Numerical Solution of Two-Point Boundary Value Problems

B.S.c. Thesis

by

Gabriella Sebestyén

Mathematics B.S.c., Mathematical Analyst

Supervisor:

István Faragó

Professor at the

Department of Applied Analysis and Computational Mathematics

Eötvös Loránd University

Budapest

2011

Contents

1 Introduction

The object of my dissertation is to present the numerical solution of two-point boundary value problems.
In some cases, we do not know the initial conditions for derivatives of a certain order. Instead, we know initial and final values for the unknown derivatives of some order. These type of problems are called boundary-value problems.
Most physical phenomenas are modeled by systems of ordinary or partial differential equations. Usually, the exact solution of the boundary value problems are too difficult, so we have to apply numerical methods.

We used different numerical methods for determining the numerical solutions of Cauchy-problem. One of them is the Explicit Euler method, which is the simplest scheme.

The Improved Euler method is the simplest of a family of similar predictor-corrector methods following the form of a single predictor step and one or more corrector steps. One subgroup of this family are the Runge-Kutta methods which use a fixed number of corrector steps. The Improved Euler method is the simplest of this subgroup.

All the methods given can be applied to higher of ordinary differential equations, provided it is possible to write an explicit expression for the highest order derivative and the system has a complete set of initial conditions.

Motivation

The following example describes a physical task what we can use in practice and it originates from a two-point boundary value problem.

Shooting Problem

We launch a cannonball from a fixed place. Let $y(t)$ be the height of the cannonball and $x(t)$ the distance from the fixed place at $t \geq 0$ moment.
Moreover we suppose that the horizontal speed is constant, the vertical distance depends only on the gravitation, so we dispense with the drag co-efficient.
The goal is to determine the angle of the launch, if the cannonball is located in place $x = L$!

The continous mathematical model of the problem is the following:

$$\begin{aligned} x'(t) = v,\ y''(t) = -g, \\ x(0) = 0,\ y(0) = 0. \end{aligned} \tag{1}$$

The solution of the second equation with the initial condition is:

$$x(t) = vt,$$
which means that
$$t = \tfrac{x}{v}.$$

Let us introduce a new function $Y(x)$ as follows:

$$y(t) = y(\tfrac{x}{v}) = Y(x).$$

Then, using the chain rule, we get

$$\begin{aligned} y'(t) &= \frac{dY}{dx}\frac{dx}{dt} = \frac{dY}{dx}v, \\ y''(t) &= v\frac{d^2Y}{dx^2}v = v^2\frac{d^2Y}{dx^2}. \end{aligned} \tag{2}$$

Hence, we obtain the problem:

$$\begin{aligned} Y''(x) &= -\frac{g}{v^2}, x \in (0, L), \\ Y(0) &= 0,\ Y(L) = 0. \end{aligned} \tag{3}$$

This problem can be easily solved and the solution is:

$$Y(x) = \frac{gx}{2v^2}(L - x). \tag{4}$$

Then we can determine the angle of the launch from the following relation:

$$\tan(\alpha) = Y'(0) = \frac{gL}{2v^2}. \tag{5}$$

Summarize the shooting problem, according to (5) the solution always exists. It means we can lauch the cannonball to every distance. But it is a paradox in the reality.

2 Solvability of Boundary Value Problems

In this section, we appoit the two-point boundary value problem generally.

Definition 1 *Let* $f : \mathbb{R}^3 \to \mathbb{R}$ *given function and* α, β *are given numbers. The problem*

$$u'' = f(t, u, u'),\ t \in (a, b), \tag{6}$$

$$u(a) = \alpha,\ u(b) = \beta \tag{7}$$

is called two-point boundary value problem.

When we analyze the Cauchy problem for the ordinary differential equation of first order in the form

$$u'(t) = g(t, u), \tag{8}$$

$$u(t_0) = u_0. \tag{9}$$

we have seen that the solvability depends only on the function f. The following theorem gives sufficient condition of the existance. The theorem connect to this problem:

Theorem 1 *Suppose* $g : [t_0 - \alpha, t_0 + \alpha] \times \overline{B(a, \beta)} \to \mathbb{R}$ *is continous and bounded by M. Suppose, furthermore, that* $g(t, \cdot)$ *is Lipschitz countinous with Lipschitz constant L for every* $t \in [t_0 - \alpha, t_0 + \alpha]$*. Then the problem (8), (9) has a unique solution* $u(t)$ *defined on* $[t_0 - b, t_0 + b]$*, where* $b = min\left\{\alpha, \frac{\beta}{M}\right\}$.

If we analyze boundary value problems (6), (7) the situation is different. As we will see, both the function f and the boundary value determine the result together.

Example

Let $f(t, u, u') = 1 - u$, hence the equation (6) has the form

$$u'' + u = 1.$$

The arbitrary solution of this differental equation is $u(t) = c_1 \cos t + c_2 \sin t + 1$, where c_1, c_2 are constants. We analyze different boundary conditions in (7), which should define these constans.

- Firstly, let us put $a = \frac{\pi}{4}, \alpha = 2, b = \pi, \beta = 2$. The solution is unique: $c_1 = -1, c_2 = \sqrt{2} + 1$.
  Hence, the unique solution is $u(t) = -1 \cos t + (\sqrt{2} + 1) \sin t + 1$.

- If we put $a = \frac{\pi}{4}, \alpha = 2, b = \frac{5\pi}{4}, \beta = 2$, then the problem has no solution, because do not exist such constants c_1, c_2 for which the boundary value is true.

The following theorem gives sufficient condition.

Theorem 2 *Suppose that,* $T = (t_1, s_1, s_2) : t \in [a, b], s_1, s_2, \in \mathbb{R}$ *and* $f : \mathbb{R}^3 \to \mathbb{R}$ *is a given function with the properties*

- $f \in C(T)$,

- $\partial_1 f, \partial_2 f \in C(T)$,

- $\partial_2 f > 0$ *on* T,

- *there exists a nonnegative constant* M *such that* $|\partial_3 f| \leq M$ *on* T.

Under these conditions the two-point boundary value problem (6), (7) has unique solution.

Corollary

Let us consider the special case when f is linear,

$$u'' = f(t, u, u') = p(t)u' + q(t)u + r(t), \ t \in (a, b),$$
$$u(a) = \alpha, \ u(b) = \beta$$

where $p, q, r \in C[a, b]$ are continous functions. If $q(t) > 0$ for all $t \in [a, b]$, then the linear boundary value problem has a unique solution, because the conditons of Theorem 2 mean the following

- $p(t)s_1 + q(t)s_2 + r(t) \in C(T)$,
- $\partial_1 f = r(t), \partial_2 f = q(t) \in C(T)$,
- $\partial_2 f = q(t) > 0$,
- $\partial_3 f = p(t)$, i.e $|p(t)|$ is bounded.

