On finite substructures of certain stable structures

Author: Zalán Gyenis

Thesis Advisor: Gábor Sági

MTA Rényi Institute

Eötvös Lóránd University

Faculty of Sciences

Institute of Mathematics

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1 Introduction

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1.1 Introduction

According to some classical results of mathematical logic if a structure has infinitely many elements it cannot be described, up to isomorphism, by first order properties. Indeed, an ultrapower $\mathcal{B}$ of an infinite structure $\mathcal{A}$ may have arbitrarily large cardinality and still satisfies exactly the same formulas as $\mathcal{A}$. Hence it is natural to take a closer look on those theories which determine their models *up to isomorphism and cardinality*: if κ is a cardinality then a theory T is defined to be κ-categorical if and only if, up to isomorphism, T has a unique model of cardinality κ.

Since the middle of the 20th century, there has been a considerable effort to study infinitely categorical theories. $\aleph_0$-categorical structures had been described early on by Svevonius and Ryll-Nardzewski. Later on, Morley showed in his celebrated theorem that a (countable, first order) theory T is $\aleph_1$-categorical if and only if it is κ-categorical for all uncountable κ, see e.g. [11] or Theorem 7.1.14 of [1].

Morley's theorem implies that from the structure theoretic point of view $\aleph_1$ categorical theories are the simplest ones: their models can be identified with a single cardinality parameter, namely with the cardinality of their universe. Vaught asked whether infinitely categorical theories are simple from the axiomatic point of view. This became one of the most important questions of model theory which finally had been solved independently by Zilber (see [14] and [15]) and by Cherlin-Lachlan-Harrington (see [3]). Both proofs are long and overtake serious technical difficulties. The answer is negative: an infinitely categorical theory cannot be finitely axiomatized (hence, it is necessarily complicated in some sense). For more recent related results we refer to Cherlin-Hrushovski, [2].

These results are based on studying finite substructures of stable structures. It turned out that if the original structure is "very stable" then it has finite structures with a highly transitive group of automorphisms and this contradicts finite axiomatizability. Constructing highly homogeneous finite structures (i.e. finite structures with highly transitive automorphism groups) seems interesting as an independent combinatorial problem, for results in this direction we refer to [9], [6] and [7].

In this work we establish a new method to construct highly homogeneous finite substructures of certain structures. This method is based on extending partially elementary mappings of ultraproducts of finite structures in a way that the resulted extension is decomposable, that is, it acts coordinatewise (in a sense). Throughout this procedure, stability (or rather the non-existence of long splitting chains) plays an important role. We present two applications of our method. These applications are the main results of this work and are as follows.

First we extend Morley's categoricity theorem to finite cardinalities: under certain additional technical conditions, if T is $\aleph_1$-categorical then its large enough finite fragments have a unique n-element model for all finite n (thus these finite fragments are n-categorical for some sense for all $n \in \omega$). This finitary extension of Morley's theorem is completely new and seem to be related to, and applicable in complexity theoretical aspects of finite model theory; see subsection 4.3.

As a second application we provide a simple approach related to a special case of the Zilber–Cherlin-Lachlan-Harrington theorem: again under some additional technical conditions, non-finite axiomatizability of infinitely categorial (even an $\aleph_0$-categorical, $\aleph_0$-stable) theories are equivalent with the existence of highly homogeneous finite models. We are also trying to push the limits of our approach beyond $\aleph_0$-stability; in this respect our results are considered to be new ones. For both applications we refer to [5] and [4].

The structure of the paper is as follows. In the rest of the introduction we summarize our notations and the needed basic definitions and facts from model theory. In Section 2 we investigate first the connection between stability and splitting chains, and then we show how to use stability to "understand the big from the small." A new method to extend certain mappings is described in Section 3. Finally we give the applications of the method in Section 4.

1.2 Notation

In this section we are summing up our system of notation.

Sets

Throughout ω denotes the set of natural numbers and for every $n \in \omega$ we have $n = \{0, 1, \ldots, n-1\}$. If A and B are sets, then ${}^{A}B$ denotes the set of functions from A to B, $|A|$ denotes the cardinality of A, and $\mathcal{P}(A)$ denotes the power set of A. In addition for a cardinal κ we use $[A]^{\kappa}$ and $[A]^{<\kappa}$ to denote the sets $\{x \in \mathcal{P}(A) : |x| = \kappa\}$ and $\{x \in \mathcal{P}(A) : |x| < \kappa\}$, respectively. We use the standard notations for cardinals, i.e. $\aleph_0 = |\omega|$ and $\mathfrak{c} = 2^{\aleph_0}$.

Sequences of variables or elements will be denoted by overlining, for instance $\bar{x}$ denotes an n-tuple $\langle x_0, x_1, \ldots, x_{n-1}\rangle$ for a given $n \in \omega$. The length of the sequences (in this case n) will always be clear from the context. For a function f let us denote the domain and the range of f by $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$, respectively. For simplicity, by a slight abuse of notation, we will write $\bar{x} \in A$ in place of $\mathrm{ran}(\bar{x}) \subseteq A$, particularly $\bar{x} \in \mathrm{dom}(f)$ expresses that f is defined on every member of $\bar{x}$, that is $\mathrm{ran}(\bar{x}) \subseteq \mathrm{dom}(f)$. For a subset $X \subseteq \mathrm{dom}(f)$, we define $f[X] = \{f(x) : x \in X\} \subseteq \mathrm{ran}(f)$.

Languages

Let L be a first order language. Throughout we assume that L contains finitely many relation symbols and no function symbols, however we note that many of the theorems remain true for an arbitrary (but countable) language. The set of all formulas of L is denoted by $\mathrm{Form}(L)$. In addition $\mathrm{At}(L)$ and $\mathrm{Form}_{\Delta}(L)$ are the sets of atomic and quantifier-free formulas, respectively. We use $\varphi(\bar{v})$ to denote a formula φ all of whose free variables occur among $\bar{v}$. For a given set X we can extend our language with constant symbols denoting elements of X. This extended language is L_X.

Structures

By a structure we understand a set $\mathcal{A} = \langle A, f_i^{\mathcal{A}}, r_j^{\mathcal{A}}\rangle_{i\in I, j\in J}$, where $f_i^{\mathcal{A}}$ and $r_j^{\mathcal{A}}$ are the interpretations of the function and relation symbols f_i and r_j, respectively. If L is a first order language containing the relation and function symbols r_i and f_j, then $\mathcal{A}$

is said to be an L-structure. We will use the convention, that structures (models) are denoted by calligraphic letters, and the underlying set of a given structure (the universe of a given model) is always denoted by the same latin letter. We use the standard validity relation $\models$, that is, for a formula $\varphi(\bar{v})$ and $\bar{a} \in A$, the statement $\mathcal{A} \models \varphi[\bar{a}]$ means that φ is true in $\mathcal{A}$ under the valuation $\bar{v} = \bar{a}$. φ is valid in $\mathcal{A}$ if it is true under all valuations. This last assertion is denoted by $\mathcal{A} \models \varphi$. The set of all valid formulas in $\mathcal{A}$ is $\mathrm{Th}(\mathcal{A})$. Conversely, if Σ is a set of formulas then the class of structures in which all the members of Σ are valid is denoted by $\mathrm{Mod}(\Sigma)$.

If $\mathcal{A}$ is a model for a language L and $R_0, \ldots, R_{n-1}$ are relations on A then $\langle \mathcal{A}, R_0, \ldots, R_{n-1} \rangle$ denotes the expansion of $\mathcal{A}$ whose similarity type is expanded by n new relation symbols (with the appropriate arities) and the interpretation of the new symbols are $R_0, \ldots, R_{n-1}$, respectively. Similarly if $L' \subseteq L$ and $\mathcal{A}$ is an L-structure, then $\mathcal{A}|_{L'}$ is the structure obtained by forgetting the functions and relations of $L \smallsetminus L'$. This is called the L'-reduct of $\mathcal{A}$.

Two structures $\mathcal{A}$ and $\mathcal{B}$ are elementary equivalent ($\mathcal{A} \equiv_e \mathcal{B}$) if $\mathrm{Th}(\mathcal{A}) = \mathrm{Th}(\mathcal{B})$. When $\mathcal{A}$ is isomorphic to $\mathcal{B}$ we write $\mathcal{A} \cong \mathcal{B}$.

1.3 Basic definitions and preliminaries

This section overviews some background. Since the mentioned results are well known, we do not present proofs here.

1.3.1 Types and the Stone topology

Let $\mathcal{A}$ be a structure, $X \subseteq A$ and $\bar{a} \in A$. We would like to understand the connection between $\bar{a}$ and X. In first order logic all we can express are sentences in the extended language L_X which are true in $\mathcal{A}$ substituing $\bar{a}$ onto the free variables. This motives the following definition.

Definition 1.1 *Let $\bar{v} = \langle v_0, \ldots \rangle$ be a sequence of variables, $\mathcal{A}$ a structure, and $X \subseteq A$. Then $p \subseteq \mathrm{Form}(L_X)$ is said to be a $\bar{v}$-type over X in $\mathcal{A}$, iff the following stipulations hold:*

- *all the free variables of formulas from p are in $\bar{v}$;*
- $\mathcal{A} \models \exists v_0 \exists v_1 \ldots \wedge p_0$ *for all* $p_0 \in [p]^{<\omega}$*;*

- *if φ is a formula with free variables only from $\bar{v}$ then $\varphi \in p$ or $\neg\varphi \in p$.*

The set of $\bar{v}$-types over X in $\mathcal{A}$ is denoted by $\mathrm{S}^{\mathcal{A}}_{\bar{v}}(X)$.

The type of $\bar{a}$ in $\mathcal{A}$ over $X \subseteq A$ is defined as usual:

$$\mathrm{tp}^{\mathcal{A}}(\bar{a}/X) = \{\varphi(\bar{v}) \in Form(L_X) : \mathcal{A} \models \varphi[\bar{a}]\}.$$

If $X = \emptyset$ we omit it, if $\mathcal{A}$ is clear from the context then we also omit it. Clearly $\mathrm{tp}^{\mathcal{A}}(\bar{a}/X) \in \mathrm{S}^{\mathcal{A}}_{\bar{v}}(X)$ and this is all we can say about the connections between a and X in $\mathcal{A}$.

