 + 280 = 880$ (in the case C buys the horse).
This model can be simplified if we only look at the players' actual income in the coalitions compared to their starting situation, as it is shown in the table below:

S	A	B	C	AB	AC	BA	ABC
v(S)	0	0	0	80	100	0	100

For example, the amount of extra value A, B and C received together is: $880 - (200 + 280 + 300) = 100$.

2.0.7 Example. (bankruptcy problem from the Talmud, Aumann-Maschler, 1985) In the Babylonian Talmud (a 2000 years-old book that contains the teachings and opinions of thousands of rabbis) there is a law, which treats the case when the bank has a given amount of money, E, and three players, who each have the right claims of $d_1 = 100$, $d_2 = 200$, $d_3 = 300$, but $E < d_1 + d_2 + d_3$. This is called a bankruptcy problem. In the old document there is a table which shows the divisions for three different value of E:

	$d_1 = 100$	$d_2 = 200$	$d_3 = 300$
$E = 100$	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
$E = 200$	50	75	75
$E = 300$	50	100	150

When $E = 100$, equal division makes good sense. The case $E = 300$ is based on the different principle of proportional division. The figures for $E = 200$ looks mysterious, it is not obvious why the money is divided like this. Later we will have the answer for this question too.

The problem can be generalized: in the n-player $(E; d_1, \ldots, d_n)$ game the characteristic function for each coalition is

$$v(S) = \max\{0, E - d(N \setminus S)\}, \text{ where } d(N \setminus S) = \sum_{i \in N \setminus S} d_i.$$

2.1 Superadditive games

It is a natural thought that for each player, joining bigger coalitions is worth more than taking part in smaller ones. There is a big category of games that have this property called superadditive games.

2.1.1 Definition. *The coalitional game (N, v) is:*

- ***subadditive****, if* $v(S) + v(T) \geq v(S \cup T)\ \forall\ S, T \subset N,\ S \cap T = \emptyset$.
- ***additive****, if* $v(S) = \sum_{i \in S} v(\{i\})\ \forall\ S \subset N$.
- ***superadditive****, if* $v(S) + v(T) \leq v(S \cup T)\ \forall\ S, T \subset N,\ S \cap T = \emptyset$.

Subadditive games are the less realistic type of games. Here, the agents are best off when they are on their own, cooperation is not desirable. In additive games, none of collaborations results in a countable advantage, the coalitional function is defined by the functions of the 1-player coalitions. They are also called inessential. Superadditivity means that the value of a coalition cannot be improved by splitting up into two smaller coalitions. Two coalitions joining together results more value, than staying as separate groups.

2.1.2 Example. All the three examples presented before (horse traders, glove market, bankruptcy problem) are superadditive.

2.1.3 Proposition. *Every positive superadditive game is* ***monotone*** *i.e.*

$$S \subseteq T \Rightarrow v(S) \leq v(T)\ \forall\ S, T \subseteq N.$$

Proof. Let $S \subseteq T$. $S \cap (T \setminus S) = \emptyset$, so from the definition of superadditivity we get $v(S) + v(T \setminus S) \leq v(T)$. The game is positive, so $v(T \setminus S) \geq 0$, and hence the statement is proven. ☐

2.1.4 Remark. Note that the converse is not necessarily true. For example if we consider the three-player game where $v(\{i\}) = 1$ and $v(\{i, j\}) = v(\{1, 2, 3\})$, then the game is monotone, but it is not superadditive.

2.1.5 Proposition. *A game is superadditive if and only if for all coalitions $S \subseteq N$ and for any partition $\{T_1, \ldots, T_k\}$ of S: $v(S) \geq \sum_{i=1}^{k} v(T_i)$.*

Proof. For each T_i, T_j from the partition, we have $T_i \cap T_j = \emptyset$, from superadditivity $v(T_i \cup T_j) \geq v(T_i) + v(T_j)$. $S = \cup_{i=1}^{k} T_i$, using induction the property is proven. For the other direction, the $k = 2$ case is equivalent to the superadditivity property. ☐

Since in superadditive games bigger coalitions are worth more than smaller ones, the players are willing to form bigger and bigger coalitions, for the greater profit. Therefore it is possible for players to form the grand coalition, N. Of course, this is not entirely trivial, not even in superadditive games, but here we are not going to deal with this problem.

Chapter 3

Imputations

Hereafter we consider the grand coalition formed. The question now is, how can we pay the players fairly for their contribution in the game.

3.0.1 Definition. *A payoff vector* $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ *(the player* i *receives* x_i*) is called an* ***imputation****, if:*

- $\sum_{i=1}^{n} x_i = v(N)$ *(**efficiency**)*
- $x_i \geq v(\{i\})\ \forall\ i = 1, \ldots, n$ *(**individually rationality**).*

Both conditions are logical. We can hand out the whole amount of money received by the grand coalition, and if a smaller group of player can not receive at least what they can achieve on their own, they will not take part in the grand coalition.

3.1 The core

The imputations are satisfying for the 1-player coalitions, but if there exists a coalition S whose total payoff is less than what the coalition can achieve acting by itself, then this coalition will not take part in bigger coalitions, so the coalition N might not form. Because of this we need a new definition which assures that the problem above will not happen.

3.1.1 Definition. *An imputation* $x \in \mathbb{R}^n$ *is in the* **core** *of the game, if it is* ***group rational*** *i.e.*

$$\sum_{i \in S} x_i \geq v(S),\ \forall\ S \subset N.$$

3.1.2 Example. In the glove market, if $|L| = |R|$ (the market is balanced), then the imputations in the core are imputations in which the players of L get the payoff p, and the players of R get $1-p$, where $p \in [0,1]$. If one of the sets have more items than the other, then the oversupplied product becomes depreciated ($p \in \{0,1\}$). In that case the core contains a single payoff vector.

3.1.3 Example. The core of the horse-traders game is the set of vectors $x = (x_A, x_B, x_C)$, where:

$$\begin{aligned} x_A + x_B + x_C &\geq 100 \\ x_A + x_B &\geq 80 \\ x_A + x_C &\geq 100 \\ x_B + x_C &\geq 0 \\ x_A, x_B, x_C &\geq 0 \end{aligned}$$

The core is non-empty if and only if $\min x_A + x_B + x_C$ does not reach $v(ABC) = 100$. By solving the LP-problem above, we get that the core is $\{(x_A, 0, 100 - x_A),\ 80 \leq x_A \leq 100\}$. This means that trader A sells the horse to C for an amount of money between 80 and 100, and player B gets nothing.

3.1.4 Remark. A good question is, can we characterize the core of a game, decide if a given imputation belongs to the core. However, this is not possible: Faigle, Fekete, Hochstättler and Kern had shown that it is coNP-complete to decide if a vector x is in the core or not.

3.1.1 Convex games

When we are looking for good imputations for all players, the question arises what kind of games have non-empty core. In this section we will see that there is a class of coalitional games that guarantees the core to be non-empty called convex games.

3.1.5 Definition. *The characteristic function v is called **supermodular**, if*

$$\forall\ S, T \subset N : v(S) + v(T) \leq v(S \cup T) + v(S \cap T).$$

*A game is **convex**, if its characteristic function is supermodular.*

3.1.6 Example. Every bankruptcy game is convex. (The proof of this theorem will be not presented here due to its length, and because bankruptcy games are not the main theme of the thesis.)

3.1.7 Remark. Additive game $\Rightarrow$ convex game $\Rightarrow$ superadditive game, but none of the implications is true backwards.

Supermodularity can be characterized in other ways too ([5]).

3.1.8 Theorem. *The following three statements are equivalent:*

1. *v is supermodular*
2. *$\forall\ S \subset T \subseteq N$ and $Z \subset N \setminus T$: $v(S \cup Z) - v(S) \leq v(T \cup Z) - v(T)$*
3. *$\forall\ S \subset T \subseteq N$ and $i \in N \setminus T$: $v(S + i) - v(S) \leq v(T + i) - v(T)$.*

Proof. Knowing that $v(S \cap Z) = v(T \cap Z) = 0$, the following chain of equivalences leads to the $1 \Leftrightarrow 2$ equivalency:

$$v(S) + v(Z) \leq v(S \cup Z) \text{ and } v(S) + v(Z) \leq v(S \cup Z)$$

By extracting these two inequalities:

$$v(T) - v(S) \leq v(T \cup Z) - v(S \cup Z)$$

And by regrouping it properly:

$$v(S \cup Z) - v(S) \leq v(T \cup Z) - v(T).$$

The $2 \Rightarrow 3$ implication is trivial ($Z = \{i\}$).
For $3 \Rightarrow 2$ we are going to use induction on the size of Z. If $|Z| = 1$ then statement 3 comes immediately. In the general step suppose that $i \in Z$ and that the statement 2 is true for S, T and $Z - \{i\}$:

$$v(S \cup (Z \setminus \{i\})) - v(S) \leq v(T \cup (Z \setminus \{i\})) - v(T) \tag{3.1}$$

Now, if we use the inequality 3 for $S \cup (Z \setminus \{i\})$ and $T \cup (Z \setminus \{i\})$ we get:

$$v(S \cup T) - v(S \cup (Z \setminus \{i\})) \leq v(T \cup Z) - v(T \cup (Z \setminus \{i\})) \tag{3.2}$$

By adding these two inequalities we get the statement 2. □

Now we show the most important property of convex games, namely that their core is non-empty ([6]).