Obviously, tehere conditons are sutisfied.

3 Shooting Method

3.1 Introduction

Consider this two-point boundary value problem:

$$u'' = f(t, u, u'), \ t \in [a, b] \tag{10}$$
$$u(a) = \alpha, u(b) = \beta.$$

When we use the Shooting Method, we transform (10) into the Cauchy-problem of the form

$$u'' = f(t, u, u') \tag{11}$$
$$u(a) = \alpha, u'(a) = c. \tag{12}$$

where c is an unknown number.

Obviously, the solution of the problem (11), (12) depends on the choice of the unknown parameter c. Therefore we use the notation $u(t, c)$ for the solution of this problem. Our aim is to define c such that $u(b, c) = \beta$.

Let $G : \mathbb{R} \ \to \mathbb{R}$ function, defined as $G(c) = u(b, c) - \beta$.

Hence, we search for c^* defined as:

$$G(c^*) = 0. \tag{13}$$

Therefore, the realization of the Shooting Method needs the solution of some (usually nonlinear) equation. Finding the root needs to apply some numerical method. Its realization can be in different ways.

3.2 Intervallum-Bisection Method

The Intervallum-Bisection Method is one of the best simple method. We search for two values c_1 and c_2 such that $G(c_1)G(c_2) < 0$, so we can determine the root by we consecutive halving the intervallum.

The Shooting Method Intervallum-Bisection's algorithm

1. We fix any value, c.

2. We solve an initial value problem with (3), (4) some known numerical method an the interval $[a, b]$.

3. In the first step we search for two c values, for which the $t = b$ point values will be different signs. These values are notated as c_1 and c_2.

4. Execute second step with $c = 0.5(c_1 + c_2)$.

5. We define the sign of $G(c)$.

6. We reselect new c_1 or c_2 value and contiunue the method.

7. When the two end points between the distance is less then $\epsilon > 0$ value, we stop the iteration.

3.3 Chord Method

The Shooting Chord Method's algorithm

1. Let $c^{(0)}$ and $c^{(1)}$ be any two given values.

2. We solve the initial value problem (3), with a numerical method on interval $[a, b]$ with $c^{(0)}$ and $c^{(1)}$, respectivelly.

3. Hence $G(c^{(0)}) = u(b, c^{(0)})$ and $G(c^{(1)}) = u(b, c^{(1)})$.

4. We use linear interpolation between the points $(c^{(0)}, G(c^{(0)}))$ and $(c^{(1)}, G(c^{(1)}))$, in order to determine the point $c^{(2)}$, where this approximation turns into zero:

$$c^{(2)} = c^{(1)} - \left(\frac{c^{(1)} - c^{(0)}}{G(c^{1}) - G(c^{(0)})} \right) G(c^{(1)})$$

5. We define $G(c^{(2)})$.

6. For any steps, we use the iteration:

$$c^{(k)} = c^{(k-1)} - \left(\frac{c^{(k-1)} - c^{(k-2)}}{G(c^{k-1}) - G(c^{(k-2)})} \right) G(c^{(k-1)}), k = 2, 3...$$

and after that, we determine $G(c^{(k)})$ value.

7. When $\left|G(c^{(k)})\right|$ is less then some fixed $\epsilon > 0$ value, we stop the iteration.

3.4 Newton Method

The Newton Method for the solution of the nonlinear equation has the form. We assume that G is differentiable and $G' \neq 0$. Then

$$c_{n+1} = c_n - \frac{G(c_n)}{G'(c_n)}, \;\; n = 0, 1, 2, ...$$

Problem: how to define the derivative $G'(u)$?

Let $u = u(t, c)$ the solution. Then from (10) we have

$$u''(t,c) = f(t, u(t,c), u'(t,c))), \;\; t \in (a,b) \tag{14}$$

$$u(a,c) = \alpha, \;\; u'(a,c) = c. \tag{15}$$

Let us define the derivative of (14), (15) with respect to the paratmeter c.

$$\frac{\partial u''}{\partial c}(t,c) = \partial_2 f(t, u(t,c), u'(t,c))\frac{\partial u}{\partial c}(t,c) + \partial_3 f(t, u(t,c), u'(t,c))\frac{\partial u'}{\partial c}(t,c) \tag{16}$$

$$\frac{\partial u(a,c)}{\partial c} = 0, \;\; \frac{\partial u'(a,c)}{\partial c} = 1.$$

If we define the new function:

$$w(t) = \frac{\partial u}{\partial c}(t,c), \tag{17}$$

then we obtain

$$w''(t) = \partial_2 f(t, u(t,c), u'(t,c))w(t) + \partial_3 f(t, u(t,c), u'(t,c))w'(t), \tag{18}$$

$$w(a) = 0, \;\; w'(a) = 1.$$

The problem (18) is a Cauchy problem, which can be solved numerically by some standard method.
Hence $G'(c) = w(b)$, therefore it will be the solution of the task.

We summarize the Newton Method's algorithm

1. Let $c \in \mathbb{R}$.

2. We solve the $u'' = f(t, u, u'), u(a) = \alpha, u'(a) = c_n$.

3. We determine the solution of this differential equation:

$$w'' = \partial_2 f(t, u, u')w + \partial_3 f(t, u, u')w'$$
$$w(a) = 0, \; w'(a) = 1.$$

4. If $|u(b) - \beta| < \epsilon \to$ stop.
   If $|u(b) - \beta| \geq \epsilon$, then $c_{n+1} = c_n - \dfrac{u(b) - \beta}{w(b)}$ and we continue from the step 2 by putting $c = c_{n+1}$.

Summary

As we can see a large variety of different methods can be used to solve two-point boundary value problems. The intervallum-bisection method and chord-method are of first oreder method.

The chord-method's advantage in front of the intervallum-bisection method. We do not search two initial points, where values will have different signs. We can start with any given $c^{(0)}$ and $c^{(1)}$ values. On the other hand, the convergence of the method can not be provided.

The Newton-method is second order method. Hence, it is more efficient, than intervallum-bisection method and chord-method. Otherwise, the Newton-method is needed to calculate the derivative. If the root is too far from the initial value, the Newton method does not converge.

For this reason, in most practical applications we determine the maximum number of the iterations.

In some cases, the numerical problems can be badly conditioned. It is not possible to solve these with standard shooting method, so we have to use multiple shooting-methods. (We suggest to read more about book [3].)

4 Finite Difference Schemes

As we have seen in the last section, the use of the shooting method is demanding.