The definition of types fairly resembles to that of the ultrafilters. This is not a coincidence, as we will see it. Consider a structure $\mathcal{A}$ a subset $X \subseteq A$ and a formula $\varphi(\bar{v}) \in \mathrm{Form}(L_X)$. Then φ defines a relation on A in a natural way, namely the relation $||\varphi||^{\mathcal{A}} = \{s \in {}^{\bar{v}}A : \mathcal{A} \models \varphi[s]\}$. Clearly these kind of relations form a Boolean algebra $\mathcal{B} = \langle\{||\varphi||^{\mathcal{A}} : \varphi \in \mathrm{Form}(L_X)\}, \cap, \cup\rangle$. Now the types $p \in \mathrm{S}^{\mathcal{A}}_{\bar{v}}(X)$ can be identified with the ultrafilters of $\mathcal{B}$. Hence types form a filter-space which is called *Stone-space*. It will be important in the latter sections that Stone-spaces are compact, Hausdorff topological spaces with the clopen base $\{N_\varphi : \varphi \in \mathrm{Form}(L_X)\}$, where $N_\varphi = \{p \in \mathrm{S}^{\mathcal{A}}_{\bar{v}}(X) : \varphi \in p\}$.

Similarly we may define types restricted to a given set of formulas. For instance if Φ is a set of formulas then $\mathrm{tp}^{\mathcal{A}}_{\Phi}(\bar{a}/X) = \{\varphi(\bar{v}, \bar{c}) \in p : \varphi(\bar{v}, \bar{w}) \in \Phi\}$. When $\Phi = \mathrm{Form}_{\Delta}(L)$ we speak about *quantifier-free types.*

1.3.2 Saturation and ultraproducts

A type $p \in \mathrm{S}^{\mathcal{A}}_{\bar{v}}(X)$ is said to be *realizable* if $p = \mathrm{tp}^{\mathcal{A}}(\bar{a}/X)$ for some $\bar{a} \in A$. For example, the statements

$$\forall y(y^2 < 2 \implies y < x) \text{ and } \forall y((y > 0 \wedge y^2 > 2) \implies y > x)$$

describe the square root of 2. This set of formulas extends to a type not realized in the model of arithmetic consisting of the rational numbers, but is realized in the reals. We formulate this phenomena in the next definition.

Definition 1.2 *Let $\mathcal{A}$ be a structure and κ a cardinal. Then $\mathcal{A}$ is κ-saturated iff for all $X \in [A]^{<\kappa}$ any type $p \in S^{\mathcal{A}}(X)$ can be realized in $\mathcal{A}$. We say that $\mathcal{A}$ is saturated if it is $|A|$-saturated.*

Saturated models exist: for instance $\langle \mathbb{Q}, < \rangle$ and the countable random graph are saturated (in fact, they are $\aleph_0$-categorical). We note that if A is infinite, then the set $\{v \neq a : a \in A\}$ of formulas is finitely satisfiable, hence it extends to a type over A in $\mathcal{A}$. Clearly this type is not realizable in $\mathcal{A}$, thus it follows that an infinite structure $\mathcal{A}$ can not be $|A|^+$-saturated. We also note that a structure is κ-saturated for all cardinal κ if and only if it is finite.

We would like to construct highly-saturated structures. For this, we use ultraproducts. First recall the definitions.

Definition 1.3 *Let $\langle \mathcal{A}_i : i \in I \rangle$ be L-structures. Then $\mathcal{B} = \Pi_{i \in I} \mathcal{A}_i$ is the direct product structure iff*

- $B = \Pi_{i \in I} A_i$;
- $(f^{\mathcal{B}}(\bar{b}))_i = f^{\mathcal{A}_i}(\bar{b}(i))$ *for all* $f \in L$ *and* $i \in I$;
- $\bar{b} \in r^{\mathcal{B}} \iff (\forall i \in I) \bar{b}(i) \in r^{\mathcal{A}_i}$ *for all* $r \in L$.

Definition 1.4 *Let $\mathcal{B} = \Pi_{i \in I} \mathcal{A}_i$ and let $\mathcal{F}$ be an ultrafilter over I. Then $a, b \in B$ are equivalent iff $\{i \in I : a(i) = b(i)\} \in \mathcal{F}$ (in symbols: $a \equiv_{\mathcal{F}} b$). The equivalence class of $a \in B$ is $a/\mathcal{F} = \{b \in B : a \equiv_{\mathcal{F}} b\}$. Now, $\mathcal{A} = \Pi_{i \in I} \mathcal{A}_i / \mathcal{F}$ is an ultraproduct if the followings hold:*

- $A = \{a/\mathcal{F} : a \in B\}$;
- $f^{\mathcal{A}}(\bar{a}/\mathcal{F}) = f^{\mathcal{B}}(\bar{a})/\mathcal{F} = \langle f^{\mathcal{A}_i}(\bar{a}(i)) : i \in I \rangle / \mathcal{F}$;
- $\bar{a}/\mathcal{F} \in r^{\mathcal{A}} \iff \{i \in I : \bar{a}(i) \in r^{\mathcal{A}_i}\} \in \mathcal{F}$.

Forming ultraproducts is a basic tool constructing a “bigger” model from several “smaller” ones in such a way that the obtained structure reflects the average properties. The following Lemma makes this idea precise.

Theorem 1.5 *(Łoś lemma) For every $\varphi \in Form(L)$ we have*

$$\Pi_{i \in I} \mathcal{A}_i / \mathcal{F} \models \varphi \iff \{i \in I : \mathcal{A}_i \models \varphi\} \in \mathcal{F}.$$

Let $\mathcal{F}$ be an ultrafilter over I and let κ be a cardinal. A function $f : [\kappa]^{<\omega} \to \mathcal{F}$ is said to be *monotone* iff for all $s_1, s_2 \in [\kappa]^{<\omega}$ one has $s_1 \subseteq s_2 \Rightarrow f(s_1) \supseteq f(s_2)$ and f is said to be *additive* iff $f(s_1 \cup s_2) = f(s_1) \cap f(s_2)$. In addition if $f, g : [\kappa]^{<\omega} \to \mathcal{F}$ are functions, then f is defined to be *smaller* then g iff for all $s \in [\kappa]^{<\omega}$ one has

$f(s) \subseteq g(s)$. The ultrafilter $\mathcal{F}$ is defined to be λ-*good* iff for every $\kappa < \lambda$ and for every monotone $f : [\kappa]^{<\omega} \to \mathcal{F}$ there is an additive $g : [\kappa]^{<\omega} \to \mathcal{F}$ such that g is smaller than f.

We recall the following facts from [1]. By Theorem 6.1.4 of [1], there are countably incomplete $|I|^+$-good ultrafilters over every infinite set I; in addition, by Theorem 6.1.8 of [1], ultraproducts modulo countably incomplete, κ-good ultrafilters are κ-saturated. We note that this is true for any ultraproduct, particulary, if we extend the language of our structures with one new relation symbol then the ultraproduct (modulo a countably incomplete κ-good ultrafilter) of these extended structures still remain κ-saturated.

Turning back to ultraproducts, we need the following definitions (see also [12]).

Definition 1.6 *Let $\langle A_i : i \in I\rangle$ be a sequence of sets, $\mathcal{F}$ an ultrafilter on I, and $R_i \subseteq {}^k A_i$ given relations. Then $\Pi_{i\in I} R_i/\mathcal{F}$ is defined as:*

$$\Pi_{i\in I} R_i/\mathcal{F} = \{s/\mathcal{F} \in {}^k\big(\Pi_{i\in I} A_i/\mathcal{F}\big) : \{i \in I : s_i \in R_i\} \in \mathcal{F}\}.$$

Definition 1.7 *We say that a relation $R \subseteq {}^k\big(\Pi_{i\in I} A_i/\mathcal{F}\big)$ is decomposable iff*

$$R = \Pi_{i\in I} R_i/\mathcal{F}$$

for some $R_i \subseteq {}^k A_i$.

1.3.3 Categoricity and stability

Let T be a set of formulas (T is called a theory) in a first order language L. How does T determine its models? On the one hand by the Löwenheim-Skolem theorems it follows that if T has an infinite model, then for all $\kappa \geq \aleph_0$ it has a model with cardinality κ. On the other hand we might ask whether T has a unique model on a given cardinal? In more detail, let $I(T,\kappa)$ be the number of non-isomorphic models of T, having cardinality κ. A theory T is said to be κ-*categorical*, if $I(T,\kappa) = 1$. The classical example is the theory of algebraically closed fields of a given characteristic: these are categorical of size $\aleph_1$. The spectrum problem is to describe the possible behaviours of $I(T,\kappa)$ as a function of κ. Several results are known: for example $I(T,\omega)$ is finite, or equals to $\aleph_0$ or $\aleph_1$ or $\mathfrak{c}$. By results of Ryll-Nardzewski, Svevonius and others (see Theorem 2.3.13 of [1]), the case $I(T, \aleph_0) = 1$ is completely characterized in terms of the automorphims of $\mathcal{A}$. For larger cardinals, the key is in stability theory, which became a separate part of mathematical logic by works of Morley, Shelah, Hrushovski, Pillay, Lascar, Zilber and Baldwin.

Definition 1.8 *Let T be a consistent theory, and let $\lambda \geq \aleph_0$ be a cardinal. Then*

(i) T is λ-stable if for all $\mathcal{A} \models T$ and $X \in [A]^\lambda$ one has $|\,\mathrm{S}_1^{\mathcal{A}}(X)| \leq \lambda$;

(ii) T is stable if it is λ-stable for some cardinal λ;

(iii) T is superstable if T is λ-stable for all $\lambda > \mu$ for some μ;

(iv) T is unstable if it is not stable.