3.1.9 Theorem. *The core of a convex coalitional game is nonempty.*

Proof. Let (N, v) be a convex coalitional game. Now, let us define a payoff vector x in the following way: $x_1 = v(\{1\})$, and for $i \geq 2$: $x_i = v(\{1, 2, \dots, i\}) - v(\{1, 2, \dots, i - 1\})$. In other words, the payoff of the i-th player is its marginal contribution to the coalition of the previous players ($\{1, 2, \dots, i\}$).

$$\begin{aligned}
&x_1 = v(\{1\}) \\
&x_2 = v(\{1, 2\}) - v(\{1\}) \\
&\dots \\
&x_i = v(\{1, 2, \dots, i\}) - v(\{1, 2, \dots, i - 1\}) \\
&\dots \\
&x_n = v(\{1, 2, \dots, n\}) - v(\{1, 2, \dots, n - 1\}) = v(N) - v(\{1, 2, \dots, i - 1\}).
\end{aligned}$$

If we sum up the equations above, we get: $\sum_{i=1}^{n} x_n = v(N)$, which is the condition of efficiency.
By convexity, using the inequality (3.1) for $S = \emptyset$, $T = \{i\}$ and $Z = \{1, 2, \ldots, i\}$, we get:

$$v(\{1, 2, \ldots, i-1\}) - v(\emptyset) \leq v(\{1, 2, \ldots, i\}) - v(\{i\}).$$

Ordering this:

$$v(\{i\}) \leq v(\{1, 2, \ldots, i\}) - v(\{1, 2, \ldots, i-1\}) = x_i$$

hence the vector is individually rational.
Lastly, we have to prove that the vector is group rational too. Let $S \subseteq N$, $S = \{a_1, \ldots, a_k\}$, and suppose that $a_1 < \ldots < a_k$. It is obvious that $S \subseteq \{1, 2, \ldots, a_k\}$. Using the second statement of Theorem 3.1.8 we obtain the following:

$$\begin{aligned}
&v(\{a_1\}) - v(\emptyset) \leq v(\{1, \ldots, a_1\}) - v(\{1, \ldots, a_1 - 1\}) = x_{a_1} \\
&v(\{a_1, a_2\}) - v(\{a_1\}) \leq v(\{1, \ldots, a_2\}) - v(\{1, \ldots, a_2 - 1\}) = x_{a_2} \\
&\ldots \\
&v(\{a_1, \ldots a_i\}) - v(\{a_1, \ldots a_{i-1}\}) \leq v(\{1, \ldots, a_k\}) - v(\{1, \ldots, a_i - 1\}) = x_{a_i} \\
&\ldots \\
&v(\{a_1, \ldots a_k\}) - v(\{a_1, \ldots a_{k-1}\}) \leq v(\{1, \ldots, a_k\}) - v(\{1, \ldots, a_k - 1\}) = x_{a_k}
\end{aligned}$$

By summing these inequalities we get:

$$v(S) = v(\{a_1, \ldots a_k\}) \leq \sum_{i=1}^{k} x_{a_k} = x(S)$$

which is the group rationality condition. So the imputation x above is always in the core of the game, and therefore that is nonempty. □

3.1.10 Remark. The converse of the theorem is not true, there are games with nonempty core, that are not convex. The convexity of a coalitional game is too strong condition, many real-world games are not convex. There is a weaker condition called balancedness.

3.1.2 Balanced games

3.1.11 Definition. *The **indicator function** of the set S is $I_S : N \to \{0, 1\}$, where:*

$$I_S(i) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise.} \end{cases}$$

3.1.12 Definition. *A map $\alpha : 2^N \to [0, 1]$ is **balanced map** if*

$$\forall\ i \in N : \sum_{S \subseteq N} \alpha(S) I_S(i) = 1.$$

3.1.13 Definition. *A game (N, v) is **balanced** if for any balanced map α:*

$$\sum_{S \subseteq N} \alpha(S)v(S) \leq v(N).$$

Now, let us see what is the connection between a balanced game, and the existence of a nonempty core. With the following theorem, we have a complete characterization of games with nonempty core.

3.1.14 Theorem (Bondareva-Shapley). *A coalitional game has a nonempty core if and only if it is balanced.*

The following proof is taken from [7]. This theorem is usually proved with linear programming techniques, using the inequalities that define the core. Instead of this, we will prove the theorem in a different way. We use a two-player non-cooperative zero-sum game, which can be generated from any cooperative game. Then we prove the connection between the Nash-equilibrium of the non-cooperative game and the imputations in the core.

Firstly, we define the non-cooperative game $\mathrm{G}(N, v)$:

- The set of strategies of the first player is $N = \{1, \ldots n\}$ (the set of players in the original coalitional game).
- The set of strategies of the second player is the set of coalitions formed by at least one player ($= \{S \subseteq N : S \neq \emptyset\}$).
- The payoff matrix of the first player is $A = (a_{ij}) \in \mathbb{R}^{n \times (2^n - 1)}$, where:
$$a_{ij} = \begin{cases} \frac{v(N)}{v(S_j)} & \text{if } i \in S_j \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The second player's payoff matrix is $-A$, since the game is zero-sum. Note that if $v(S_j) = 0$, then a_{ij} is undefined. If in this case we assign a very large number to a_{ij}, then the second player will never choose that strategy (column j), so we can restrict the strategies to the subsets S_j, where $v(S_j) > 0$. The first player only gets a positive payoff, if she chooses a player (from the original game) belonging to the chosen coalition of the second player. The amount received by the first player is inversely proportional to the worth of the coalition chosen by the second player. Let (s, t) be one of the Nash-equilibriums of the game $G(N, v)$. We denote $w(G(N, v))$ the payoff of the first player in the case of Nash-equilibrium, $w(G(N, v)) = s^T At$. This is the value of the game which means using the right strategies, this value is always reachable for the first player, and the second player can only lose more than this (these two strategies are called the best public strategies of both player). From Neumann-theorem, this value is always well defined (and it is also a Nash-equilibrium).

3.1.15 Lemma. $0 < w(G(N, v)) \leq 1$.

Proof. All the columns of the matrix A have at least one nonzero entry. If the first player plays $x = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)$, then all components of the vector $x^T A$ are positive. By multiplying this with any column of A, we get $w > 0$. The entries of the last column are all ones $\left(\frac{v(N)}{v(N)}\right)$, so the second player can always prevent the first player from getting a payoff greater than 1 by playing the last column. Therefore we get the $w \leq 1$ inequality too. □

3.1.16 Theorem (Aumann). *If the game (N, v) is balanced, then $w(G(N, v)) \geq 1$.*

Proof. For simplicity, assume that the game is 0-1 normalized, i.e. $v(\{i\}) = 0$ and $v(N) = 1$. By contradiction, assume that $w(G(N, v)) = a$, where $0 < a < 1$. Then there is a mixed strategy y for the second player, that guarantees that the payoff of the first player will be at most a. Denote with F the family of coalitions S where $v(S) > 0$, and let $y_S > 0$ for all $S \in F$. We have that for all rows i:

$$\sum_{S \in F, i \in S} \frac{y_S}{v(S)} \leq a, \text{ thus } \sum_{S \in F, i \in S} \frac{y_S}{av(S)} \leq 1.$$

We now define $w(S) = \frac{y_S}{av(S)}$ for all $S \in F$. Then

$$\sum_{S \in F, i \in S} w(S) \leq 1.$$

Now let $w_i = 1 - \sum_{S \in F, i \in S} w(S)$. Considering the family of coalitions T consisting of F and all the $\{i\}$-s, we receive

$$\sum_{S \in T, i \in S} w(S) = \sum_{S \in F, i \in S} w(S) + w_i = 1,$$

showing that T is a balanced family with balanced weights $w(S)$.
Since the game is balanced we have $\sum_{S \in T} w(S) \leq v(N)$, and from $v(\{i\}) = 0$ we obtain $\sum_{S \in F} w(S) v(S) \leq v(N)$. This implies

$$\sum_{S \in F} \frac{y_S}{a} \leq v(N) = 1, \text{ i. e. } \sum_{S \in F} y_S \leq a < 1,$$

which contradicts that y is a mixed strategy for the second player. □

3.1.17 Remark. As a consequence of the previous two theorems we have the equality $w(G(N, v)) = 1$, but in the proof of Bondareva-Shapley theorem we are going to use $w(G(N, v)) \geq 1$.