In the this section, we describe the numerical solutions of two-point boundary value problems by using finite difference method.

The finite difference techniques are based upon the approximations that allow to replace the differential equations by finite difference equations. These finite difference approximations are in algebraic form, and the unknowns solutions are related to grid points. Thus, the finite difference solution basically involves three steps:

1. We define the sequence of the meshes on the solution domain $[a, b]$.
2. We approximate the given differential equation by the system off difference equations that relates the solutions to grid points.
3. We solve the above algebraic system of the equations.

4.1 Finite Differences

Henceforward, we use these relations:

$$u'(t) = \frac{u(t+h) - u(t)}{h} - \frac{1}{2}hu''(\xi_1) \tag{19}$$

$$u'(t) = \frac{u(t) - u(t-h)}{h} + \frac{1}{2}hu''(\xi_2) \tag{20}$$

$$u'(t) = \frac{u(t+h) - u(t-h)}{2h} - \frac{1}{6}h^2u'''(\xi_3) \tag{21}$$

$$u''(t) = \frac{u(t+h) - 2u(t) + u(t-h)}{h^2} - \frac{1}{12}h^2u^{(4)}(\xi_4) \tag{22}$$

where $\xi_i \in (t, t+h)(i = 1, 2, 3, 4)$ are fixed numbers.
These relations can be verified easily by use of the Taylor expansion with remainder in Lagrange form.

If the function $u(t)$ is smooth enough, we can approximate function $u(t)$ in some fixed point t^*.

This gives the idea to approximate the derivative of $u(t)$(smooth enough) function in a fixed point t^* as follows

$$u'(t^*) \cong \frac{u(t^*+h)-u(t^*)}{h},\ u'(t^*) \cong \frac{u(t^*)-u(t^*-h)}{h}, \tag{23}$$

$$u'(t^*) \cong \frac{u(t^*+h)-u(t^*-h)}{2h},\ u''(t^*) \cong \frac{u(t^*+h)-2u(t^*)+u(t^*-h)}{h^2}$$

4.2 Solution with Finite Differences Method

Let

$$\overline{\omega_h} = \left\{ t_i = a+ih, i=0,1,....,N+1, h=\frac{b-a}{N+1} \right\} \tag{24}$$

be an equisdistante mesh, where $N \in \mathbb{R}\, h$ is the step-size of mesh, i.e, .

Let ω_h denote the inside points of the mesh,

$$\omega_h = \left\{ t_i = a+ih, i=1,....,N, h=\frac{b-a}{N+1} \right\}, \tag{25}$$

finally γ_h is the boundary-points of the mesh:

$$\gamma_h = \{t_0 = a, t_{N+1} = b\}. \tag{26}$$

Assume that, (9),(10) has unique solution, that is there exists a function $u(t) \in C^2[a,b]$ such that

$$u''(t) = f(t, u(t), u'(t)),\ t \in (a,b), \tag{27}$$

$$u(a) = \alpha,\ u(b) = \beta.$$

Hence, substituting the mesh-points $t_i \in \omega_h$ into (27), we obtain the relation:

$$u''(t_i) = f(t_i, u(t_i), u'(t_i)), i=1,2,...,N, \tag{28}$$

$$u(t_0) = \alpha,\ u(t_{N+1}) = \beta.$$

When f is some given smooth function, for it's derivatives we will use the forward-difference, backward-difference, and central-difference formulas respectively.

The forward-difference formula, which is based on (19) is the following:

$$y_{x,i} = \tfrac{y_{i+1}-y_i}{h}.$$

The backward-difference formula, which is based on (20) is the following:

$$y_{\overline{x},i} = \tfrac{y_i - y_{i-1}}{h}.$$

The central-difference formula, which is based on (21) is the following:

$$y_{x^\circ,i} = \tfrac{y_{i+1}-y_{i-1}}{2h}.$$

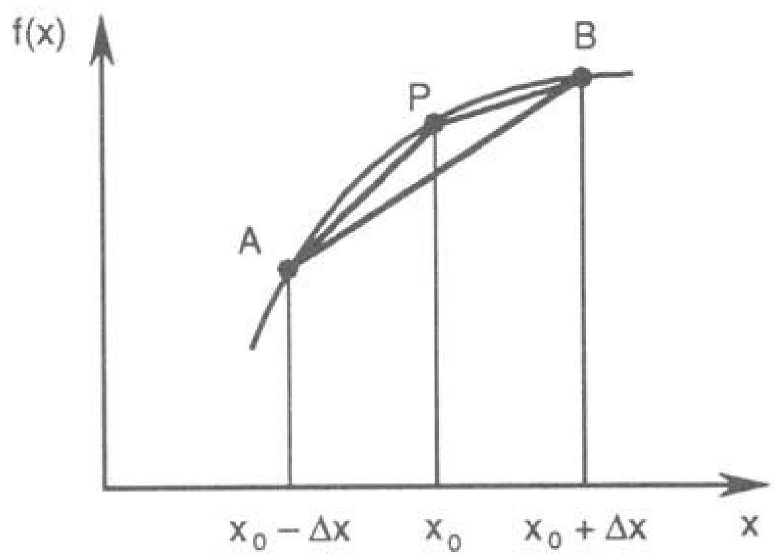
If we use these relations we get

$$y_{\overline{x}x,i} = f(t_i, y_i, y_{x,i}),\ i = 1, 2, ..., N, \tag{29}$$
$$y_0 = \alpha,\ y_{N+1} = \beta.$$

$$y_{\overline{x}x,i} = f(t_i, y_i, y_{\overline{x},i}),\ i = 1, 2, ..., N, \tag{30}$$
$$y_0 = \alpha,\ y_{N+1} = \beta.$$

$$y_{\overline{x}x,i} = f(t_i, y_i, y_{x^\circ,i}),\ i = 1, 2, ..., N, \tag{31}$$
$$y_0 = \alpha,\ y_{N+1} = \beta.$$

This picture shows the numerical derivatives to the function f at the point P, with different formulas: forward, backward, and central differences. The forward-difference and backward-difference are first order, the central-difference is second order approximations for the first derivative in x_0 point.



If we use this relation for the derivatives, we will have equations for the unknown value y_i as follows from the (29) relation:

$$f(t_i, y_i, \frac{y_{i+1} - y_i}{h}) = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}, i = 1, 2, ..., N, \tag{32}$$

$$y_0 = \alpha,\ y_{N+1} = \beta.$$

We can solve simultaneously these equations, where $y_0, y_1, ..., y_{N+1}$ are the unkonown values.

But, we do not know the answer for the following questions:

- Does an unique solution exist for the system (32)?
- How can we provide a solution for the system efficiently?
- Is the approximation y_i far from the exact solution $u(t_i)$?