A structure $\mathcal{A}$ is said to be *stable* (*unstable*) iff $\mathrm{Th}(\mathcal{A})$ is stable (unstable). Examples of both stable and unstable structures are well known. For example algebraically closed fields (of a given characteristic) are $\aleph_0$-stable. We note that a stable structure is also stable for all cardinals with the property $\lambda = \lambda^{|Form(L)|}$. This fact is established by the spectrum-theorem, see Theorem I.2.2 (1)-(8) of [13]. It follows that for a countable language, stable structures are $\mathfrak{c}$-stable which will be important for us. Now, we draw up some well known implications on stability and categoricity; for the proofs, we refer to [1].

Theorem 1.9 *Let T be a theory in a countable language.*

(i) If T is $\aleph_0$-stable then it is stable for all infinite cardinals (see Lemma 7.1.3. of [1]);

(ii) If T is λ-categorical for $\lambda \geq \aleph_1$ then it is $\aleph_0$-stable (see Lemma 7.1.4 of [1]);

(iii) If T is $\aleph_0$-stable then T has a κ-saturated model with cardinality λ, for $\mathrm{Form}(L) \leq \kappa \leq \lambda$, κ is regular (see Lemma 7.1.6 of [1]).

Theorem 1.10 *(Upward-Morley, see Lemma 7.1.7 and Theorem 7.1.14 of [1] or Theorem 6.1.1 of [10]). Let T be an $\aleph_1$-categorical theory. Then*

(1) Every non-countable models of T are saturated;

(2) T is κ-categorical for all $\kappa > \aleph_0$.

Two structures $\mathcal{A}$, $\mathcal{B}$ are defined to be a *Vaughtian pair* iff $\mathcal{B}$ is a nontrivial elementary substructure of $\mathcal{A}$ and there exists an infinite definable relation in B

which cannot be realized in $A \setminus B$. By a theorem of Baldwin and Lachlan, a structure $\mathcal{A}$ is uncountably categorical iff it is $\aleph_0$-stable and its models does not contain Vaughtian pairs. For the details we refer to Theorem 6.1.18 of [10]. Actually we will only make use of the "easier" direction of the Baldwin-Lachlan theorem: if a theory T is uncountably categorical then its models does not contain Vaughtian pairs - otherwise, by a standard two cardinals theorem (see e.g. Theorem 3.2.9 of [1]) one could construct two non-isomorphic models of T with cardinalities $\aleph_1$.

1.3.4 Homogeneity

By a partial isomorphism between two relational structures $\mathcal{A}$ and $\mathcal{B}$ (with the same similarity type) we mean a partial function $f : A \to B$ which is an isomorphism between the substructures corresponding to the domain and the range of f. Clearly such a function preserves all the quantifier-free formulas, that is for all $\bar{a} \in \mathrm{dom}(f)$ one has $\mathrm{tp}^{\mathcal{A}}_{\Delta}(\bar{a}) = \mathrm{tp}^{\mathcal{B}}_{\Delta}(f(\bar{a}))$. If a (partial) function f preserves all the formulas of $\mathrm{Form}(L)$ then it is said to be an *elementary* mapping. Similarly if $\Phi \subseteq \mathrm{Form}(L)$ is a set of formulas, then f is called to be Φ-*elementary* iff it preserves all the formulas from Φ, that is $\mathrm{tp}^{\mathcal{A}}_{\Phi}(\bar{a}) = \mathrm{tp}^{\mathcal{B}}_{\Phi}(f(\bar{a}))$ for all $\bar{a} \in \mathrm{dom}(f)$. The difference between elementary functions and partial isomorphisms is that the latter are not sensitive of the connections with elements outside of the domain.

In the latter sections we will deal with extending certain types of partial functions.

Definition 1.11 *Let κ be a cardinal, let $\mathcal{A}$ be a structure, and let $X \in [A]^{<\kappa}$. Then $\mathcal{A}$ is defined to be*

- *partially κ-homogeneous iff for every partial isomorphism $f : X \to A$ and $a \in A$ there is a partial isomorphism g which extends f and $a \in \mathrm{dom}(g)$;*
- *κ-homogeneous iff every partial isomorphism $f : X \to A$ extends to an automorphism of $\mathcal{A}$;*
- *homogeneous iff it is $|A|$-homogeneous;*
- *strongly κ-homogeneous iff every $f : X \to A$ partial elementary mapping extends to an automorphism of $\mathcal{A}$;*
- *strongly homogeneous iff it is strongly $|A|$-homogeneous.*

There is a strong connection between saturatedness and homogeneity which we now recall from [1] (see Proposition 5.1.9 of [1]).

Proposition 1.12 *Every saturated structure is strongly homogeneous, in fact, a structure is saturated if and only if it is universal and strongly homogeneous.*

2 More on stability

In this section we examine first the connection between stability and splitting chains, later we introduce the mirroring principle which will be used in extending decomposable mappings.

2.1 Splitting chains

Definition 2.1 *Let $p \in \mathrm{S}^{\mathcal{A}}(X)$ and $Y \subseteq X$. Then p splits over Y if there exist $\bar{a}, \bar{b} \in X$ and $\varphi \in \mathrm{Form}(L)$ such that $\mathrm{tp}^{\mathcal{A}}(\bar{a}/Y) = \mathrm{tp}^{\mathcal{A}}(\bar{b}/Y)$, but $\varphi(v, \bar{a}) \in p$ and $\neg\varphi(v, \bar{b}) \in p$.*

Lemma 2.2 *(see Lemma I.2.7 of [13]). Let $\mathcal{A}$ be a λ-stable structure. Then there does not exist an increasing sequence $\langle A_i : i \leq \lambda \rangle$ and $p \in \mathrm{S}^{\mathcal{A}}(A_\lambda)$ such that $p|_{A_{i+1}}$ splits over A_i for all $i < \lambda$.*

The non-existence of long splitting chains play a central role in our method of extending decomposable mappings. By the previous lemma we may conclude that if a structure is stable then there are no long splitting chains in it. It is natural to ask whether the converse is true. We show that from the assumption that there is no long splitting chain, stability follows.

Proposition 2.3 *Suppose $\mathcal{A}$ is a structure such that there does not exists an increasing sequence $\langle A_i : i \leq \lambda \rangle$ and $p \in \mathrm{S}^{\mathcal{A}}(A_\lambda)$ such that $p|_{A_{i+1}}$ splits over A_i for all $i < \lambda$. Then $\mathcal{A}$ is 2^{2^λ}-stable.*

Proof. Let $X \subseteq A$ with $|X| \leq 2^{2^\lambda}$. We have to show that $|\,\mathrm{S}^{\mathcal{A}}(X)| \leq 2^{2^\lambda}$. Taking an $|X|^+$-saturated elementary extension of $\mathcal{A}$ we may assume that every type over X is realized. Using the assumption that there is no long splitting chain, for all $a \in A$, first we construct a set $A(a) \subseteq X$ by transfinite recursion such that the following two conditions hold:

(1) $|A(a)| \leq \lambda$;

(2) $\mathrm{tp}(a/X)$ does not split over $A(a)$.

To this end let $a \in A$ be fixed, let $p = \mathrm{tp}(a/X)$, and set $A_0 = \emptyset$. Let β be an ordinal and suppose for all $\alpha < \beta$ that $A_\alpha \subseteq X$ such that $|A_\alpha| \leq |\alpha| + \aleph_0$.

I. In the first case suppose β is successor, say $\beta = \alpha + 1$. If p splits over A_α, then by definition there exist $\bar{x}, \bar{y} \in X$ and $\varphi \in \mathrm{Form}(L)$ such that $\mathrm{tp}(\bar{x}/A_\alpha) = \mathrm{tp}(\bar{y}/A_\alpha)$, but $\varphi(v, \bar{x}) \in p$ and $\neg\varphi(v, \bar{y}) \in p$. In this case let $A_\beta = A_\alpha \cup \{\bar{x}, \bar{y}\}$. If p does not split over A_α then the construction is complete.

II. In the second case suppose β is a limit ordinal. Then let $A_\beta = \cup_{\alpha<\beta} A_\alpha$.

Now observe that $p|_{A_{\alpha+1}}$ splits over A_α for all $\alpha < \beta$, consequently by our assumption the transfinite construction stops at a level $\beta < \lambda$. Finally let $A(a) = A_\beta$.

Similarly there exists a set $B(a) \subseteq X$ with

(3) $A(a) \subseteq B(a)$;

(4) $|B(a)| \leq 2^\lambda$;

(5) for all $\bar{x} \in X$ there exists $\bar{y} \in B(a)$ such that $\mathrm{tp}(\bar{x}/A(a)) = \mathrm{tp}(\bar{y}/A(a))$.

For this, choose an arbitrary realization of each type over $A(a)$ and let their collection be $B(a)$. Then (3) and (5) clearly holds. To show that (4) holds observe that by (1) we have $|\,\mathrm{S}_i(A(a))| \leq 2^\lambda$, and thus

$$|B(a)| \leq \aleph_0 \cdot |\bigcup_{i\in\omega} \mathrm{S}_i(A(a))| \leq \aleph_0 \cdot \sum_{i\in\omega} 2^\lambda = \aleph_0^2 \cdot 2^\lambda = 2^\lambda$$

To end the proof we only have to show that

($\star$) if $a_0, a_1 \in A$ with $B(a_0) = B(a_1)$ and $\mathrm{tp}(a_0/B(a_0)) = \mathrm{tp}(a_1/B(a_1))$ then $\mathrm{tp}(a_0/X) = \mathrm{tp}(a_1/X)$.

It is sufficient to establish ($\star$) since there are only $(2^{2^\lambda})^{2^\lambda} = 2^{2^\lambda}$ possibilities to choose $B(a_0)$, and thus by ($\star$) there are only 2^{2^λ} many types over X.