Now we can easily prove the Bondareva-Shapley theorem.

Proof. First we show that if the core is non-empty then the game is balanced: Let x be in the core, $\sum_{i=1}^{n} x_i = v(N)$, $\sum_{i \in S} x_i \geq v(S)\ \forall\ S \subseteq N$, and consider a balanced map α. We have:

$$\alpha(S) \sum_{i \in S} x_i \geq \alpha(S) v(S)\ \forall\ S \subseteq N, \text{ from which:}$$

$\sum_{S \subseteq N} \alpha(S) \sum_{i \in S} x_i \geq \sum_{S \subseteq N} \alpha(S) v(S)$, and since α is a balanced map:

$$\sum_{S \subseteq N} \alpha(S) \sum_{i \in S} x_i = \sum_{i=1}^{n} x_i = v(N) \geq \sum_{S \subseteq N} \alpha(S) v(S)$$

we get that the game is balanced. For the other direction, we use Theorem 3.1.15. By balancedness, $w(G(N,v)) \geq 1$. Then there is a mixed strategy for the first player, x, which gives a payoff at least one, independently of the pure strategy chosen by the second player. Then we have:

$$\sum_{i \in S} x_i \frac{v(N)}{v(S)} \geq 1, \text{ i.e.} \sum_{i \in S} x_i v(N) \geq v(S) \ \forall \ S \subseteq N.$$

Therefore, the vector $x^* = xv(N)$ is in the core of (N, v), so that is nonempty.

□

Of course, not every game is balanced. For these games, where the core is empty, we need alternative solutions to define the right payoff vector. The most important among these is the Shapley-value and the nucleolus.

3.2 The Shapley value

3.2.1 Definition. *A **value function** ϕ is a function that assigns a value to each possible characteristic function of a n-player game, $\phi(v) = (\phi_1(v), \phi_2(v), \ldots, \phi_n(v))$, $\phi_i(v) \in \mathbb{R}$.*

We now define the concept of fairness with the help of the following four axioms:

3.2.2 Definition. *The **Shapley-axioms** for $\phi(v)$:*

1. ***Efficiency:*** $\sum_{i \in N} \phi_i(v) = v(N)$.
2. ***Symmetry:*** *If i, j are such that $v(S \cup \{i\}) = v(S \cup \{j\}) \ \forall \ S \subset N$ where $i, j \notin S$, then $\phi_i(v) = \phi_j(v)$.*
3. ***Dummy action:*** *If i is such that $v(S) = v(S \cup \{i\}) \ \forall \ S \subset N$ where $i \notin S$, then $\phi_i(v) = 0$.*
4. ***Additivity:*** *If u and v are characteristic functions, then $\phi(u + v) = \phi(u) + \phi(v)$.*

Efficiency, symmetry and dummy action are natural assumptions. Additivity is needed when we want our game to be replayed, and considered together as one game. It is surprising, but there is only one function that satisfies the Shapley-axioms, and it also can be given with a close formula. Proof of this is put together from [1], [5] and [6].

3.2.3 Theorem (Shapley). *There exists a unique function $\phi(v)$ satisfying the Shapley axioms.*

Proof. For a nonempty set $S \subset N$ let w_S represent the special characteristic function defined for all $T \subset N$ as follows:

$$w_S(T) = \begin{cases} 1, & \text{if } S \subset T \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

Before we continue the proof of the Shapley theorem, we show that every characteristic function can be represented as a weighted sum of characteristic functions of the form (3.3):

3.2.4 Lemma. *Every characteristic function v can be written as $v = \sum_{S \subset N} c_S w_S$, where w_S is the characteristic function defined above, and c_S is a suitable constant.*

Proof. Let $c_\emptyset = 0$, and define by induction on the number of elements in T, $\forall\ T \subset N$:

$$c_T = v(T) - \sum_{S \subset N} c_S.$$

Each c_T is defined in terms of c_S for all $S \subset T$. Then:

$$\sum_{S \subseteq N} c_S w_S(T) = \sum_{S \subseteq T} c_S = c_T + \sum_{S \subset T} c_S = v(T).$$

Hence, $v = \sum_{S \subset N} c_S w_S$ as was to be shown. □

From axiom 1, $\sum_{i \in N} \phi_i(w_S) = w_N(S) = 1$. From axiom 2, if $i \in S$, and $j \in S$, then $\phi_i(w_S) = \phi_j(w_S)$. From axiom 3, $\phi_i(w_S) = 0$, if $i \neq S$. Combining these three together we have: $\phi_i(w_S) = \frac{1}{|S|}$, if $i \in S$. Applying this similar to the characteristic function cw_S with a given constant c, we find:

$$\phi_i(cw_S) = \begin{cases} \frac{c}{|S|}, & \text{if } i \in S \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

Using the lemma above and the axiom 4, if a value function exists, it must be in the form of:

$$\phi_i(v) = \sum_{S \subseteq N, i \in S} \frac{c_S}{|S|}. \tag{3.5}$$

This works even when some of the constants c_S are negative, because of the axiom 4 (u, v and $u - v$ are characteristic functions, $u = (u - v) + v$, and from this we get that $\phi(u-v) = \phi(u) - \phi(v)$). Lastly, we have to prove that (3.5) with c_S given in (3.4) satisfies the Shapley-axioms. That could be done directly too, but since there is a closed formula for the Shapley-value (presented in Theorem 3.2.6), from unicity we have that it should be the same as (3.5). For unicity, suppose that there are two sets of constants, c_S and c'_S such that

$$v(T) = \sum_{S \subseteq N} c_S w_S = \sum_{S \subseteq N} c'_S w_S, \forall\ T \subseteq N.$$

By induction we show that $c_S = c'_S$ for all $S \subset T$. If $|T| = 1$, then all $w_S(T)$ vanish except for $S = \{i\}$, so we have $c_{\{i\}} = c'_{\{i\}}$ for all $i \in N$. Now, let Z be an arbitrary coalition, and suppose that $c_S = c'_S$, $\forall\ S \subset Z$. If $T = Z$ in the equation above, then all terms cancel except term with $S = Z$, so we have $c_Z = c'_Z$. □

3.2.5 Remark. Superadditivity was not needed in the proof above, so the Shapley-value is proper even if the game is not superadditive. However, our main aim is to work with superadditive games, so this property is not needed further.

3.2.6 Theorem. *The Shapley-value can be written in the form:*

$$\begin{aligned}\phi_i(v) &= \sum_{S \subset N \setminus \{i\}} \frac{|S|!(n-|S|-1)!}{n!} \left(v(S \cup \{i\}) - v(S)\right) \\ &= \frac{1}{n} \sum_{S \subset N \setminus \{i\}} \frac{1}{\binom{n-1}{|S|}} (v(S \cup \{i\}) - v(S)).\end{aligned}$$

Before we turn to the proof of the theorem, let us see what the Shapley-value intuitively means. Suppose that the players arrive in a random order, that means we choose one by the all $n!$ possible orders. Then the quantity $v(S) - v(S \cup \{i\})$ is the amount by which the value of coalition S increases when player i joins it. The probability that when i enters, he will find the coalition S there already, is

$$\frac{(|S|-1)!(n-|S|-1)!}{n!} = \frac{1}{n\binom{n-1}{|S|}}.$$

Therefore $\phi_i(v)$ is the expected value of the players i contribution to the grand coalition, if the players sequentially form this coalition in a random order.