4.3 The Solution of Linear Boundary Value Problems with Finite Differences

In this section we apply the difference method to find the solution of linear boundary value problems of the form:

$$u'' = p(t)u' + q(t)u + r(t), t \in [a, b], \tag{33}$$

$$u(a) = \alpha, \quad u(b) = \beta.$$

We have seen in the chapter 2, that this problem has a unique solution when: $p, q, r \in C[a, b]$ and $q(t) > 0$ at intervall $[a, b]$.

Let us denote by $p_i = p(t_i)$, $q_i = q(t_i)$ and $r_i = r(t_i)$.

Theorem 3 *The finite difference discretization of the linear boundary value probelm (33) results in a system of linear algebraic equations of the form*

$$a_i y_{i-1} + d_i y_i + c_i y_{i+1} = -r_i,\ i = 1, 2, ..., N \tag{34}$$

$$y_0 = \alpha,\ y_{N+1} = \beta$$

where the coefficients have

$$|d_i| - |a_i| - |c_i| = q_i > 0. \tag{35}$$

Proof

The discretization for the (31) problem is:

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = p_i \frac{y_{i+1} - y_i}{h} + q_i y_i + r_i,\ i = 1, 2, ..., N, \tag{36}$$

$$y_0 = \alpha, \quad y_{N+1} = \beta.$$

It is a linear system of equations.

If we choose:

$$a_i = -\frac{1}{h^2}, d_i = \frac{2}{h^2} + q_i - \frac{1}{h}p_i, c_i = -\frac{1}{h^2} + \frac{1}{h}p_i, \tag{37}$$

we will get the (34) task.

When h is small enough:

$$|d_i| - |a_i| - |c_i| = \frac{1}{h^2}((2 + h^2 q_i - hp_i) - 1 - (1 - hp_i)) = q_i. \tag{38}$$

If we use the discretization (30), we will be able to determine a_i, d_i, c_i easily:

$$a_i = -\frac{1}{h^2} - \frac{1}{h}p_i,\ d_i = \frac{2}{h^2} + q_i + \frac{1}{h}p_i,\ c_i = -\frac{1}{h^2}. \tag{39}$$

We can get another relations, when we use the discretization (29):

$$a_i = \frac{-1}{h^2} - \frac{1}{2h}p_i,\ d_i = \frac{2}{h^2} + q_i,\ c_i = \frac{-1}{h^2} + \frac{1}{2h}p_i. \tag{40}$$

Let e_i be the difference between the exact and the approximate values at the point $t = t_i$. It means: $e_i = u_i - y_i$.
Then we introcude the error function as follows:

$$e_h(t_i) = e_i$$

Definition 2 *A numerical method is called convergent, if the error function tends to zero, when $h \to 0$, that is:*

$$\lim_{h \to 0} e_h = 0. \tag{41}$$

If $e_h = O(h^p)$, we call the numerical method of pth-order method.

Theorem 4 *The (37), (39), (40) finite differenc method is convergent of the form to the solution of the linear boundary value problems,*

1. *the method of (40) is second-order,*
2. *the (37) and (39) are first-order methods.*

Proof

1., Firstly we prove the second statement.
If we use (21) and (22) relation in $t = t_i$ point:

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \frac{1}{12}h^2 u^{(4)}(\xi_4^{(i)}) = p_i \left(\frac{u_{i+1} - u_{i-1}}{2h} - \frac{1}{6}h^2 u'''(\xi_3^{(i)}) \right) + q_i u_i + r_i. \tag{42}$$

From the last relation we get:

$$\left(-\frac{1}{h^2} + \frac{p_i}{2h} \right) u_{i+1} + \left(\frac{2}{h^2} + q_i \right) u_i + \left(-\frac{1}{h^2} - \frac{p_i}{2h} \right) u_{i-1} = -r_i - h^2 g_i, \tag{43}$$

where

$$g_i = \frac{1}{12} u^{(4)} \left(\xi_4^{(i)} \right) - \frac{p_i}{6} u'''(\xi_3^{(i)}). \tag{44}$$

Hence we can write as

$$a_i u_{i-1} + d_i u_i + c_i u_{i+1} = -r_i - h^2 g_i. \tag{45}$$

If we use the relation (32), we will get a linear system of equations:

$$a_i e_{i-1} + d_i e_i + c_i e_{i+1} = -h^2 g_i, \;\; i = 1, 2, ..., N, \;\; e_0 = e_{N+1} = 0 \tag{46}$$

The ith equation of (46) can be writtem as

$$d_i e_i = -a_i e_{i-1} - c_i e_{i+1} - h^2 g_i. \tag{47}$$

We estimate (46) in the absolult value

$$|d_i||e_i| \leq |a_i||e_{i-1}| + |c_i||e_{i-1}| + h^2|g_i| \leq (|a_i| + |c_i|)||e_h||_\infty + h^2||g||_\infty \tag{48}$$

Let i_0 be the index, for which $||e_h||_\infty = |e_{i_0}|$. Since this statement is true for all $i = 1, 2, ..., N$, we have

$$|d_{i_0}|||e_h||_\infty \leq (|a_{i_0}| + c_{i_0})||e_h||_\infty + h^2||g||_\infty \tag{49}$$

We can get from (49)

$$(|d_{i_0}| - |a_{i_0}| - |c_{i_0}|)||e_h||_\infty \leq h^2||g||_\infty \tag{50}$$

If we use the property (35)

$$q_{i_0}||e_h||_\infty \leq h^2||g||_\infty \tag{51}$$

Use the $q_{min} \geq 0$ relation, we get:

$$||e_h||_\infty \leq h^2 \frac{||g||_\infty}{q_{min}} \leq \widetilde{C} h^2, \tag{52}$$

where

$$\widetilde{C} = \left(\frac{M_4}{12} + \frac{p_{max} M_3}{6}\right)/q_{min},\ M_j = max|u^{(j)}|,\ p_{max} = max|p|.$$

It means, that in case $h \to 0$, the method have second-order convergence.