To see ($\star$), let $\bar{c} \in X$ be arbitrary. By (5) there exists $\bar{c}' \in B(a_0)$ such that $\mathrm{tp}(\bar{c}/A(a_0)) = \mathrm{tp}(\bar{c}'/A(a_0))$. Now $\mathrm{tp}(a_0/X) \ni \varphi(v, \bar{c}) \iff \mathcal{A} \models \varphi(a_0, \bar{c}) \iff \mathcal{A} \models \varphi(a_0, \bar{c}') \overset{(2)}{\iff} \mathcal{A} \models \varphi(a_1, \bar{c}') \iff \mathcal{A} \models \varphi(a_1, \bar{c}) \iff \varphi(v, \bar{c}) \in \mathrm{tp}(a_1/X)$. ∎

2.2 Mirroring principle

We will deal with extending partial elementary mappings of certain ultraproducts. It is well known that every saturated structure $\mathcal{A}$ is strongly homogeneous: every elementary mapping f of $\mathcal{A}$ with $|f| < |A|$ can be extended to an automorphism of $\mathcal{A}$; for more details, we refer to Proposition 5.1.9 of [1]. The basic idea of the proof of this theorem is that by saturatedness, if $f : A \to A$ is a "small" elementary mapping, and $a \notin \mathrm{dom}(f)$, then the type $f[\mathrm{tp}^{\mathcal{A}}(a/\mathrm{dom}(f))]$ can be realized outside of $\mathrm{ran}(f)$. The problem is that it is not only the "small" mappings which we would like to extend. For instance if $\mathcal{A}$ is an ultraproduct and f is decomposable then $|f|$ might be as big as $|A|$, and since $\mathcal{A}$ can not be $|A|^+$-saturated we can not hope anything like above.

However, stability saves the situation. We show that one can choose a subset, which is small enough for our intentions and even catches all the first order properties. More precisely, our Lemma 2.5 guarantees sets $A(a)$ and $B(a)$, which determine the type of a over certain subsets of A. This technique may be considered as a kind of mirroring: we understand the connections of an element and a subset by mirroring the properties onto a smaller subset. First we need the following proposition.

Proposition 2.4 *Suppose $\mathcal{A}$ is a stable structure, $D \subset A$ and $\langle \mathcal{A}, D\rangle$ is $\mathfrak{c}^+$-saturated. Then there exist $A_D \subseteq D$, $p_D \in S(A_D)$, and $a_D \in A \setminus D$, such that $|A_D| < \mathfrak{c}$, a_D realizes p_D, and if $c \in A \setminus D$ realizes p_D then $\mathrm{tp}^{\mathcal{A}}(c/D)$ does not split over A_D.*

Proof. We apply transfinite recursion. Let $a_0 \in A \setminus D$ be arbitrary, $A_0 = \emptyset$ and $p_0 = \mathrm{tp}^{\mathcal{A}}(a_0/A_0)$. Let β be an ordinal and suppose for all $\alpha < \beta$ that a_α, $A_\alpha \subseteq D$, and p_α are already defined, such that $p_\alpha \in \mathrm{S}(A_\alpha)$, $|A_\alpha| \le |\alpha| + \aleph_0$, and a_α realizes p_α.

I. β is successor, say $\beta = \alpha+1$. First, suppose there exists $c \in A \setminus D$ which realizes p_α but $\mathrm{tp}^{\mathcal{A}}(c/D)$ splits over A_α (it may happen that $c = a_\alpha$). Then by definition there exist $\bar{d}_0, \bar{d}_1 \in D$ and φ such that $\mathrm{tp}^{\mathcal{A}}(\bar{d}_0/A_\alpha) = \mathrm{tp}^{\mathcal{A}}(\bar{d}_1/A_\alpha)$, but $\varphi(v, \bar{d}_0) \in \mathrm{tp}^{\mathcal{A}}(c/D)$ and $\varphi(v, \bar{d}_1) \notin \mathrm{tp}^{\mathcal{A}}(c/D)$. Let $A_\beta = A_\alpha \cup \{\bar{d}_0, \bar{d}_1\}$, $p_\beta = \mathrm{tp}^{\mathcal{A}}(c/A_\beta)$, and $a_\beta = c$. If there are no $c \in A \setminus D$ with $\mathrm{tp}^{\mathcal{A}}(c/D)$ splitting over A_α, then A_β, p_β and a_β are undefined, and the transfinite construction is complete.

II. β is a limit ordinal. Let $A_\beta = \cup_{\alpha<\beta} A_\alpha$ and $p_\beta = \cup_{\alpha<\beta} p_\alpha$. By assumption $\langle \mathcal{A}, D\rangle$ is $\mathfrak{c}^+$-saturated hence there exists $a_\beta \in A \setminus D$ which realizes p_β.
III. Clearly, for each α, $p_{\alpha+1}$ splits over A_α, hence by Lemma 2.2 this construction stops at a level $\beta < \mathfrak{c}$. Let $A_D = A_\beta$, $p_D = p_\beta$, and $a_D = a_\beta$. ∎

Lemma 2.5 *Let $\mathcal{A}$ be stable, and $D \subset A$ such that $\langle \mathcal{A}, D\rangle$ is a $\mathfrak{c}^+$-saturated structure. Then there exist $a \in A \setminus D$ and sets $A(a) \subseteq B(a) \subseteq D$ such that*

(1) $|A(a)| \leq \mathfrak{c}$ and $\mathrm{tp}^{\mathcal{A}}(a/D)$ does not split over $A(a)$;

(2) $|B(a)| \leq \mathfrak{c}$ and every type over $A(a)$ can be realized in $B(a)$;

(3) for all $b \in A \setminus D$ the following holds:

$$\mathrm{tp}^{\mathcal{A}}(a/B(a)) = \mathrm{tp}^{\mathcal{A}}(b/B(a)) \Longrightarrow \mathrm{tp}^{\mathcal{A}}(a/D) = \mathrm{tp}^{\mathcal{A}}(b/D).$$

Proof. (1) Let A_D, p_D and a_D be as in Proposition 2.4, and let $A(a) = A_D$ and $a = a_D$. Then $\mathrm{tp}^{\mathcal{A}}(a/D)$ does not split over $A(a)$.

(2) Choose an arbitrary realization of each type over $A(a)$, and let their collection be $B(a)$. By (1) we know that $|A(a)| \leq \mathfrak{c}$, hence by stability

$$|B(a)| \leq \aleph_0 \cdot |\bigcup_{i\in\omega} \mathrm{S}_i^{\mathcal{A}}(A(a))| \leq \aleph_0^2 \mathfrak{c} = \mathfrak{c}.$$

Clearly $A(a) \subseteq B(a)$, and every type over $A(a)$ can be realized in $B(a)$.

(3) We prove that $B(a)$ fulfills (3). Suppose $\mathrm{tp}^{\mathcal{A}}(a/B(a)) = \mathrm{tp}^{\mathcal{A}}(b/B(a))$ and $\varphi(v, \bar{d}) \in \mathrm{tp}^{\mathcal{A}}(a/D)$. We have to show $\varphi(v, \bar{d}) \in \mathrm{tp}^{\mathcal{A}}(b/D)$. By (2) there exists $\bar{d}' \in B(a)$ such that $\mathrm{tp}^{\mathcal{A}}(\bar{d}/A(a)) = \mathrm{tp}^{\mathcal{A}}(\bar{d}'/A(a))$. By (1) $\mathrm{tp}^{\mathcal{A}}(a/D)$ does not split over $A(a)$ hence $\varphi(v, \bar{d}') \in \mathrm{tp}^{\mathcal{A}}(a/B(a)) = \mathrm{tp}^{\mathcal{A}}(b/B(a))$. Since b realizes p_D, Proposition 2.4 implies that $\mathrm{tp}^{\mathcal{A}}(b/D)$ does not split over $A(a)$ as well. Therefore $\varphi(v, \bar{d}) \in \mathrm{tp}^{\mathcal{A}}(b/D)$, as desired. ∎

3 Extending decomposable mappings

In this section we are presenting a method for constructing so called *decomposable* isomorphisms between certain ultraproducts. A function $f : \Pi_{i\in I}A_i/\mathcal{F} \to \Pi_{i\in I}B_i/\mathcal{F}$ is called decomposable if f "acts coordinatewise," that is, if for all $i \in I$ there are functions $f_i : A_i \to B_i$ such that $f = \Pi_{i\in I}f_i/\mathcal{F}$. Our construction will extend partial decomposable mappings through a transfinite recursion. In the inner steps we have to guarantee that we can continue the partial function with some element. To this end we formulate several principles (so called continuation principles) above.

3.1 Continuation principles

Definition 3.1 *Let I be an arbitrary set and $\mathcal{F}$ an ultrafilter over I. Then*

$$\langle \Phi_i : i \in I\rangle$$

is a finite distribution of formulas iff the following stipulations hold:

(i) $(\forall i \in I)\ \Phi_i \in [\mathrm{Form}(L)]^{<\omega}$;

(ii) for every $i \in I$ if φ is a subformula of an element of Φ_i then $\varphi \in \Phi_i$;

(iii) for every formula φ we have $\{i \in I : \varphi \in \overline{\Phi}_i\} \in \mathcal{F}$, where $\overline{\Phi}$ is the set of all formulas equivalent to a suitable boolean combination of members of Φ.