Proof. We have to prove that the formula above satisfies the Shapley-axioms, since we have already shown that there is at most one such function. Axioms 2, 3 and 4 come directly from the formula (for example, if $v(S) = v(S \cup \{i\})$, then $v(S) - v(S \cup \{i\}) = 0$, thus $\phi_i(v) = 0$). Since in each realization of forming the grand coalition, exactly $v(N)$ is given to the players, the average amount of given is also $v(N)$, hence axiom 1 is proven too. □

3.2.7 Example. The Shapley-value of the horse-traders game is: $\phi = (63\frac{1}{3}, 13\frac{1}{3}, 23\frac{1}{3})$. It can be clearly seen that in this case the Shapley-value does not belong to the core, where player B got nothing for its contribution, although he was important for the seller to get the price of the horse up. This is another proof that the core may be the set of stable imputations, a payoff allocation from it is not necessary fair.

3.2.8 Example. In the glove market, if $|L| = |R|$, then the Shapley-value is when every player gets $\frac{1}{2}$, that is a payoff from the core. But if the market is not balanced, the Shapley-value will not be the single value in the core where players get 0 or 1, every player gets some money, even if they have the oversupplied product. For example, if $|L| = 1$, and $|R| = 2$, then we get: $\phi(l, r_1, r_2) = \left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$.

3.2.9 Example. In the bankruptcy game of Talmud, the Shapley-value int the $E = 100$, and $E = 300$ cases is the same as the payoff defined in the book. If $E = 200$, the Shapley-value is: $\phi = (33\frac{1}{3}, 83\frac{1}{3}, 83\frac{1}{3})$, which is also similar to the imputation in the book.

Now, let us see some important properties of the Shapley-value ([6]).

3.2.10 Theorem. *For superadditive games, the Shapley-value is an imputation.*

Proof. Let (N, v) be a superadditive game. By definition we have:

$$v(S \cup \{i\}) - v(S) \geq v(\{i\}) \; \forall \; i \in N, S \subseteq N \setminus \{i\}$$

. This means that for each marginal vector the player i gets at least $v(\{i\})$. Since the Shapley-value is the average over all marginal vectors, it will be also individually rational. □

3.2.11 Theorem. *If the game is convex, then the Shapley-value belongs to the core.*

Proof. Let (N, v) be a convex game. In the proof of Theorem 3.1.9. we used a marginal vector to prove the core is nonempty, and hence all marginal vectors are in the core. The core is a convex set, therefore the Shapley-value (which is the average of the marginal vectors, finite points in the set) is also in the core set. □

3.3 The nucleolus

The core of the game can be empty, or can be too big for us to decide which imputation is the best. Another problem is, that the unique Shapley-value does not always belong to the core. We need an imputation which is as stable as possible, it always exists and is unique.

The nucleolus is an imputation that minimizes the dissatisfaction of the players. In other words, in this payoff the maximal dissatisfaction among the players is minimal ([8]).

3.3.1 Definition. *Let x be an imputation. The* ***excess*** *of the coalition S is:*

$$E(S, x) = v(S) - \sum_{i \in S} x_i.$$

3.3.2 Remark. Naturally, the imputations in the core are the payoffs whose excesses are negative or zero for all coalitions.

Let now $E(x) \in \mathbb{R}^{2^n - 2}$ be the vector whose elements are the excesses for all coalitions excluding the empty and the grand coalition (we don't have to deal with these, because both excesses are zero for all imputations), in a non-increasing order: $E_i(x) \geq E_j(x)$ for all $i < j$. We now have to define the lexicographic order of vectors.

3.3.3 Definition. *Let u and v be d-dimensional vectors. $u \leq_{lex} v$, if $\exists\ k > 0$ such that $\forall\ i < k : u_i < v_i$ and $u_k \leq v_k$.*

Now, if $\mathbb{E} = \{E(x),\ x$ is an imputation$\}$, then the nucleolus will be the imputation x corresponding to the lexicographically smallest $E(x)$ of $\mathbb{E}$ ($E^*(x)$).

3.3.4 Example. In the glove market, if the market is balanced (|L| = |R|), then every player gets $p = \frac{1}{2}$ (In the unbalanced case, the nucleolus is the single element of the core).

3.3.5 Example. In the horse-traders game the nucleolus is the vector $(90, 0, 10)$, which is the same-weighted average of the best payoff for A (100, 0, 0) and C (80, 0, 20).

3.3.6 Example. In the bankruptcy game, the nucleolus of the game is the same as in the Talmud document, in the $E = 200$ case too, where we get $n = (50, 75, 75)$. It is quite interesting how the creators of the law "guessed right" this mathematically correct imputation not only in the two easier cases, but in the third one too.

Now, we present some properties of the nucleolus ([9]).

3.3.7 Theorem. *The nucleolus always exists and it is unique.*

Proof. For simplicity, assume that the game is superadditive (if not, it has to be shown, that the nucleolus lives in a compact set). Let X represent the set of all imputations. Consider the following problem:

$$\max_{x \in X} \left(\min_S (x(S) - v(S)) \right).$$

The function $\min_S x(S) - v(S)$ is continuous, and the set X is compact, so the set of maximizers is non-empty and compact. Denote this by $X_1 \subseteq X$. Now, write the problem:

$$\max_{x \in X_1} \left(\min_S^2 (x(S) - v(S)) \right),$$

where $\min^2$ represents the second worst treated coalitions. By the same arguments, the set of maximizers, X_2 is non-empty and compact.
We have finite number of coalitions, so this process can be repeated only a finite number of times ($\min^3$, $\min^4$, ...). By induction, the nucleolus (that is considered as a set now) is non-empty.
Now we have to prove the uniqueness, namely that the set is a singleton. Suppose that x and y are in the nucleolus, and by contradiction, $x \neq y$. $E^*(x) = E^*(y)$ by definition, denote $S_1, \ldots, S_m$ ($m = 2^n - 2$) the list of coalitions as arranged in $E^*(x)$. Since $x \neq y$, there exists k to be the first in the order for which $E^*_k(x) \neq E(S_k, y)$, namely $E^*_k(x) < E(S_k, y)$. Also, for all $i > k$ in the order, $E^*_i(x) \geq E^*_k(x)$ and $E(S_i, y) = E^*_k(x)$. If we consider the allocation $z = \frac{x+y}{2}$, then $E^*_j(z) = E^*_j(x)$ for all $j < k$, $E^*_k(z) > E^*_k(x)$ and $E^*_i(z) > E^*_k(x)$. Thus $E(z) >_{lex} E(x)$, which is a contradiction, so the nucleolus must be unique. $\square$

3.3.8 Theorem. *If the game is superadditive, then the nucleolus is an imputation.*

Proof. The efficiency is trivial, so only the individual rationality has to be proven. By contradiction, let x be the nucleolus, and suppose that $\exists\ j \in N : x_j < v(\{j\})$. Let i be the player whose individual excess is the smallest: $x_i - v(\{i\}) = \min_{j\in N}\left(x_j - v(\{j\})\right)$. Denote by M the set of coalitions whose excess is the smallest at x, it must be the case that $i \in S\ \forall\ S \in M$. To see this, suppose not: $\exists\ S \in M, i \neq S$. Then we have $x(S \cup \{i\}) \leq x_i + x(S) - v(S) - v(\{i\}) < x(S) - v(S)$, which is contradictory. Thus, $i \in S$ for every $S \in M$.
Now, let y be the following allocation: $y_1 = x_1 + \epsilon$, and for all $j \neq i : y_j = x_j - \frac{\epsilon}{n-1}$ ($\epsilon > 0$ is a given constant). It is clear that $\forall\ S \in M : E(S,x) < E(S,y)$, and by choosing ϵ arbitrary small, we receive that $\forall S \in M, T \notin M : E(S,x) < E(S,y)$. Thus, $E^*(y) >_{lex} E^*(x)$, which contradicts that x is in the nucleolus. □

3.3.9 Theorem. *The nucleolus belongs to the core, if that is nonempty.*

Proof. We have to prove only that the nucleolus is group rational, since the efficiency is trivial. The core is a compact set, and the imputations in the core are the payoffs whose excesses are negative or zero for all coalitions. If the core is non-empty, then there is a lexicographically minimal element in it, and that will be the nucleolus. □

3.3.10 Remark. We could be saying, that the nucleolus (as its name represents too) is the core of the core, its innermost point.