Now we prove the first statement
If we use (19) and (22) relations in the point $t = t_i$:

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \frac{1}{12} h^2 u^{(4)}(\xi_4^{(i)}) = p_i \left(\frac{u_{i+1} - u_i}{h} - \frac{1}{2} h u''(\xi_1^{(i)})\right) + q_i u_i + r_i. \tag{53}$$

We get from the last relation:

$$\left(-\frac{1}{h^2} + \frac{p_i}{h}\right) u_{i+1} + \left(\frac{2}{h^2} + q_i - \frac{1}{h} p_i\right) u_i + \left(-\frac{1}{h^2}\right) u_{i-1} = -r_i - h g_i, \tag{54}$$

where

$$g_i = -\frac{1}{12} u^{(4)}(\xi_4^{(i)}) + \frac{p_i}{2} u''(\xi_1^{(i)}). \tag{55}$$

Hence we can write with another notation as follows

$$a_i u_{i-1} + d_i u_i + c_i u_{i+1} = -r_i - h g_i. \tag{56}$$

If we use the relation (32), we will get a system of linear algebraic equations:

$$a_i e_{i-1} + d_i e_i + c_i e_{i+1} = -h g_i,\ i = 1, 2, ..., N,\ e_0 = e_{N+1} = 0. \tag{57}$$

The ith equation of (57) is

$$d_i e_i = -a_i e_{i-1} - c_i e_{i+1} - h g_i. \tag{58}$$

We estimate in absolult value this expression:

$$|d_i||e_i| \le |a_i||e_{i-1}| + |c_i||e_i - 1| + h|g_i| \le (|a_i| + |c_i|)||e_h||_\infty + h||g||_\infty \tag{59}$$

Let i_0 be an index, for which is $||e_h||_\infty = |e_{i_0}|$. Since, $\forall i = 1, 2, ..., N$ true the (58) statement, hence

$$|d_{i_0}|||e_h||_\infty \le (|a_{i_0}| + c_{i_0})||e_h||_\infty + h||g||_\infty \tag{60}$$

We can get from (58)

$$(|d_{i_0}| - |a_{i_0}| - |c_{i_0}|)||e_h||_\infty \le h||g||_\infty \tag{61}$$

If we use (35)

$$q_{i_0}||e_h||_\infty \le h||g||_\infty \tag{62}$$

Use the $q_{min} \geq 0$ condition:

$$||e_h||_\infty \leq h\frac{||g||_\infty}{q_{min}} \leq \widetilde{C}h, \tag{63}$$

where

$$\widetilde{C} = \left(\frac{M_4}{12} + \frac{p_{max}M_2}{2}\right)/q_{min},\ M_j = max|u^{(j)}|,\ p_{max} = max|p| \tag{64}$$

It means, if $h \to 0$, the method will be first-rate.

This table shows the order of the finite different approximations. Now the ∇x means h.

Table 1 Finite Difference Approximations for Φ_x and Φ_{xx}, where FD = Forward Difference, BD = Backward Difference, and CD = Central Difference

Derivative	Finite Difference Approximation	Type	Error
Φ_x	$\frac{\Phi_{i+1}-\Phi_i}{\Delta x}$	FD	$O(\Delta x)$
	$\frac{\Phi_i-\Phi_{i+1}}{\Delta x}$	BD	$O(\Delta x)$
	$\frac{\Phi_{i+1}-\Phi_{i-1}}{\Delta x}$	CD	$O(\Delta x)^2$
	$\frac{-\Phi_{i+2}+4\Phi_{i+1}-3\Phi_i}{2\Delta x}$	FD	$O(\Delta x)^2$
	$\frac{3\Phi_i-4\Phi_{i-1}+\Phi_{i-2}}{2\Delta x}$	BD	$O(\Delta x)^2$
	$\frac{-\Phi_{i+2}+8\Phi_{i+1}-8\Phi_{i-1}+\Phi_{i-2}}{12\Delta x}$	CD	$O(\Delta x)^4$
Φ_{xx}	$\frac{\Phi_{i+2}-2\Phi_{i+1}+\Phi_i}{(\Delta x)^2}$	FD	$O(\Delta x)^2$
	$\frac{\Phi_i-2\Phi_{i-1}+\Phi_{i-2}}{(\Delta x)^2}$	BD	$O(\Delta x)^2$
	$\frac{\Phi_{i+1}-2\Phi_i+\Phi_{i-1}}{(\Delta x)^2}$	CD	$O(\Delta x)^2$
	$\frac{-\Phi_{i+2}+16\Phi_{i+1}-30\Phi_i+16\Phi_{i-1}-\Phi_{i-2}}{(\Delta x)^2}$	CD	$O(\Delta x)^4$

Now we exlude the $q(t) \geq 0$ conditon and use the case, where $p(t) = 0$. Let us consider the following boundary-value problem:

$$u'' = r(t), t \in [a, b], \tag{65}$$

$$u(a) = \alpha, u(b) = \beta.$$

We use the uniform mesh of the form

$$\overline{\omega}_h = t_i = ih, i = 0, 1, ..., N+1, h = \frac{b-a}{N+1} \tag{66}$$

Using approximation for the second derivative, we obtaion for the parameter the following

$$a_i = -\frac{1}{h^2},\ d_i = \frac{2}{h^2},\ c_i = -\frac{1}{h^2}. \tag{67}$$

Hence, the task is to solve the system linear of algebraic equation of the form

$$A_h y_h = b_h, \tag{68}$$

Where $A_h \in \mathbb{R}^{N \times N}$ is given matrix in the form

$$A_h = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & \dots & 0 & 0 & -1 & 2 \end{pmatrix}$$

and $b_h \in \mathbb{R}^N$ is a vector of the form

$$b_h = \begin{pmatrix} -r(t_1) + \frac{\alpha}{h^2} \\ -r(t_i), i = 2, 3, ..., N-1 \\ -r(t_N) + \frac{\beta}{h^2} \end{pmatrix}$$

Firstly, we show that this discretization is correct.

Theorem 5 *The problem (68) has unique solution, for every $h > 0$.*

Proof

We have to show that the matrix A_h is regular, hence the $\lambda = 0$ is not its eigenvalue. Let us determine the eigenvalues of the A_h matrix. Since the A_h matrix is symmetric, the eigenvalues are real and hence $A_h v_k = \lambda_k v_k$.
Hence, $v_0 = v_{N+1} = 0$ and the v_i co-ordinate of the v_i eigenvalue:

$$\frac{-v_{i-1} + 2v_i - v_{i+1}}{h^2} = \lambda v_i,\ i = 1, 2, ..., N \tag{69}$$

We get this task:

$$v_{i-1} - 2(1 - 0.5\lambda h^2)v_i + v_{i+1} = 0,\ i = 1, 2, ..., N, \tag{70}$$

$$v_0 = 0,\ v_{N+1} = 0.$$

We search for the solution of (68) in the form

$$v_i = \sin(pt_i),\ i = 0, 1, ..., N + 1, \tag{71}$$

where, $p \in \mathbb{R}$ are some unknown numbers.