Lemma 3.2 *Let $f = \langle f_i : i \in I\rangle/\mathcal{F} : A \to B$ be a decomposable elementary mapping between $\mathcal{A} = \Pi_{i\in I}\mathcal{A}_i/\mathcal{F}$ and $\mathcal{B} = \Pi_{i\in I}\mathcal{B}_i/\mathcal{F}$. Then for every $\varphi \in \mathrm{Form}(L)$ there exists $J = J(\varphi) \in \mathcal{F}$ such that f_i preserves φ for all $i \in J$.*

Proof. Fix φ and suppose, seeking a contradiction that $\{i \in I : f_i \text{ preserves } \varphi\} \notin \mathcal{F}$. It follows that there exist $\bar{a}_i \in \mathrm{dom}(f_i)$ such that $\{i \in I : \mathcal{A}_i \models \varphi[\bar{a}_i] \not\Longleftrightarrow \mathcal{B}_i \models \varphi[f_i(\bar{a}_i)]\} \in \mathcal{F}$. Now let $\bar{a} = \langle \bar{a}_i : i \in I\rangle/\mathcal{F}$. Observe that $f(\bar{a}) = \langle f_i(\bar{a}_i) : i \in I\rangle/\mathcal{F}$. Then by Łós lemma $\mathcal{A} \models \varphi[\bar{a}] \not\Longleftrightarrow \mathcal{B} \models \varphi[f(\bar{a})]$ which contradicts the fact that f is elementary. ∎

Definition 3.3 *Let $\mathcal{A} = \Pi_{i\in I}\mathcal{A}_i/\mathcal{F}$ and $\mathcal{B} = \Pi_{i\in I}\mathcal{B}_i/\mathcal{F}$ be two ultraproducts, let $f = \Pi_{i\in I}f_i/\mathcal{F} : A \to B$ be a decomposable elementary mapping, and let $\nabla = \langle \Phi_i : i \in I\rangle$ be a finite distribution of formulas.*

• *We say that* $\langle \mathcal{A}, \mathcal{B}, f\rangle$ *has the elementary-decomposition property* (EDP) *with respect to* ∇ *iff* $\{i \in I : f_i$ *preserves* $\Phi_i\} \in \mathcal{F}$.

• *We say that* $\langle \mathcal{A}, \mathcal{B}\rangle$ *has the universal elementary-decomposition property* (UEDP) *with respect to* ∇ *iff for any decomposable elementary mapping* f*, the triplet* $\langle \mathcal{A}, \mathcal{B}, f\rangle$ *has* EDP *(w.r.t.* ∇*).*

• *We say that* $\langle \mathcal{A}, \mathcal{B}, f\rangle$ *has the strict elementary-decomposition property* (SEDP) *with respect to* ∇ *iff* $\{i \in I : \frac{\mathrm{dom}(f_i)}{\mathrm{ran}(f_i)}$ *is a* Φ_i*-elementary substructure in* $\frac{\mathcal{A}_i}{\mathcal{B}_i}\} \in \mathcal{F}$.

Lemma 3.4 SEDP *implies* EDP. *In more detail if* $f = \Pi_{i\in I} f_i/\mathcal{F} : A \to B$ *is a decomposable elementary mapping between* $\mathcal{A} = \Pi_{i\in I}\mathcal{A}_i/\mathcal{F}$ *and* $\mathcal{B} = \Pi_{i\in I}\mathcal{B}_i/\mathcal{F}$ *such that* $\langle \mathcal{A}, \mathcal{B}, f\rangle$ *has* SEDP *then* $\langle \mathcal{A}, \mathcal{B}, f\rangle$ *has* EDP *(with respect to the same distribution of formulas).*

Proof. Suppose $\langle \mathcal{A}, \mathcal{B}, f\rangle$ has SEDP with respect to $\nabla = \langle \Phi_i : i \in I\rangle$. If R is a relation symbol in our language L then denote by $\varrho(R)$ the arity (the number of variables) of R. Let $\Phi = \{R(v_0, ..., v_{\varrho(R)-1}) : R \in L \text{ is a relation symbol}\}$. By convention Φ is finite, thus by Lemma 3.2 we have $J_0 = \{i \in I : f_i \text{ preserves } \Phi\} \in \mathcal{F}$, and by SEDP, $J = \{i \in J_0 : \frac{\mathrm{dom}(f_i)}{\mathrm{ran}(f_i)}$ is a Φ_i-elementary substructure of $\frac{\mathcal{A}_i}{\mathcal{B}_i}\} \in \mathcal{F}$. Consequently, to complete the proof it is enough to show that

$$(*) \qquad f_i \text{ preserves } \Phi_i \text{ for every } i \in J.$$

To do so, we fix an $i \in I$ and we apply induction on complexity of members of Φ_i. Since $i \in J$ it follows that $(*)$ is true for atomic formulas. Now suppose that $(*)$ is true for $\varphi, \psi \in \Phi_i$ and let $\bar{a} \in \mathrm{dom}(f_i)$ be arbitrary.

• Assume $\neg\varphi \in \Phi_i$. Then $\mathcal{A}_i \models \neg\varphi(\bar{a})$ iff $\mathcal{A}_i \not\models \varphi(\bar{a})$ iff (by induction) $\mathcal{B}_i \not\models \varphi(f_i(\bar{a}))$ iff $\mathcal{B}_i \models \neg\varphi(f_i(\bar{a}))$, so $(*)$ holds for $\neg\varphi$ as well.

• Assume $\varphi \wedge \psi \in \Phi_i$. Then $\mathcal{A}_i \models \varphi(\bar{a}) \wedge \psi(\bar{a})$ iff $\mathcal{A}_i \models \varphi(\bar{a})$ and $\mathcal{A}_i \models \psi(\bar{a})$ iff (by induction) $\mathcal{B}_i \models \varphi(f_i(\bar{a}))$ and $\mathcal{B}_i \models \psi(f_i(\bar{a}))$ iff $\mathcal{B}_i \models \varphi(f_i(\bar{a})) \wedge \psi(f_i(\bar{a}))$, so $(*)$ holds for $\varphi \wedge \psi$ as well.

• Finally assume $\exists v\varphi \in \Phi_i$. Then $\mathcal{A}_i \models \exists v\varphi(v, \bar{a})$ iff there exists $b \in A_i$ with $\mathcal{A}_i \models \varphi_i(b, \bar{a})$. By the construction of J, $\mathrm{dom}(f_i)$ is a Φ_i-elementary substructure of

$\mathcal{A}_i$ so the last condition is equivalent with $(**)$ below:

$$(**) \text{ there exists } b \in \mathrm{dom}(f_i) \text{ with } \mathcal{A}_i \models \varphi(b, \bar{a}).$$

By induction $(**)$ holds if and only if $\mathcal{B}_i \models \varphi(f_i(b), f(\bar{a}))$, whence $\mathcal{B}_i \models \exists v \varphi(v, f_i(\bar{a}))$ follows. Conversely, if $\mathcal{B}_i \models \exists v \varphi(v, f(\bar{a}))$ then, since $\mathrm{ran}(f_i)$ is a Φ_i-elementary substructure of $\mathcal{B}_i$, it follows that there exists $b \in \mathrm{dom}(f_i)$ with $\mathcal{B}_i \models \varphi(f_i(b), f(\bar{a}))$. Hence, by induction $(**)$ follows, and the proof is complete. ∎

Definition 3.5 *Suppose $\mathcal{A} = \Pi_{i\in I}\mathcal{A}_i/\mathcal{F}$ and $\mathcal{B} = \Pi_{i\in I}\mathcal{B}_i/\mathcal{F}$ are two ultraproducts, ∇ is a finite distribution of formulas, and $f : A \to B$ is a decomposable elementary mapping.*

($\mathbf{CP_1}$) *$\langle \mathcal{A}, \mathcal{B}, f\rangle$ is said to admit the first continuation principle iff $A \setminus \mathrm{dom}(f) \neq \emptyset$ and there exist $a \in A \setminus \mathrm{dom}(f)$, $b \in B \setminus \mathrm{ran}(f)$ and a partial function $g : A \to B$ such that $f \cup \{\langle a, b\rangle\} \subseteq g$ and g is still elementary.*

($\mathbf{CP_2}$) *$\langle \mathcal{A}, \mathcal{B}\rangle$ is said to admit the second continuation principle iff whenever $\langle \mathcal{A}, \mathcal{B}, f\rangle$ has* EDP *and $A \setminus \mathrm{dom}(f) \neq \emptyset$ then there exist $a \in A \setminus \mathrm{dom}(f)$, $b \in B \setminus \mathrm{ran}(f)$ and a partial function $g : A \to B$ such that $f \cup \{\langle a, b\rangle\} \subseteq g$ and $\langle \mathcal{A}, \mathcal{B}, g\rangle$ still has* EDP*.*

($\mathbf{CP_3}$) *$\langle \mathcal{A}, \mathcal{B}\rangle$ is said to admit the third continuation principle iff whenever $\langle \mathcal{A}, \mathcal{B}, f\rangle$ has* SEDP *and $A \setminus \mathrm{dom}(f) \neq \emptyset$ then there exist $a \in A \setminus \mathrm{dom}(f)$, $b \in B \setminus \mathrm{ran}(f)$ and a partial function $g : A \to B$ such that $f \cup \{\langle a, b\rangle\} \subseteq g$ and $\langle \mathcal{A}, g\rangle$ still has* SEDP*.*

3.2 Conditions on continuation

In this section we provide sufficient conditions implying $\mathrm{CP}_1, \mathrm{CP}_2, \mathrm{CP}_3$ and EDP. We start by a simple case, namely we show that if $\mathcal{A} = \Pi_{i\in I}\mathcal{A}_i/\mathcal{F}$ has a finite elimination base then EDP follows.

Proposition 3.6 *Let $\mathcal{A} = \Pi_{i\in I}/\mathcal{A}_i\mathcal{F}$ and $\mathcal{B} = \Pi_{i\in I}\mathcal{B}_i/\mathcal{F}$ be elementarily equivalent structures having a finite elimination base, and let $f : \mathcal{A} \to \mathcal{B}$ be a decomposable elementary mapping. Then $\langle \mathcal{A}, \mathcal{B}, f\rangle$ has* EDP*.*

Proof. We shall prove, that there is a finite distribution of formulas $\langle \Phi_i : i \in I\rangle$, such that for any decomposable elementary mapping $f = \Pi_{i\in I} f_i/\mathcal{F} : A \to B$, one has $\{i \in I : f_i \text{ preserves } \Phi_i\} \in \mathcal{F}$. Let Φ be a finite elimination base, and for all $i \in I$ let Φ_i be the smallest set of formulas containing Φ and closed under subformulas (thus Φ_i does not depend on i). This will be a good distribution of formulas. According to Lemma 3.2 for every $\varphi \in \Phi$ there is a set $J(\varphi)$ such that f_i preserves φ for all $i \in J(\varphi)$. Now let $J = \bigcap_{\varphi \in \Phi} J(\varphi) \in \mathcal{F}$. Then for all $i \in J$, f_i preserves Φ_i. ∎

Actually, the previous proof establishes the following somewhat stronger observation.