3.3.11 Theorem. *The nucleolus satisfies the first three Shapley-axioms.*

Proof. Let x be the nucleolus. The axiom 1 is trivial. For axiom 2, by contradiction suppose that $v(S \cup \{i\}) = v(S \cup \{j\})$ and $x_i > x_j$. If

$$y_i = y_j = \frac{x_i + x_j}{2}, \text{ and } y_k = x_k, \text{ if } k \neq i, j.$$

then the excess corresponding to y is a lexicographically smaller than the one corresponding to x, contradiction.
Axiom 3 can be proven similarly: suppose that $v(S \cup \{i\}) = v(S)$ and $x_i > 0$. Let $\epsilon = \frac{x_i}{n}$, and y the following:

$$y_i = \epsilon, \text{ and } y_j = x_j + \epsilon, \text{ if } j \neq i$$

The excess corresponding to y is also lexicographically smaller than corresponding to x, contradiction, therefore the axiom 3 is proved. □

3.3.12 Remark. From the proposition above, we would assume that the nucleolus and the Shapley-value stay close to each other. Later we will see, that in the model presented these two imputations seem to be exactly the same.

Chapter 4

Wind energy aggregations 1st model

The source of this chapter is [8].

4.1 Problem formulation and the market model

Consider a group of n independent wind power producers, $N = \{1, \dots, n\}$. The power produced by the farm i at time t is a scalar valued random process, denote this by $w_i(t)$. Thus, the collective wind power production is represented as a vector-valued random process: $w(t) = (w_1(t), \dots, w_n(t))$.

Note that this independence has nothing to do with probabilistic independence, it only means that the farms are individually producing power without helping each other (we may call it market independence). Of course, the random processes can be correlated to each other, that is a natural thought, for example they are all dependent on the weather conditions or some other factors.

If the random process is absolute continuous, then we can use the well known cumulative distribution function and probability density function, but in all cases (for instance if we have discrete random variables) the general cumulative distribution function can be used (that is also dependent on t):

4.1.1 Definition. *The general cumulative distribution function of $w(t)$ is:*

$$F(x,t) = P(w(t) \leq x) = P(w_1(t) \leq x_1, \dots w_n(t) \leq x_n).$$

The function $F(x,t)$ has support $[0, W_1] \times \dots \times [0, W_n]$, where W_i is the capacity of the wind producer i and the corresponding probability density function is denoted by $f(x,t)$ (but we are not going to need this in most cases).

Now, we have to define the producing process if the wind producers are willing to form coalitions $S \subseteq N$ to aggregate their wind power production, and go to the electricity market together.

4.1.2 Definition. *The aggregate output corresponding to a coalition $S \subseteq N$ is:*

$$w_S(t) = \sum_{i \in S} w_i(t),$$

and similarly, the cumulative distribution function is:

$$F_S(x,t) = P(w_S(t) \leq x).$$

We can calculate this CDF using the joint distribution. In the example we are going to elaborate, one coalition's distribution function depends on the correlation of the different random processes.

Distributions are here defined always on the interval $[t_0, t_f]$, with length of $T = t_f - t_0$. Of course, we don't want our functions to always depend on the exact time, but only the interval in which energy is produced.

4.1.3 Definition. *The* ***time-averaged*** *CDF and PDF are the following:*

$$\begin{aligned} F_S(x) &= \frac{1}{T}\int_{t_0}^{t_f} F_S(x,t)dt, \\ f_S(x) &= \frac{1}{T}\int_{t_0}^{t_f} f_S(x,t)dt. \end{aligned} \tag{4.1}$$

Now, let us see how the electricity market behaves in this model. Consider a market system that consists of a single *ex-ante forward market*, and an *ex-post imbalance penalty* for scheduled contract deviations. Contracts are structured as power levels that are constant over time intervals of a specific duration (typically an hour long). In the absence of energy storage capabilities, the decision of how much constant power to offer over any individual time interval is independent of the decision for other intervals. Hence, our analysis focuses on optimizing a *constant power contract* C over a single time interval $[t_0, t_f]$.

The *clearing price* in the forward market is denoted by p, and q represents the *imbalance penalty price* for contract shortfalls ($/ MW-hour). We also need the proportion of these two measures, $\gamma = \frac{p}{q}$.

Prices p and q can be also random variables, but in this model they are assumed to be fixed and known.

The profit acquired strongly depends on these two constants, the energy shortfalls and the amount of energy curtailed by the coalition S over the time interval $[t_0, t_f]$. Before we define these mathematically, we need a notation: let $x^+ = \max(x, 0)$.

4.1.4 Definition. *The* ***profit acquired****, the* ***amount of energy curtailed or spillage*** *and the* ***energy shortfall*** *of coalition S are:*

$$\begin{aligned}
\Pi(C, w_S) &= \int_{t_0}^{t_f} pC - q(C - w_S(t))^+ dt \\
\Sigma_+(C, w_S) &= \int_{t_0}^{t_f} (w_S(t) - C)^+ \\
\Sigma_-(C, w_S) &= \int_{t_0}^{t_f} (C - w_S(t))^+
\end{aligned} \tag{4.2}$$

Because we are dealing with random variables, we will be concerned with the expected profit, expected amount of curtailed wind energy, and the expected energy shortfall. We would like to have the **maximum expected profit** for a coalition S, so we have the following optimization problem:

$$\psi(w_S) = \max \mathrm{E}(\Pi(C, w_S)) \tag{4.3}$$

The **profit maximizing contract** corresponding to the same coalition S is:

$$C_S^* = \arg\max \mathrm{E}(\Pi(C, w_S)) \tag{4.4}$$

Further, to find these optimal values, we are going to need a generalized inverse function, as $F_S(x)$ is not an injection in most cases. Of course, the definition below can apply for these functions too.

4.1.5 Definition. *The* ***generalized inverse function*** *of the function F is:*

$$F^{-1}(y) = \inf\{x \in D_F : y \leq F(x)\}$$

.

4.1.6 Theorem. *Define the time-averaged distribution function $F_S(x)$ as in 4.1, and let γ be given. Then:*

1. *An optimal contract is given by: $C_S^* = F_S^{-1}(\gamma)$ (here we have $D_F = [0, W_S]$).*
2. *The expected profit, the spillage and the shortfall are given by:*

$$\begin{aligned}
E(\Pi(C, w_S)) &= qT \int_0^{\gamma} F_S^{-1}(y) dy \\
E(\Sigma_+(C, w_S)) &= T \int_0^{\gamma} (C_S^* - F_S^{-1}(y)) dy \\
E(\Sigma_-(C, w_S)) &= T \int_0^{\gamma} (F_S^{-1}(y) - C_S^*) dy
\end{aligned} \tag{4.5}$$

4.1.7 Theorem. *Let $\{C_1, \ldots, C_n\}$ be a set of n individual contracts. Then almost surely:*

$$\Pi(\sum_{i \in N} C_i, w_N) \geq \sum_{i \in N} \Pi(C_i, w_i).$$

Proof. We are going to use the subadditivity of x^+ and the properties of the integrate operator:

$$\Pi(\sum_{i\in N} C_i, w_N) = \int_{t_0}^{t_f} (p\sum_{i\in N} C_i - w_N(t))^+ dt \le \sum_{i\in N} \int_{t_0}^{t_f} (pC_i - w_i(t))^+ dt = \sum_{i\in N} \Pi(C_i, w_i).$$

□

Since we are in possession of the suitable game-theoretical background, we are able to analyze the cooperative game formalized above.

Firstly, we are going to have a lemma in connection with the map ψ defined in (4.3).

4.1.8 Lemma. *The map $\psi(x)$ is homogenous and superadditive, i.e.*

1. *for any $\lambda \in \mathbb{R}, \lambda \ge 0 : \psi(\lambda x) = \lambda\psi(x)$*
2. *$\psi(x) + \psi(y) \le \psi(x + y)$.*

Proof. For 1, let λ be fixed. Let $g_\alpha(x, t)$ denote the marginal CDF for the random process αx. Observe that:

$$g_\alpha(x, t) = P(\alpha x(t) \le x) = F\left(\frac{x}{\alpha}, t\right).$$

Then the time-averaged CDF, $G_\alpha(x)$ is similary given by:

$$G_\alpha(x) = \frac{1}{T}\int_{t_0}^{t_f} F\left(\frac{x}{\alpha}, t\right) dt = F\left(\frac{x}{\alpha}\right),$$

and the generalized inverse of G_α is in the form of:

$$G_\alpha^{-1}(y) = \alpha F^{-1}(y).$$

From this and theorem 4.1.6 we get the homogeneity instantly.
For 2, let x and y be stochastic processes. Then:

$$\psi(x) + \psi(y) = \max_{C_x \le 0} E(\Pi(C_x, x)) + \max_{C_y \le 0} E(\Pi(C_y, y)) = E(\Pi(C_x^*, x)) + E(\Pi(C_y^*, y)).$$