Subsituting in (71) into (69), we get:

$$\sin(p(t_i - h)) - 2(1 - 0.5\lambda h^2)\sin(pt_i) + \sin(p(t_i + h)) = 0,\ i = 1, 2, ..., N \tag{72}$$

Using these relation
$\sin(p(t_i - h)) + \sin(p(t_i + h)) = 2\sin(pt_i)\cos(ph)$ for (69), we obtain

$$2\sin(pt_i)\cos(ph) - 2(1 - 0.5\lambda h^2)\sin(pt_i) = 0, i = 1, 2, ..., N \tag{73}$$

We can rewrite the last result in the form

$$(2\cos(ph) - 2(1 - 0.5\lambda hr))\sin(pt_i) = 0, i = 1, 2, ..., N \tag{74}$$

Since, $\sin(pt_i) = v_i$ and the eigenvalue v is not zero, at least for one index i we have: $v_i \neq 0$. Hence,

$$2\cos(ph) - 2(1 - 0.5\lambda h^2) = 0. \tag{75}$$

From the relation (75) we have

$$\lambda = \frac{2}{h^2}(1 - \cos(ph)) = \frac{4}{h^2}\sin^2\frac{ph}{2}. \tag{76}$$

From (70) and (71) we get $v_0 = \sin(p \cdot 0) = 0, v_{N+1} = \sin(p \cdot l) = 0$ coherences wil be true. Because of this, $pl = k\pi$ therefore $p = \frac{k\pi}{l}$.
Using this relation in (76), we get the eigenvalues of matrix A_h :

$$\lambda_k = \frac{4}{h^2}\sin^2\frac{kh\pi}{2l},\ k = 1, 2, ..., N. \tag{77}$$

Otherwise, we can write the v_k eigenvector from (71)

$$v_k = \left(\sin\frac{k\pi ih}{l}\right)_{i=1}^{N}, k = 1, 2, ..., N. \tag{78}$$

Since, λ_k depends on h, we use the notation $\lambda_k = \lambda_k(h)$!
Obviously, $\lambda_k(h) > 0$ for every $k = 1, 2, ..., N$, so we can determine the least eigenvalue as

$$\min\lambda_k(h) = \lambda_1(h) = \frac{4}{h^2}sin^2\frac{\pi h}{2l}. \tag{79}$$

Let s denote a new variable:

$$s = \frac{\pi h}{2l}. \tag{80}$$

Since $h \leq l/2$, we can write this relation for the least eigenvalue in the form

$$\lambda_1(s) = \frac{\pi^2}{l^2}(\sin s/s)^2, s \in \left(0, \frac{\pi}{4}\right] \tag{81}$$

The function $\frac{sinx}{x}$ is monoton decreasing at intervall $(0, \frac{\pi}{4}]$, so

$$inf\lambda_1(h) = \lambda_1\left(\frac{l}{2}\right) = \frac{16}{l^2}sin^2\frac{\pi}{4} = \frac{8}{l^2}. \tag{82}$$

Hence, for all $l : \lambda_1(h) \geq \frac{8}{l^2} > 0$. Therefore $\lambda = 0$ is not an eigenvalue.

Example

In the following example we show the numerical solution of boundary value problems by use of finite differences.

$$u''(t) = 6t, \tag{83}$$

$$su(0) = 0, u(2) = 8.$$

We use the uniform mesh with $N = 4$,

$$\overline{\omega_h} = \left\{t_i = ih, i = 0, 1, 2, 3, 4, 5,\ h = \tfrac{2}{4+1} = 0,4\right\}.$$

Hence, the mesh points are

$$t_0 = 0,\ t_1 = 0.4,\ t_2 = 0.8,\ t_3 = 1.2,\ t_4 = 1.6,\ t_5 = 2.$$

By subsituting the last values, the matrix A_h has the form

$$A_h = \begin{pmatrix} 12.5 & -6.25 & 0 & 0 \\ -6.25 & 12.5 & -6.25 & 0 \\ 0 & -6.25 & 12.5 & -6.25 \\ 0 & 0 & -6.25 & 12.5 \end{pmatrix}$$

and the vector b_h is

$$b_h = \begin{pmatrix} -2.4 \\ -4.8 \\ -7.2 \\ 40.4 \end{pmatrix}$$

We solve these system $A_h y_h = b_h$. Hence

$$y_h = \begin{pmatrix} 0.064 \\ 0.512 \\ 1.728 \\ 4.096 \end{pmatrix}$$

The exact solution at these points are:

$$\begin{pmatrix} 0.064 \\ 0.512 \\ 1.728 \\ 4.096 \end{pmatrix}$$

The solution of the problem (83) is the function $u(t) = t^3$.
Therefore $M_4 = max\left|u^4(t)\right| = 0$. Since for this problem $p(t) = 0$, therefore $\widetilde{C} = 0$ in (64). The results in that the finite difference method is exact for this problem (83).

This example shows tha case, where $M_4 = max\left|u^4(t)\right| \neq 0$ and the result is not exact for this problem.

$$u''(t) = 24t^2,$$
$$u(0) = 0, u(2) = 32.$$

We use the uniform mesh with N=4,

$$\overline{\omega_h} = \left\{t_i = ih, i = 0, 1, 2, 3, 4, 5,\ h = \tfrac{2}{4+1} = 0,4\right\}.$$

Hence, the mesh points are

$$t_0 = 0,\ t_1 = 0.4,\ t_2 = 0.8,\ t_3 = 1.2,\ t_4 = 1.6,\ t_5 = 2.$$

By subsituting the last values, the matrix A_h has the form

$$A_h = \begin{pmatrix} 12.5 & -6.25 & 0 & 0 \\ -6.25 & 12.5 & -6.25 & 0 \\ 0 & -6.25 & 12.5 & -6.25 \\ 0 & 0 & -6.25 & 12.5 \end{pmatrix}$$

and the vector b_h is

$$b_h = \begin{pmatrix} -3.84 \\ -15.36 \\ -34.56 \\ 138.56 \end{pmatrix}$$

We solve these system $A_h y_h = b_h$.

Hence

$$y_h = \begin{pmatrix} 0.2560 \\ 1.1264 \\ 4.4544 \\ 13.3120 \end{pmatrix}$$

The exact solution at these points are:

$$\begin{pmatrix} 0.0512 \\ 0.8192 \\ 4.1472 \\ 13.1072 \end{pmatrix}$$

5 Numerical solution with MATLAB

5.1 Why MATLAB?

In the following section we use MATLAB to solve boundary value-problems. We have seen the theory of the numerical solution.

Having a closer look on the numerical example we can see that by increasing the number of the grid points the size of the equation system increasing rapidly. Unfortunatelly, - as the most of the mathematical problems - it very soon surpass the limit what can be managed by hand. This is the main reason why I have used MATLAB.