Corollary 3.7 *Let $\mathcal{A} = \Pi_{i\in I}/\mathcal{A}_i\mathcal{F}$ and $\mathcal{B} = \Pi_{i\in I}\mathcal{B}_i/\mathcal{F}$ be elementarily equivalent structures having a finite elimination base. Then $\langle \mathcal{A}, \mathcal{B}\rangle$ has* UEDP. ∎

Theorem 3.8 *Suppose $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent, their common theory is uncountably categorical, $f : A \to B$ is an elementary mapping such that $D = \mathrm{dom}(f) \neq A$, $R = \mathrm{ran}(f)$ and $\langle \mathcal{A}, D\rangle$, $\langle \mathcal{B}, R\rangle$ are $\mathfrak{c}^+$-saturated. Then there exists an elementary mapping f' strictly extending f.*

The point here is, that our statement may also apply to cases when $|\mathrm{dom}(f)| = |A|$, so ordinary saturation cannot be used.

Proof. We distinguish two cases.

Case 1: $D = \mathrm{dom}(f)$ is not an elementary substructure of $\mathcal{A}$. Then by the Łoś-Vaught test, there is a formula ψ, and constants $\bar{d} \in D$, such that $\mathcal{A} \models \exists v\psi(v, \bar{d})$, but there is no such $v \in D$. Since $\mathcal{A}$ is uncountably categorical, it is $\aleph_0$-stable. Hence, the isolated types over D are dense in $\mathrm{S}_1^{\mathcal{A}}(D)$. Consequently, there is an isolated type $p \in \mathrm{S}^{\mathcal{A}}(D)$ containing $\psi(v, \bar{d})$. Let $a \in A$ be a realization of p (such a realization exists since p is isolated). Then $\mathcal{A} \models \psi(a, \bar{d})$, so $a \notin D$. Let $b \in B$ be a realization of $f[p]$ in $\mathcal{B}$. Again, since $f[p]$ is isolated, b exists. Finally let $f' = f \cup \{\langle a, b\rangle\}$. Clearly, f' is an elementary mapping strictly extending f.

Case 2: $\mathcal{D} \prec \mathcal{A}$ is an elementary substructure. Let $a \in A \setminus D$, $A(a) \subseteq B(a) \subseteq D$ as in Lemma 2.5. It is enough to show that $p = f[\mathrm{tp}^{\mathcal{A}}(a/B(a))]$ can be realized in $B \setminus \mathrm{ran}(f)$ because if b realizes p in $B \setminus \mathrm{ran}(f)$ then $f' = f \cup \langle\{a, b\}\rangle$ is the required elementary mapping strictly extending f.

Adjoin a new relation symbol R to the language of $\mathcal{B}$ and interpret it in $\mathcal{B}$ as $\operatorname{ran}(f)$. By saturatedness it is enough to show that each $\phi \in p$ can be realized in $B \setminus R$. Let $\phi \in p$ be arbitrary, but fixed. By assumption, $\mathcal{D}$ is an elementary substructure of $\mathcal{A}$, so it follows that a is not algebraic over D. Hence, because of f is elementary, the relation defined by ϕ in $\mathcal{B}$ is infinite as well. In addition, $\mathcal{B}$ is uncountably categorical, consequently $\langle \mathcal{B}, f[\mathcal{D}]\rangle$ is not a Vaughtian pair (see, for example, Theorem 6.1.18 of [10]). Thus the relation defined by ϕ in $\mathcal{B}$ can be realized in $B \setminus R$, therefore $\neg R(v) \wedge \phi(v)$ can be satisfied in $\mathcal{B}$, for all $\phi \in p$. ∎

Now we turn to provide a sufficient condition for CP_3. Recall, that we have fixed a first order language L containing finitely many relation symbols and does not contain function-symbols. Here we also fix an enumeration $\langle \varphi_n : n \in \omega\rangle$ of first order formulas of L. If $\mathcal{A}$ is an L-structure and $\mathcal{B}$ is a substructure of it, then

$$\varepsilon^{\mathcal{A}}(\mathcal{B}) = \sup\{n \in \omega : \mathcal{B} \text{ is a } \varphi_k\text{-elementary substructure of } \mathcal{A} \text{ for all } k \leq n\}.$$

Roughly speaking, ε measures that $\mathcal{B}$ is how close to being an elementary substructure of $\mathcal{A}$. Clearly, $\varepsilon^{\mathcal{A}}(\mathcal{B}) = \omega$ if and only if $\mathcal{B}$ is an elementary substructure.

Theorem 3.9 *Suppose $\mathcal{A} = \Pi_{i\in I}\mathcal{A}_i/\mathcal{F}$ and $\mathcal{B} = \Pi_{i\in I}\mathcal{B}_i/\mathcal{F}$ are elementarily equivalent, uncountably categorical structures, $\mathcal{F}$ is $\mathfrak{c}^+$-good, $f = \langle f_i : i \in I\rangle/\mathcal{F} : A \to B$ is a decomposable elementary mapping and $a \in A \setminus \operatorname{dom}(f), b \in B \setminus \operatorname{ran}(f)$ with $f[\operatorname{tp}^{\mathcal{A}}(a/\operatorname{dom}(f))] = \operatorname{tp}^{\mathcal{B}}(b/\operatorname{ran}(f))$. Suppose, in addition, that the following stipulations hold.*

(1) For every $n \in \omega$ we have $\{i \in I : \varepsilon^{\mathcal{A}_i}(\operatorname{dom}(f_i)) \geq n\} \in \mathcal{F}$ and

(2) there is a natural number N such that for every $i \in I$ there exists $C_i \subseteq A_i$ with $|C_i \setminus \operatorname{dom}(f_i)| \leq N$, $\operatorname{dom}(f_i) \subseteq C_i, a_i \in C_i$ and $\varepsilon^{\mathcal{A}_i}(\mathcal{C}_i) \geq \varepsilon^{\mathcal{A}_i}(\operatorname{dom}(f_i))$.

Then there exists a decomposable elementary mapping $g = \langle g_i : i \in I\rangle/\mathcal{F} : A \to B$ extending f with $\operatorname{dom}(g_i) = C_i$ for every $i \in I$.

Proof. Let $D = \operatorname{dom}(f)$ and let $C = \Pi_{i\in I}C_i/\mathcal{F}$. By assumptions (1) and (2), C determines an elementary substructure of $\mathcal{A}$ (because $\{i \in I : \mathcal{C}_i$ is n-elementary in $\mathcal{A}_i\} \in \mathcal{F}$ holds for every $n \in \omega$). Applying N times consecutively Theorem 3.8

to $\mathcal{C}, \mathcal{B}$ and f, it follows, that f can be extended to an elementary mapping g with $\operatorname{dom} g = C$.

To complete the proof, we have to show that g is decomposable. First observe, that by assumption (2) we have $|C - D| \leq N$. Consequently, $g \setminus f$ is finite and therefore is decomposable. Hence $g = f \cup (g \setminus f)$ itself is also decomposable. ∎

Definition 3.10 *Let $\mathcal{A} = \Pi_{i\in I}\mathcal{A}_i/\mathcal{F}$ and $\mathcal{B} = \Pi_{i\in I}\mathcal{B}_i/\mathcal{F}$ be two ultraproducts. The pair $\langle \mathcal{A}, \mathcal{B}\rangle$ is defined to be relatively κ-homogeneous iff the following holds: for every decomposable elementary mapping $f : A \to B$, for every $a \in A \setminus \operatorname{dom}(f)$ and for every $C \in [\operatorname{dom}(f)]^{<\kappa}$ the type $f[\operatorname{tp}^{\mathcal{A}}(a/C)]$ can be realized in $B \setminus \operatorname{ran}(f)$.*

This concept can be considered as an adaptation of κ-homogeneity (in which instead of decomposability of f we require $|\operatorname{dom}(f)| < \kappa$ and $C = \operatorname{dom}(f)$; for more details we refer e.g. to the Remark before Proposition 5.1.9 of [1]).

Proposition 3.11 *Suppose $\mathcal{A} = \Pi_{i\in I}\mathcal{A}_i/\mathcal{F}$ and $\mathcal{B} = \Pi_{i\in I}\mathcal{B}_i/\mathcal{F}$ are stable structures such that $\langle \mathcal{A}, \mathcal{B}\rangle$ is relatively $\mathfrak{c}^+$-homogeneous. Let $f = \Pi_{i\in I} f_i/\mathcal{F} : A \to B$ be a decomposable elementary mapping such that $A \setminus \operatorname{dom}(f) \neq \emptyset$. Then $\langle \mathcal{A}, \mathcal{B}, f\rangle$ admits the continuation principle* CP_1.

Proof. Let $D = \operatorname{dom}(f)$ and $R = \operatorname{ran}(f)$. As supposed, $D = \langle D_i : i \in I\rangle$ is decomposable, hence $\langle \mathcal{A}, D\rangle = \Pi_{i\in I}\langle \mathcal{A}_i, D_i\rangle/\mathcal{F}$ is $\mathfrak{c}^+$-saturated. So there exist $a \in A \setminus D$, $A(a)$ and $B(a)$, satisfying the conclusion of Lemma 2.5, and by relative $\mathfrak{c}^+$-homogeneity, there exists a corresponding $b \in B \setminus R$, such that b realizes $f\big[\operatorname{tp}^{\mathcal{A}}(a/B(a))\big]$.

First we claim that $f \cup \{\langle a, b\rangle\}$ is elementary. Suppose $\mathcal{A} \models \psi(a, \bar{d})$ for $\bar{d} \in D$. Consider $\operatorname{tp}^{\mathcal{A}}(\bar{d}/A(a))$. By Lemma 2.5 (2), there exist $\bar{d}' \in B(a)$ such that $\operatorname{tp}^{\mathcal{A}}(\bar{d}'/A(a)) = \operatorname{tp}^{\mathcal{A}}(\bar{d}/A(a))$. Now $\mathcal{A} \models \psi(a, \bar{d}')$, because by Lemma 2.5 (1), $\operatorname{tp}^{\mathcal{A}}(a/D)$ does not split over $A(a)$. By the choice of b, $\mathcal{A} \models \psi(a, \bar{d}') \longleftrightarrow \mathcal{B} \models \psi(b, f(\bar{d}'))$.