From Theorem 4.1.7 we have:

$$E(\Pi(C_x^*, x)) + E(\Pi(C_y^*, y)) \le E(\Pi(C_x^* + C_y^*, x + y)) \le E(\Pi(C_{x+y}^*, x + y) = \psi(x + y)$$

hence the superadditivity is proven. □

We are now able to define the coalitional game (N, v) according to the model above, where:

$$v(S) = \psi(w_S). \tag{4.6}$$

4.1.9 Theorem. *The coalitional game (N, v) for aggregation formation of wind producers is superadditive.*

Proof. The value function is defined as $v(S) = \psi(w_S)$, which is superadditive from the lemma above. More specifically, if $S, T \subseteq N, S \cap T = \emptyset$:

$$v(S) + v(T) = \psi(w_S) + \psi(w_T) \leq \psi(w_S + w_T) = v(S \cup T).$$

□

4.1.10 Theorem. *The game (N, v) is balanced, and therefore has a nonempty core.*

Proof. Let $\alpha : 2^N \to [0, 1]$ be an arbitrary balanced map.

$$\begin{aligned}
\sum_{S \subseteq N} \alpha(S) v(S) &= \sum_{S \subseteq N} \alpha(S) \psi(w_S) \\
&= \sum_{S \subseteq N} \psi(\alpha(S) w_S) \text{ from homogenity of } \psi \\
&\leq \psi(\sum_{S \subseteq N} \alpha(S) w_S) \text{ from superadditivity of } \psi \\
&= \psi(\sum_{i \in N} \sum_{S \subseteq N} \alpha(S) I_S(i) w_i) \\
&= \psi(\sum_{i \in N} w_i) = v(N) \quad \text{from balancedness of } \alpha.
\end{aligned}$$

□

For a convex game, Shapley-value provides an allocation that belongs to the core. The primary advantage of the Shapley-value is that it is given by a closed-form expression, and that makes its computation easy. In the case of nucleolus, we have difficulties in computation, because the process involves the lexicographical minimization of 2^n excess values, which is extremely costly. Unfortunately, our class of cooperative games is not always convex, as it will be shown in the next example.

4.1.11 Example. Consider three independent wind producers, $N = \{1, 2, 3\}$, $t \in [0, 1]$, each producers capacity is 2, the random processes w_1 and $w_2 = w_3$ are independent with the following probability distribution: $w_{1,2}(t) = \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ 2 & \text{w.p. } \frac{1}{2}. \end{cases}$ Suppose that the clearing price and the penalty for shortfall are $p = 0.5$, and $q = 1$. Then we have:

$$\begin{aligned}
&v(\{i\}) = \psi(w_i) = 0.5, \\
&v(\{1, 2\}) = v(\{1, 3\}) = \psi(w_1 + w_i) = 1.25 \\
&v(\{2, 3\}) = \psi(w_2 + w_3) = 1 \\
&v(\{1, 2, 3\}) = \psi(w_1 + w_2 + w_3) = 1.75
\end{aligned}$$

Note that $v(\{1, 2, 3\}) - v(\{1, 2\}) = 0.5 < v(\{1.3\}) - v(\{1\}) = 0.75$, which contradicts the supermodularity property, so the game is not convex.

4.2 Discussion of an example

In this section we are going to present in detail the 5.1 example in the article [8]. Our point of departure is the original example, but later we are going to change the cumulative distribution functions and the number of players. We are going to calculate the important game-theoretical terms like the core, the Shapley-value and the nucleolus.

Consider three wind power producers with unit capacity each. Their power outputs are independent random processes, which are given the table below. $x \in \Omega$ represents a fraction of the power capacity, and $P(x)$ is the probability of its occurence:

$x \in \Omega$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$P(x)$	0.05	0.18	0.26	0.18	0.10	0.08	0.05	0.04	0.03	0.02	0.01

The cumulative distribution function and the probability density function of the discrete random variable for $y \in [0,1]$ are defined as:

$$f(y) = \sum_{x\in\Omega} P(x)\delta(y-x) \tag{4.7}$$

$$F(y) = \sum_{x\in\Omega} P(x)H(y-x), \tag{4.8}$$

where:

$$\delta(x) = \begin{cases} 1, x = 0 \\ 0, x \neq 0 \end{cases} \text{ and } H(x) = \begin{cases} 1, x \leq 0 \\ 0, x < 0. \end{cases}$$

The time interval is $[0,1]$, the shortfall penalty is $q = 1$, the clearing price is $p \in [0,1]$, and therefore we have $\gamma = \frac{p}{q} = p$.

The maximum expected profit for each aggregation can be obtained with the expressions (4.1), (4.6) and (4.8), and it is given by the following: The maximum expected profit for each aggregation can be obtained with the expressions (4.1), (4.6) and (4.8), and it is given by the following:

$$v(S) = pF_S^{-1}(p) - \sum_{x\in\Omega_S} P(x)(F_S^{-1}(p) - x)^+ \tag{4.9}$$

We are going to work with the coalitions: $S_1 = \{1\}$, $S_2 = \{1,2\}$ and $S_3 = \{1,2,3\}$. Ω_S denote the sample space of the random process w_S, so we have $\Omega_{S_1} = [0,1]$, $\Omega_{S_2} = [0,2]$ and $\Omega_{S_3} = [0,3]$.

The sum of two random variables is given by convolution, so using this and the expression above we are able to compute the CDF and the maximum expected profit for aggregations S_1, S_2 and S_3. We were using Matlab to compute and show these functions.

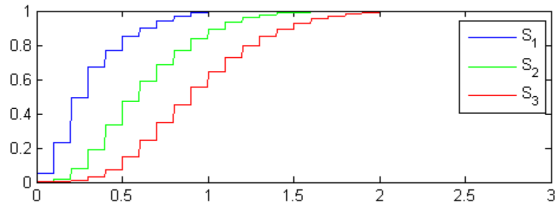

Figure 4.1: The cumulative distribution function for different aggregations

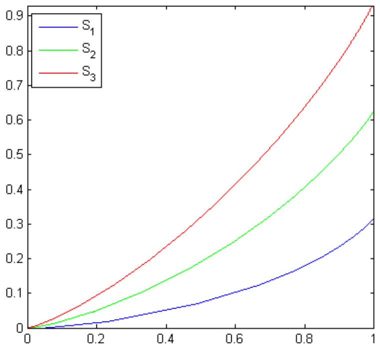

Figure 4.2: Expected profit

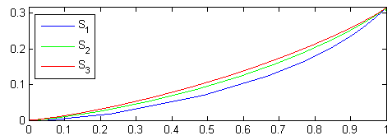

Figure 4.3: Normalized expected profit

The expected profit is normalized dividing by the capacity of the aggregation, and it is always increasing except when $\gamma = p = 0$ and $\gamma = p = 1$. We are going to ignore these limiting cases because there are computational issues around these, the marginal errors are too high.

Defining the characteristic function as the expected profit for aggregations, we have different coalitional games for each value of $\gamma \in [0, 1]$. The core here is two-dimensional, so we can not visualize it in each point, since the function we got goes from one dimension to 2-D. Instead we are going to show the core in three different points, for $\gamma = 0.2$, $\gamma = 0.5$ and $\gamma = 0.8$, presented in Figure 4.4. We used the program Polymake for computing and visualizing the core, and note that we multiplied all values of imputation by 10.

The game is symmetric i. e. each coalition with the same size has the same characteristic function. Thus, the polytopes representing the core are also symmetric, and the Shapley-value, just like the nucleolus, is at the exact center of the figures, with a payoff allocation where every player gets the same amount of money, namely $\frac{v(S_3)}{3}$.

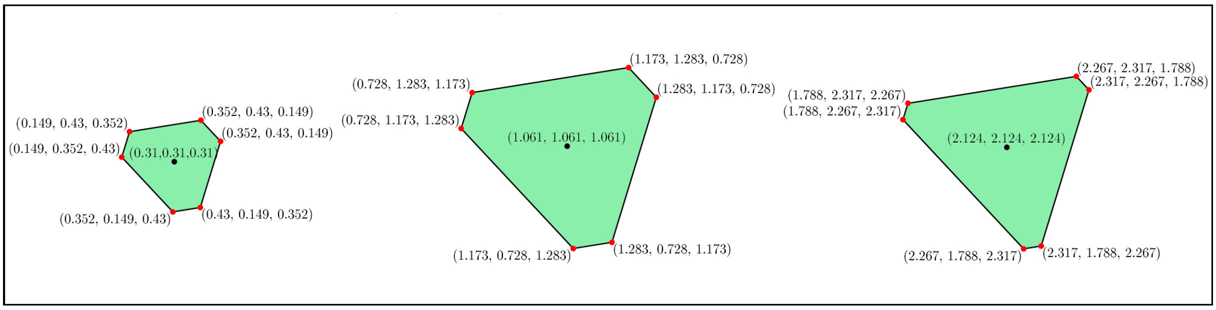

Figure 4.4: Imputations in the core and the Shapley-value/nucleolus for three value of γ

The superadditivity of the game is clearly visible from the normalized expected profit, which is even more noticable, if there are more players. The case of seven players is presented at Figure 4.6.