MATLAB is - coming from MATrix LABoratory - very effective, interactive system, what was developed for technological calculation and visual display of calculation results. It contains tools which make capable to solve the most common tasks of numerical analysis, matrix algebra, signal processing and graphycal imagery.

One of it's greatest benefit is in MATLAB the matrix is basic data structure. With MATLAB we can solve complex numerical problems by describing the problem as we would do in math. Because of these we can determine the solutions very quickly and in a very effective way.

5.2 Solving the example with MATLAB

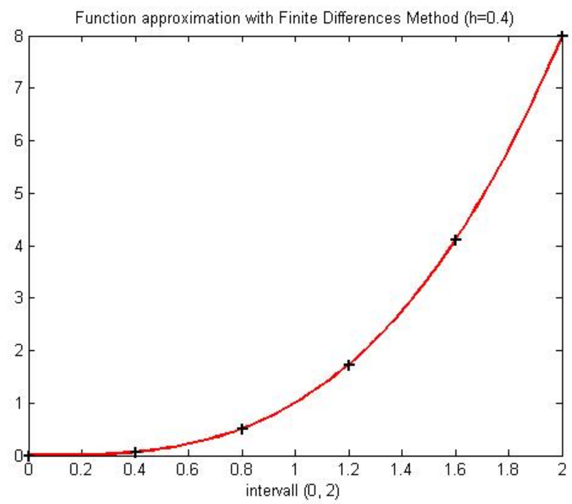
5.2.1 Finite Differences Method

The example is the same as in the section 4. It shows how to use MATLAB to solve boundary value probelems with finite difference method. Firstly, we define the golobal values: α, β, h, l. Afther that, we make the vector b and the matrix A. In he last step we solve this equation system: $A_h y_h = b_h$.
The h value shows the size of the step. If we increase the numbers of h, we will get better approximation.

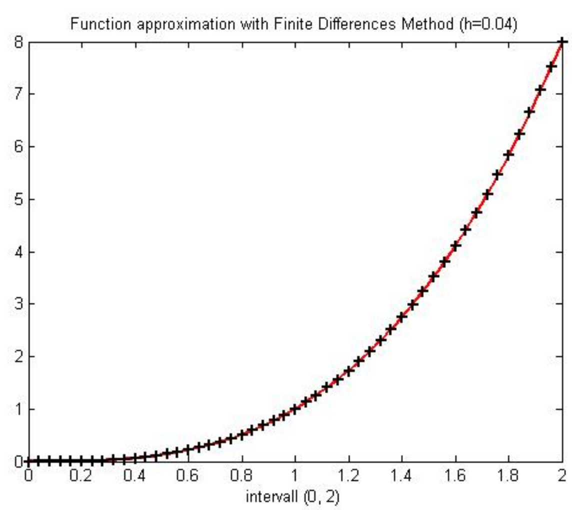
```
% Finite Difference Method
% u"(t)=r(t), r(t)=6t
% u(0)=alfa , u(l)=beta

% The definate of the global values
global alfa beta h l;
alfa = 0;
beta = 8;
l = 2;
n = 4;
h = l/(n+1);
```

```
% Making of the vector b
b = zeros(n, 1);
b(1) = -6*1*h + (alfa/h^2);
b(n) = -6*n*h + (beta/h^2);
for i = 2:n-1
    b(i) = -6*i*h;
end

% Making of the matrix A
A = zeros(n, n);
A(1, 1) = 2; A(1, 2) = -1;
A(n, n-1) = -1; A(n, n) = 2;
for i = 2:n-1
    A(i, i-1) = -1;
    A(i, i) = 2;
    A(i, i+1) = -1;
end

A = (1/h^2) * A;

% The solution of the system
y = A\b;
ym = linspace(0, 8, n+2);
ym(2:n+1) = y;

% The correct solution's presentment
% u(t) = t^3
delta = linspace(0, 2, 100);
u = delta.^3;
p = plot(delta, u, 'k');
xlabel('intervall (0, 2)')
title('Function approximation with Finite Differences
       Method (h=0.4)')
set(p, 'Color', 'red', 'LineWidth', 2)
hold on;

% Presentment of the approximation
intervall = linspace(0, 2, n+2);
plot(intervall, ym, 'k+', 'LineWidth', 2);
```

The picture shows the correct and the approximation solutions with $h = 0.4$.



The picture shows the correct and the approximation solutions with $h = 0.04$.



5.2.2 Shooting Method

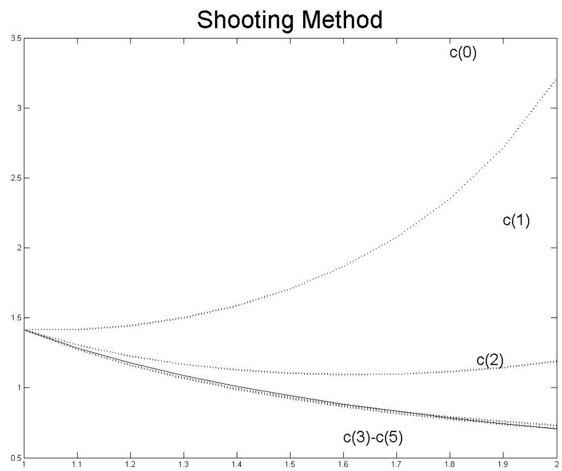
The example is the same as in the section 3. It shows how to use MATLAB to solve boundary value probelems with shooting method.

```
% Shooting Method for
% u''(t) = (u(t))^3   on interval [1,2]
% u(a) = alpha = sqrt(2), u(b) = beta = sqrt(2)/2
% boundary value problem

alpha = sqrt(2);
beta = 0.5*sqrt(2);
interval = [1, 2];
epsilon = 1e-6;
h = 0.001;
error = Inf;
i = 0;
c = 0;

% finding the c value, see section 3.
while ((error > epsilon) && (i < 10))
    U = [];
    ode1 = @(t,u)(u(2));
    ode2 = @(t,u)(u(1)^3);
    [t, U] = EulerMethod(ode1, ode2, interval, [alpha, c],
                         U, h, 'System');
    ode3 = @(t,z)(z(2));
    ode4 = @(t,z)(3*z(1));
    [t, Z] = EulerMethod(ode3, ode4, interval, [0, 1],
                         U(:, 1), h, 'Derivative');
    plot(t,U(:,1),'k:');
    hold on;
    F = U(end, 1) - beta;
    error = abs(F);
    c = c - F/Z(end, 1);
    i = i + 1;
end

i;
U1 = U(:,1);
Y = sqrt(2)*ones(length(t),1)./t;

plot(t,Y,'k');
title('Shooting Method');
```