If $\operatorname{tp}^{\mathcal{B}}(b/R)$ does not split over $f[A(a)]$, then $\mathcal{B} \models \psi(b, f(\bar{d}')) \longleftrightarrow \psi(b, f(\bar{d}))$, hence $f\big[\operatorname{tp}^{\mathcal{A}}(a/D)\big] = \operatorname{tp}^{\mathcal{B}}(b/R)$, that is $f \cup \{\langle a, b\rangle\}$ is elementary, as promised. So it remained to show that $\operatorname{tp}^{\mathcal{B}}(b/R)$ does not split over $f[A(a)]$.

Suppose, seeking a contradiction, that there exist $f(\bar{c}), f(\bar{c}') \in f[D]$ and φ, such that $\mathrm{tp}^{\mathcal{B}}(f(\bar{c})/f[A(a)]) = \mathrm{tp}^{\mathcal{B}}(f(\bar{c}')/f[A(a)])$, but $\varphi(v, f(\bar{c})) \in \mathrm{tp}^{\mathcal{B}}(b/R)$ and $\varphi(v, f(\bar{c}')) \notin \mathrm{tp}^{\mathcal{B}}(b/R)$. Now, f^{-1} is also elementary, $b \notin \mathrm{dom}(f^{-1})$, hence again by relative $\mathfrak{c}^+$-homogeneity, $f^{-1}\big[\mathrm{tp}^{\mathcal{B}}(b/f[B(a)] \cup \{\bar{c}, \bar{c}'\})\big]$ can be realized by $a' \notin \mathrm{ran}(f^{-1}) = D$. Clearly, $\mathrm{tp}^{\mathcal{A}}(a'/B(a)) = \mathrm{tp}^{\mathcal{A}}(a/B(a))$, but $f^{-1}(\bar{c}), f^{-1}(\bar{c}')$ and φ shows, that $\mathrm{tp}^{\mathcal{A}}(a'/D) \neq \mathrm{tp}^{\mathcal{A}}(a/D)$, which contradicts the choice of a, and Lemma 2.5 (3). ∎

There is a special case when we can also guarantee CP_2. Namely if $\langle \mathcal{A}, \mathcal{B} \rangle$ has UEDP. According to Corollary 3.7 this is the situation if there exists a finite elimination base (e.g. if the common theory of the structures admits quantifier elimination). We formulate this fact for the future purposes. The point here is that we can extend a decomposable mapping coordinatewise keeping EDP.

Proposition 3.12 *Suppose $\mathcal{A} = \Pi_{i\in I}\mathcal{A}_i/\mathcal{F}$ and $\mathcal{B} = \Pi_{i\in I}\mathcal{B}_i/\mathcal{F}$ are elementarily equivalent stable structures, and also assume that $\langle \mathcal{A}, \mathcal{B} \rangle$ is relatively $\mathfrak{c}^+$-homogeneous and has* UEDP. *Then $\langle \mathcal{A}, \mathcal{B} \rangle$ admits the second continuation principle* CP_2.

Proof. Let $f : A \to B$ be a decomposable elementary mapping with $A \setminus \mathrm{dom}(f) \neq \emptyset$. By Proposition 3.11 there exists $a \in A$ and $b \in B$ such that $f \cup \{\langle a, b \rangle\}$ is also elementary.

As supposed f is decomposable, and $a \notin \mathrm{dom}(f)$, $b \notin \mathrm{ran}(f)$, therefore

$$J = \{i \in I : a_i \notin \mathrm{dom}(f_i),\, b_i \notin \mathrm{ran}(f_i)\} \in \mathcal{F}.$$

Now let

$$h_i = \begin{cases} f_i \cup \{\langle a_i, b_i \rangle\} & \text{if } i \in J \\ f_i & \text{otherwise} \end{cases}.$$

Then $\Pi_{i\in I} h_i/\mathcal{F}$ is elementary, thus by UEDP, $J_1 = \{i \in I : g_i \text{ preserves } \Phi_i\} \in \mathcal{F}$. Now setting the function

$$g_i = \begin{cases} f_i \cup \{\langle a_i, b_i \rangle\} & \text{if } i \in J \cap J_1 \\ f_i & \text{otherwise} \end{cases}$$

it preserves Φ_i. ∎

3.3 Extending decomposable mappings

Theorem 3.13 *Let $\mathcal{A} = \Pi_{i\in I}\mathcal{A}_i/\mathcal{F}$ and $\mathcal{B} = \Pi_{i\in I}\mathcal{B}_i/\mathcal{F}$ be two ultraproducts and let $g : A \to B$ be a decomposable elementary mapping. Suppose $|A_i| = |B_i| < \aleph_0$ for all $i \in I$. Suppose in addition that either (1) or (2) below hold.*

(1) $\langle \mathcal{A}, \mathcal{B}, g\rangle$ has EDP *and $\langle \mathcal{A}, \mathcal{B}\rangle$ has* CP_2*;*

(2) $\langle \mathcal{A}, \mathcal{B}, g\rangle$ has SEDP *and $\langle \mathcal{A}, \mathcal{B}\rangle$ has* CP_3

(according to the finite distribution of formulas $\nabla = \langle \Phi_i : i \in I\rangle$).

Then g extends to a decomposable isomorphism.

Proof. We will distinguish two cases: in the first case we assume (1), that is, we assume $\langle \mathcal{A}, \mathcal{B}\rangle$ has CP_2 while in the second case we assume (2), that is, we assume $\langle \mathcal{A}, \mathcal{B}\rangle$ has CP_3. During this proof, we will handle these cases simultaneously.

We apply transfinite recursion. Let $\mathcal{A}$, $\mathcal{B}$ and g be as in the theorem. By assumption, g is decomposable, that is, there is a sequence of representatives $\langle g_i : i \in I\rangle$ such that $g = \Pi_{i\in I} g_i/\mathcal{F}$. By EDP in the first case, $J = \{i \in I : g_i$ preserves $\Phi_i\} \in \mathcal{F}$. Similarly, by SEDP in the second case $J = \{i \in I : {}^{\mathrm{dom}(g_i)}_{\mathrm{ran}(g_i)}$ is a Φ_i-elementary substructure of ${}^{\mathcal{A}_i}_{\mathcal{B}_i}\} \in \mathcal{F}$. Let $f^0 = g$ and

$$f_i^0 = \begin{cases} g_i & \text{if } i \in J \\ \emptyset & \text{otherwise.} \end{cases}$$

Let β be an ordinal, and suppose $f^\alpha = \langle f_i^\alpha : i \in I\rangle/\mathcal{F}$ have already been defined for every $\alpha < \beta$ such that the following stipulations hold:

• $f^\alpha : A \to B$ is a decomposable elementary mapping;

• $f_i^\gamma \subseteq f_i^\nu$ for $\gamma < \nu$ and all $i \in I$;

• in the first case f_i^α preserves all the members of Φ_i for all $i \in I$ (particularly, $\langle \mathcal{A}, \mathcal{B}, f^\alpha\rangle$ has EDP);

• in the second case ${}^{\mathrm{dom}(f_i^\alpha)}_{\mathrm{ran}(f_i^\alpha)}$ is a Φ_i-elementary substructure of ${}^{\mathcal{A}_i}_{\mathcal{B}_i}$, for all $i \in I$ (particularly, $\langle \mathcal{A}, \mathcal{B}, f^\alpha\rangle$ has SEDP).

I. Successor case

(Extending) Suppose $\beta = \alpha^+$. For simplicity, denote f^α, $\mathrm{dom}(f^\alpha)$ and $\mathrm{ran}(f^\alpha)$ by f, D and R, respectively. We may assume $A \smallsetminus D \neq \emptyset$, since otherwise f^α would be a decomposable isomorphism extending g. According to CP_2 or CP_3, there exist

$a \in A \setminus D$, $b \in B \setminus R$ and a partial function $h : A \to B$ such that $f^\alpha \cup \{\langle a, b\rangle\} \subseteq h$ and

- in the first case $\langle \mathcal{A}, \mathcal{B}, h\rangle$ has EDP;
- in the second case $\langle \mathcal{A}, \mathcal{B}, h\rangle$ has SEDP.

(Giving representants) We would like to keep all the stipulations given in the beginning of our transfinite construction. Hence we define f^β via its representatives $\langle f_i^\beta : i \in I\rangle$. As supposed, h is decomposable, say $h = \langle h_i : i \in I\rangle/\mathcal{F}$ and $\langle \mathcal{A}, \mathcal{B}, h\rangle$ has EDP (respectively, $\langle \mathcal{A}, \mathcal{B}, h\rangle$ has SEDP). In the first case $J = \{i \in I : f_i \subseteq h_i$ and h_i preserves $\Phi_i\} \in \mathcal{F}$, while in the second case $J = \{i \in I : f_i \subseteq h_i$ and $\frac{\mathrm{dom}(h_i)}{\mathrm{ran}(h_i)}$ is a Φ_i-elementary substructure of $\frac{\mathcal{A}_i}{\mathcal{B}_i}\} \in \mathcal{F}$. Now let

$$f_i^\beta = \begin{cases} h_i & \text{if } i \in J \\ f_i^\alpha & \text{otherwise.} \end{cases}$$

Then $f^\beta = \Pi_{i \in I} f_i^\beta/\mathcal{F}$ is as desired.