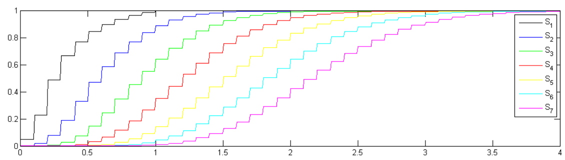

Figure 4.5: CDF for the 7-players game

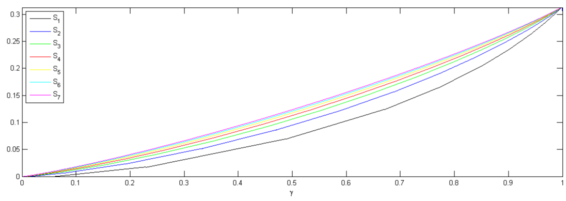

Figure 4.6: Normalized expected profit for 7 players

The dimension of the core is always $n-1$, since the equality $v(N) = \sum_{i=1}^{n} v(\{i\})$ defines a hyperplane that has to contain all the points. In the case of four players we are receiving 3-dimensional polyhedrons as it is shown below the case $\gamma = 0.5$ (Figure 4.7).

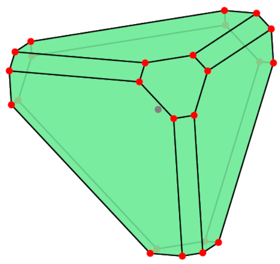

Figure 4.7: The core and the Shapley-value/nucleolus for 4 players

While there are some cases where farmers get wind energy from the same distribution, it is more natural to assume that the distributions are different for each player. Furthermore we define similar, but not equal distributions, and see what changes in the game. The rule of how the distribution works is the same as before, but we have three different distribution now, which are shown in the table below:

$x \in \Omega$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$P_1(x)$	0.05	0.18	0.26	0.18	0.10	0.08	0.05	0.04	0.03	0.02	0.01
$P_2(x)$	0.05	0.15	0.15	0.15	0.10	0.10	0.06	0.04	0.03	0.02	0.01
$P_3(x)$	0.05	0.30	0.25	0.20	0.09	0.03	0.02	0.02	0.02	0.01	0.01

By using the first two distributions above we receive the CDF and expected profit for two players as it follows:

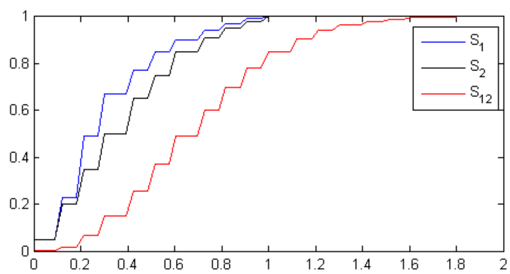

Figure 4.8: The CDF-s in the 2-player asymmetric case

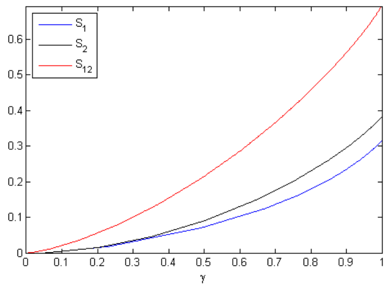

Figure 4.9: Expected profit for 2 players

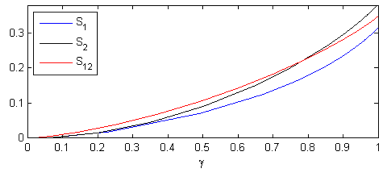

Figure 4.10: Normalized expected profit for 2 players

In the case of two players, the core is a closed interval, so we are able to show it for every $\gamma \in [0, 1]$ by marking each value with a different color. The results can be seen in the Figure 4.11.

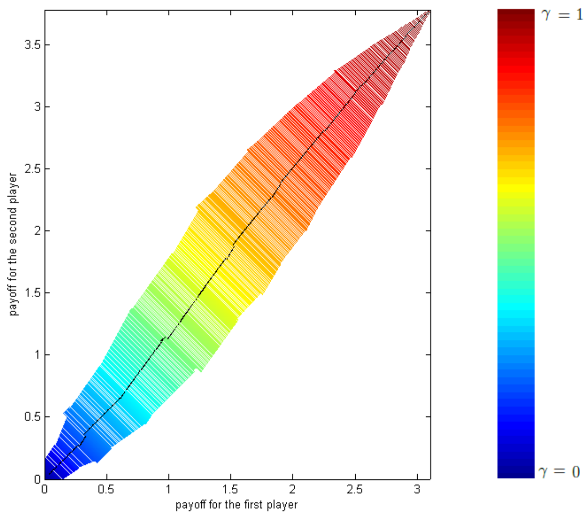

Figure 4.11: The core, Shapley-value and nucleolus for each value of γ

While when players distributed power equally, it was logical that the Shapley-value and the nucleolus were the same. It is interesting that this seems to happen in the non-equal case too. Calculating the two payoffs separately and then displaying them resulted that they are in the same spot again, marked with black at each point.

In the case of three players, all coalitions (even the ones that contain the same amount of players) are slightly different, but it can also be seen that bigger coalitions produce much power as they are supposed to.

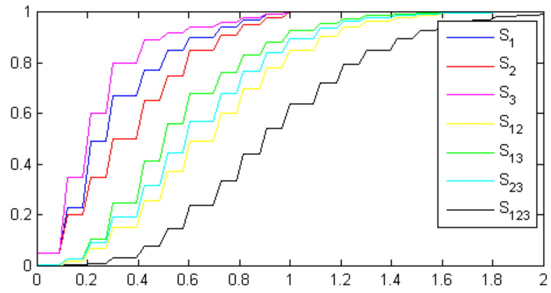

Figure 4.12: The CDF-s in the 3-player asymmetric game

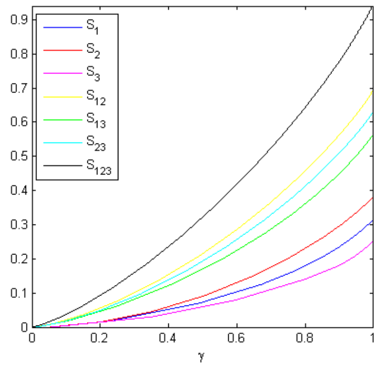

Figure 4.13: Expected profit in the 3-player asymmetric game

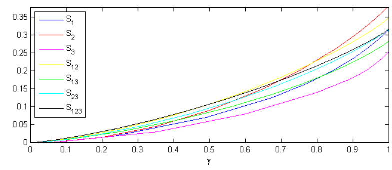

Figure 4.14: Normalized expected profit in the 3-player asymmetric game

The core of the game is 2-dimensional just like in the symmetric case. We observe the same as in the 2-player case: the Shapley-value and the nucleolus coincide again.

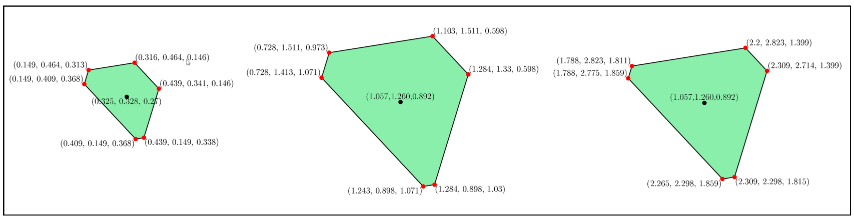

Figure 4.15: The Core and the Shapley-value/nucleolus in the 3-player asymmetric case

Observing all these, we can make the following hypothesis in connection with the game exposed above.

4.2.1 Hypothesis. The coalitional game in the Example 5.1 is convex, and the Shapley-value is equal to the nucleolus in all cases.