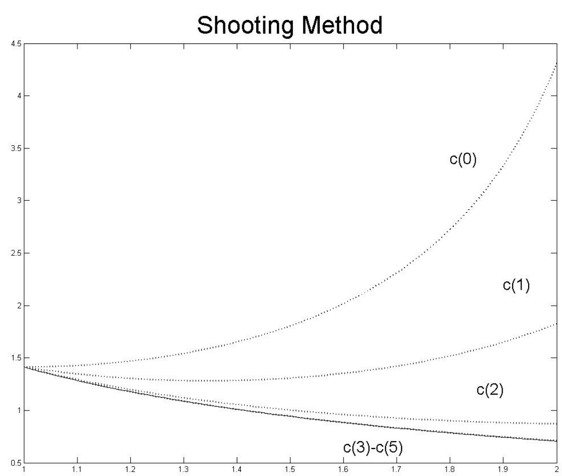
```
text(1.85,3.4,'c(0)');
text(1.9,1.6,'c(1)');
text(1.85,1,'c(2)');
text(1.8,0.65,'c(3)-c(5)');\\[20pt]
% Euler Method
function [t Y] = EulerMethod(ode1, ode2, interval, Y0,
                              U, h, Function)

% n: number of the inteval partitions
% t: the interval devided to n+1 partitons
% k: the solution of the ode in the i. step
% Y: the vector of the solution
n = (interval(2)-interval(1))/h;
t = linspace(interval(1), interval(2), n+1)';
k = zeros(1, 2);
Y = Y0;

% calculating the Y vector - the solution of the
% differential system
for i=1:n
    if strcmp('System', Function) == 1
        solution1 = feval(ode1, t(i), Y(i, :));
        solution2 = feval(ode2, t(i), Y(i, :));
        k(1, :) = [solution1, solution2];
        Y(i+1, :) = Y(i, :) + h*k(1, :);
    else
        solution1 = feval(ode1, t(i), Y(i, :));
        solution2 = feval(ode2, t(i), Y(i, :));
        k(1, :) = [solution1, solution2*(U(i, 1))^2];
        Y(i+1, :) = Y(i, :) + h*k(1, :);
    end
end
```

These pictures show the solutions. The continous line is the correct solution, the dotted lines are the approximation solutions. We can see the approximatons converge to the correct solutions.

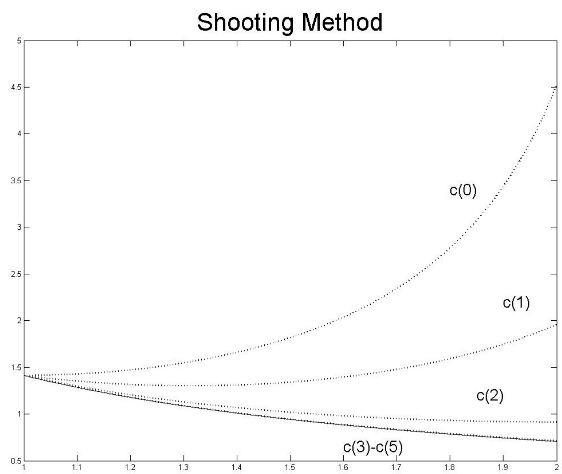
The picture shows the correct and the approximation solutions with $h = 0.1$.



The picture shows the correct and the approximation solutions with $h = 0.01$.



The picture shows the correct and the approximation solutions with $h = 0.001$.



6 Summary

In my dissertation I have showed the numerical solutions of two-point boundary value problems. The ordinary differential equations are widely used in more part of the life, for example mathematics, physics, economy, industry. Most of them are too diffucult, so we have to use numerical solutions.

As we have seen there are methods to determine the solution of the Cauchy-problem. (The Euler Method and the Improved Euler Method are one of the simplest schemes.)
In the boundary value problems there are given function f and the boundary values, hence

$$\begin{gathered} u'' = f(t, u, u'),\ t \in (a, b) \\ u(a) = \alpha,\ u(b) = \beta \end{gathered}$$

We have seen two types of numerical solutions, the Shooting Method and the Finite Differences Scheme.
When we use the Shooting Method, we transform the boundary value problem into a Cauchy-problem. Therefore, we need to solve an algebraic equation (usually a nonlinear one). The realization can be done in different ways. The intervallum-bisection method and chord-method are the simpliest, but they are first-rate. The chord-method's advantage: we do not need to search for two initial points, where the values will have different signs. We can start with any given $c^{(0)}$ and $c^{(1)}$ values. On the other hand, the convergence of the method can not be provided.

The Newton-method is second order. Hence, it is more efficient than the intervallum-bisection method and the chord-method. Otherwise, the Newton-method is needed to calculate the derivative. If the root is too far from the initial value, the Newton method does not converge.

The other chance to solve boundary value probelmes is the Finite Difference Scheme. The finite difference techniques are based upon the approximations that allow to replace the differential equations by finite difference equations. These finite difference approximations are in algebraic form, and the unknown solutions are related to grid points. Usually, we use these different formulas: forward, backward, and central differences. The forward-difference and the backward-difference are first-rate, and the central-difference is a second-rate approximation.

It is true for almost all of the numerical problems that, to solve them we have to use computer resources because by hand only small problems can be solved acceptable time. As we can see, MATLAB is one of the best choices to solve bigger tasks. With it I could declare and solve easily two-point boundary value problems. Beside the graphical capabilities MATLAB provides a simple interface for them, so the result of the tasks could be visualised in a quick and simple way.

All in all, the two-point boundary value problems are difficult, but with the numerical solutions and an appropriate computer framework they can be solved in effective way.

References

[1] István Faragó, Róbert Horváth: Numerikus módszerek, Typotex, 2011.

[2] Stoyan Gisbert: MATLAB, Typotex, 2005.

[3] R. Kress: Numerical analysis, Springer, 1998.

[4] C. F. Gerald, P. O. Wheatley, Addision: Applied Numerical Analysis, Addision Wesley Longman, Inc., 1997.

Acknowledgement

I would like to thank heartly all my teachers who have thought me.
I am thankful for my thesis supervisor István Faragó support me to develop and understanding of the subject.
This thesis wouldn't have been accomplished without the help of my parents and my fiance: Viktor Svantner.

Nyilatkozat

Név:

ELTE Természettudományi Kar, szak:

ETR azonosító:

Szakdolgozat címe:

A szakdolgozat szerzőjeként fegyelmi felelősségem tudatában kijelentem, hogy a dolgozatom önálló munkám eredménye, saját szellemi termékem, abban a hivatkozások és idézések standard szabályait következetesen alkalmaztam, mások által írt részeket a megfelelő idézés nélkül nem használtam fel.

Budapest, 20

a hallgató aláírása