Chapter 5

Wind energy aggregations 2nd model

In this chapter, we are going to show an improved model for wind energy producers presented in [10]. The basic definitions are the same as in the first model: we define the random process $w(t)$, the CDF and PDF, and the time-averaged versions of these. Instead of the generalized inverse function we use the quantile function as it follows:

5.0.1 Definition. *For any $\beta \in [0, 1]$, the β-**quantile function** of F is*

$$F^{-1}(\beta) = \inf\{x \in [0, 1] : \beta \leq F(x)\}.$$

We are going to have this quantile function for every coalition S, $F_S^{-1} : [0, 1] \to [0, \sum_{i \in S} W_i]$.

Now, let us see the new market model. We consider a competitive market with a *two-settlement structure* through which wind power producers submit offers for energy. It consists of two *ex-ante* market: a day-ahead forward markets (DA) and a real-time spot market (RT), and one *ex-post* market for penalties. The penalty price for uninstructed deviations reflects the RT price of energy, it is assumed to be unknown at the close of DA market, and not revealed until the RT market is cleared. Let C denote the *constant power contract* in the DA market, for the coalition S scheduled to deliver power continuously over the time interval $[t_0, t_f]$. We will restrict the analysis to a single contract interval as we did in the first model. The optimization problem is again to find the maximum expected profit obtained by a coalition, and to find the contract corresponding to this. The *clearing price* p and the *penalty for shortfalls* q appear here too, but we also have a price λ (\$/ MW-hour) for *positive deviations*. We make use of the following market assumptions:

1. We disregard the network structure of the power system. All wind generators are connected to a common bus in the power network, which is uncongested, and the locational marginal prices are uniform.

2. The wind power producers behave as price takers, and the forward price p is assumed fixed and known.

3. The wind power producers have a zero marginal cost of production.
4. The imbalance prices q and λ are random variables, with expectations denoted by μ_q and μ_λ. These are assumed to be statistically independent of $w(t)$.
5. The imbalance prices q and λ are assumed to be non-negative.

5.0.2 Definition. *Under this market model, the* ***profit acquired*** *by a coalition $S \subseteq N$ for a contract C on the interval $[t_0, t_f]$ is:*

$$\Pi(C, w_S, q, \lambda) = \int_{t_0}^{t_f} pC - q(C - w_S(t))^+ - \lambda(w_S(t) - C)^+ dt. \tag{5.1}$$

The expression compared to the first model is widened with the factor of positive deviations (third term). The profit is dependent on the random wind power process w_S.

5.0.3 Definition. *The* ***expected profit***, ***shortfall*** *and* ***surplus*** *relative to a contract C is:*

$$\begin{aligned} J_S(C) &= E(\Pi(C, w_S, q, \lambda)) \\ H_S^+(C) &= E(\int_{t_0}^{t_f} (w_S(t) - C)^+) \\ H_S^-(C) &= E(\int_{t_0}^{t_f} (C - w_S(t))^+) \end{aligned} \tag{5.2}$$

Similar to the first model, the maximum expected profit is:

$$\psi(w_S) = \max_{C \geq 0} J_S(C) \tag{5.3}$$

The profit maximizing contract corresponding to the same coalition S is:

$$C_S^* \in \arg\max_{C \geq 0} J_S(C) \tag{5.4}$$

5.0.4 Theorem. *Denote the time-averaged distribution $F_S(w)$. Then:*

1. *An optimal contract is given by: $C_S^* = F_S^{-1}(\gamma)$, where $\gamma = \frac{p+\mu_\lambda}{\mu_q+\mu_\lambda}$,*
2. *The optimal expected profit, the surplus and the shortfall are given by:*

$$\begin{aligned} \frac{J_S(C_S^*)}{T} &= \mu_q \int_0^\gamma F_S^{-1}(x)dx - \mu_\lambda \int_\gamma^1 F_S^{-1}(x)dx \\ H_S^+(C_S^*) &= T \int_0^\gamma (F_S^{-1}(x) - C_S^*)dx \\ H_S^-(C_S^*) &= T \int_0^\gamma (C_S^* - F_S^{-1}(x))dx \end{aligned} \tag{5.5}$$

5.1 Coalitional value-at-risk deviation

The optimal expected profit depends explicitly on a measure of statistical dispersion referred to as the conditional value-at-risk deviation measure.

5.1.1 Definition. *For any $\gamma \in (0,1)$, the **conditional value-at-risk deviation measure** of $W_S \sim F_S$ is defined as: $D_\gamma(W_S) = E(W_S) - E(W_S : W_S \leq F_S^{-1}(\gamma))$.*

The conditional value-at-risk deviation measure measures the gap between the unconditional mean and the mean in the γ-probability trail. Using algebraic manipulations we can rearrange the expected profit:

$$\frac{J_S(C_S^*)}{T} = pE(W_S) - pD_\gamma(W_S) \tag{5.6}$$

The important question is the net change induced through aggregation. Let

$$\Delta_S = \sum_{i \in S} D_\gamma(W_i) - D_\gamma(W_S) \tag{5.7}$$

denote **the reduction in dispersion** induced by aggregating the production from farms in the coalition S. It can be shown easily that $\Delta_S \geq 0$ for all $S \subseteq N$. It follows that the net benefit due to aggregation is directly attributable to the reduction in dispersion. In particular:

$$J_S(C_S^*) - \sum_{i \in S} J_i(C_i^*) = pT\Delta_S. \tag{5.8}$$

It is clear that the aggregation will improve the optimal expected profit insomuch as it reduces the statistical dispersion of the aggregate output.

5.1.2 Theorem. *Let $\Lambda = \{\lambda \in \mathbb{R}^n, \lambda \geq 0, \sum_{i \in N} \lambda_i = pT\Delta_N\}$. Then the set of imputations in the coalitional game is given by:*

$$I = \{x \in \mathbb{R}^n, x_i = v(\{i\}) + \lambda_i, \lambda \in \Lambda, \forall i \in N.$$

Proof. The individual rationality is trivial, since $\lambda_i \geq 0$. Using the expression Δ_N in 5.7, efficiency follows from:

$$\sum_{i \in N} x_i = \sum_{i \in N} J_i(C_i^*) + pT\Delta_N = J_N(C_N^*) = v(N).$$

□

Theorems 4.1.8-4.1.11 can also be applied here, with the newly introduced variables, but with the same proofs. Note that the proofs of these appear in the second article, but for coherence we presented them in the first model.

5.2 Realized Coalition Profit

It is certainly true that the expected optimal profit is non-negative. However, it may happen that it may can take negative values for some cases because of certain imbalance prices. This leads us to introduce and explore profit allocations mechanism to distribute the realized profit among the coalition members ex-post. It is expected that the payment of coalition members, averaged over an increasing number of days, approaches an imputation x^* in the core of the game.

To see this, assume that the wind power process w_N^k and imbalance prices q^k and λ^k are independent and identically distributed across days indexed by k. This implies that the optimal profit corresponding to an $S \subseteq N$ is also an independent and identically distributed sequence:

$$\Pi_S^k = \Pi(C_S^*, w_S^k, q^k, \lambda^k), \text{ where } C_S^* = F_S^{-1}(\gamma). \tag{5.9}$$

Let the allocation of the profit realized on day k be denoted by $\rho^k = (\rho_1^k, \ldots, \rho_n^k)$, where member i received ρ_1^k on day k.

5.2.1 Definition. *A profit allocation $\rho^k \in \mathbb{R}^n$ is **budget balanced** with respect to the profit realized on day k, if $\sum_{i=1}^n \rho_i^k = \Pi_N^k$.*

5.2.2 Definition. *A mechanism for daily profit allocation ρ^k is **strongly consistent** with respect to a fixed allocation $x \in \mathbb{R}^n$, if*

$$\frac{1}{K}\sum_{i=1}^{K} \rho_i^k \to x_i \text{ almost surely.}$$

Now, consider the following naive mechanism for daily profit allocation. Let x^* be an imputation in the core. Given a realization profit for the grand coalition at day k, Π_N^k, distribute the profit among the coalition members according to the following rule:

$$\rho_i^k = \beta_i \Pi_N^k, \text{ where } \beta_i = \frac{x_i^*}{\sum_{j=1} n x_j^*}. \tag{5.10}$$

5.2.3 Theorem. *The profit allocation mechanism above is both budget balanced and strongly consistent with respect to the corresponding imputation x^* in the core.*

Proof. Since $\sum_{i=1}^n \beta_i = 1$, we have the budget balancedness. On the other hand, we have independent and identically distributed random variables, so the law of large numbers is applicable, which leads to strong consistency. □

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