II. Limit case

(Extending) Set $f_i^\beta = \bigcup_{\alpha<\beta} f_i^\alpha$ for all $i \in I$, and $f^\beta = \langle f_i^\beta : i \in I\rangle/\mathcal{F}$. We show that f^β is still elementary. For this it is enough to prove, that f_i^β preserves Φ_i for all $i \in I$. Fix $i \in I$ and let $\varphi \in \Phi_i$ with parameters $\bar{d}$ from $\mathrm{dom}(f_i^\beta)$, and suppose $\mathcal{A}_i \models \varphi(\bar{d})$. By definition, there exists $\alpha < \beta$ such that $\bar{d} \in \mathrm{dom}(f_i^\alpha)$. Now observe that $f_i^\beta|_{\mathrm{dom}(f_i^\alpha)} = f_i^\alpha$. In the first case it was stipulated, that f_i^α preserves all the members of Φ_i; while in the second case we stipulated that $\mathrm{dom}(f_i^\alpha)$ and $\mathrm{ran}(f_i^\alpha)$ are Φ_i-elementary substructures, whence, by the proof of Lemma 3.4, it also follows that f_i is Φ_i-elementary. Hence $\mathcal{A}_i \models \varphi(\bar{d})$ iff $\mathcal{B}_i \models \varphi(f_i^\alpha(\bar{d}))$. Since i was arbitrary, f_i^β preserves Φ_i for all $i \in I$. By construction, f^β is decomposable. It is easy to see, that the other stipulations given in the beginning of the transfinite construction hold, as well.

III. Summing up

The construction above stops at an ordinal β, such that f^β is a decomposable elementary mapping, for which $\mathrm{dom}(f^\beta) = A$. Since each A_i is finite and has same cardinality as B_i, it follows, that f^β is a decomposable isomorphism. ∎

Now we formulate this theorem in a special case.

Corollary 3.14 *Let $\mathcal{A} = \Pi_{i\in I}\mathcal{A}_i/\mathcal{F}$ and $\mathcal{B} = \Pi_{i\in I}\mathcal{B}_i/\mathcal{F}$ be elementarily equivalent stable structures with $|A_i| = |B_i| < \aleph_0$ for all $i \in I$, where $\mathcal{F}$ is a countably incomplete $\mathfrak{c}^+$-good ultrafilter. Suppose also that $\langle \mathcal{A}, \mathcal{B}\rangle$ is relatively $\mathfrak{c}^+$-homogeneous having* UEDP*. Then any decomposable elementary mapping $g : A \to B$ extends to a decomposable isomorphism.*

Proof. By Proposition 3.12, $\langle \mathcal{A}, \mathcal{B}\rangle$ admits the second continuation principle CP_2, hence Theorem 3.13 (1) finishes the proof. ∎

4 Fruits of the approach

4.1 Morley's Theorem to the finite

As it is well known, Morley's theorem states that a theory is $\aleph_1$-categorical if and only if it is categorical on all uncountable cardinals. Our aim is to extend (a special case) of this theorem to finite cardinals as well.

We show that finite fragments of certain $\aleph_1$-categorical theories T are also categorical in the following sense: for all finite subsets Σ of T there exists a finite extension Σ' of Σ, such that up to isomorphism, Σ' can have at most one n-element model Σ'-elementarily embeddable into models of T, for all $n \in \omega$. For details, see Theorem 4.3, which is the main theorem of this section

We start by a theorem, stating, that under some additional technical conditions, an $\aleph_1$-categorical structure can be uniquely decomposed to ultraproducts of its finite substructures. Fix an enumeration $\langle \varphi_n : n \in \omega\rangle$ of first order formulas of L, and denote its n-th initial segment by Φ_n.

Definition 4.1 *A theory T is defined to be tight iff all $\mathcal{A} \models T$ satisfy the following stipulations.*

- *for every finite $X \subseteq A$ and $i \in \omega$, there exists a finite Φ_i-elementary substructure $\mathcal{A}_0$ of $\mathcal{A}$ with $X \subseteq A_0$;*
- *there exists $N \in \omega$ such that for any $n \in \omega$, any algebraically closed Φ_n-elementary substructure $\mathcal{A}_0$ of $\mathcal{A}$ containing an $\aleph_0$-saturated elementary substructure of $\mathcal{A}$ and any $a \in A$ there exists a Φ_n-elementary substructure $\mathcal{B}$ of $\mathcal{A}$ with $A_0 \subseteq B, a \in B$ and $|B \smallsetminus A_0| \leq N$.*

A structure is defined to be tight iff its theory is tight.

Theorem 4.2 *(Unique Factorization Theorem.)*
Suppose $\mathcal{C}$ is uncountably categorical, tight and $\mathcal{F}$ is countably incomplete and $\mathfrak{c}^+$-good. If the ultraproducts $\mathcal{A} = \Pi_{i \in I} \mathcal{A}_i / \mathcal{F}$ and $\mathcal{B} = \Pi_{i \in I} \mathcal{B}_i / \mathcal{F}$ are elementarily equivalent with $\mathcal{C}$, for every $i \in I$ we have $|A_i| = |B_i| < \aleph_0$ and for every $n \in \omega$

$$\{i \in I : \mathcal{A}_i \textit{ is a } \Phi_n - \textit{elementary substructure of } \mathcal{C}\} \in \mathcal{F},$$

then $\{i \in I : \mathcal{A}_i$ is isomorphic to $\mathcal{B}_i\} \in \mathcal{F}$.

Proof. We may assume that $\mathcal{C}$ is countable and $\aleph_0$-saturated. Fix an increasing sequence $\langle \mathcal{C}_n : n \in \omega \rangle$ of finite structures such that for every $n \in \omega$, $\mathcal{C}_n$ is a Φ_n-elementary substructure of $\mathcal{C}$ and $C = \cup_{n \in \omega} C_n$. Since $\mathcal{F}$ is countably incomplete, it is non-principal, in particular it is $\aleph_0$-regular whence $\mathcal{A}$ and $\mathcal{B}$ are $\aleph_1$-universal. Hence $\mathcal{C}$ can be elementarily embedded into $\mathcal{A}$ and $\mathcal{B}$; let f and g be such elementary embeddings. For every $a \in A$ and $b \in B$ fix representatives $\hat{a} \in a$ and $\hat{b} \in b$ (that is, $\hat{a} \in \Pi_{i \in I} A_i$ is such that $\hat{a}/\mathcal{F} = a$; and similarly for b and $\hat{b}$). Let $\langle I_n \in \mathcal{F} : n \in \omega \rangle$ be a descending chain with $\cap_{n \in \omega} I_n = \emptyset$ and for each $n \in \omega$ let

$$J_n = \{i \in I_n : \mathcal{A}_i \text{ is a } \Phi_n - \text{elementary substructure of } \mathcal{C} \text{ and } \{\widehat{f(x)}(i) : x \in C_n\} \text{ and } \{\widehat{g(x)}(i) : x \in C_n\} \text{ are isomorphic to } \mathcal{C}_n\}.$$

Clearly, $J_n \in \mathcal{F}$ for every $n \in \omega$. For each $i \in I$ let $\nu(i) = \max\{n \in \omega : i \in J_n\}$. Then for every $i \in I$ there exists an isomorphism f_i mapping $\{\widehat{f(x)}(i) : x \in C_{\nu(i)}\}$ onto $\{\widehat{g(x)}(i) : x \in C_{\nu(i)}\}$. Clearly, $\mathrm{dom}(f_i)$ is a $\Phi_{\nu(i)}$-elementary substructure of $\mathcal{C}$ and hence of $\mathcal{A}_i$ as well. Finally let $f = \Pi_{i \in I} f_i / \mathcal{F}$ and let $\nabla = \langle \Phi_{\nu(i)} : i \in I \rangle$ be a finite distribution of formulas. If for almost every $i \in I$ we have $\mathrm{dom}(f_i) = A_i$ then the statement of the theorem follows. So we may assume $\mathrm{dom}(f) \neq A$. Next, we claim, that

$$(*) \qquad \langle \mathcal{A}, \mathcal{B} \rangle \text{ has CP}_3 \text{ according to } \nabla.$$

To see this, let $g : \mathcal{A} \to \mathcal{B}$ be a decomposable elementary mapping such that $\langle \mathcal{A}, \mathcal{B}, g \rangle$ has SEDP and $A \setminus \mathrm{dom}(g) \neq \emptyset$. In order to keep notation simpler, throughout the proof $\mathrm{dom}(g)$ and $\mathrm{ran}(g)$ will be denoted by D and R, respectively. Since $\langle \mathcal{A}, \mathcal{B}, g \rangle$ has SEDP, it follows that D is an elementary substructure of $\mathcal{A}$, and since D is decomposable, it is $\aleph_1$-universal. Hence D contains an elementary substructure isomorphic to $\mathcal{C}$. By Theorem 3.8 there exists $a \in A \setminus D$, $b \in B \setminus R$ with $g[\mathrm{tp}(a/D)] = \mathrm{tp}(b/R)$. Tightness of $\mathcal{C}$ implies that all the conditions of Theorem 3.9 are fulfilled, hence there exists a decomposable elementary mapping g' with $g \cup \{\langle a, b \rangle\} \subseteq g'$ and $\langle \mathcal{A}, \mathcal{B}, g' \rangle$ having SEDP. Hence $(*)$ holds.

Finally applying Theorem 3.13, it follows that f (constructed in the first paragraph of the proof) can be extended to a decomposable isomorphism between $\mathcal{A}$ and $\mathcal{B}$, which completes the proof. ∎

At this point, everything is given, to prove the main result of this section: an extension of Morley's categoricity theorem to the finite.

Theorem 4.3 *Let $\mathcal{C}$ be an $\aleph_1$-categorical and tight structure, let $\mathrm{Th}(\mathcal{C}) = \{\varphi_i : i \in \omega\}$ and denote the i-th initial segment of $\mathrm{Th}(\mathcal{C})$ by Φ_i. Then there exists $N = N(\mathcal{C})$ such that $\forall n > N$, if $\mathcal{A}, \mathcal{B}$ are finite Φ_n-elementary substructures of $\mathcal{C}$ and $|A| = |B|$, then $\mathcal{A} \cong \mathcal{B}$.*

Proof. Suppose, seeking a contradiction, for all $N \in \omega$ there exist (at least) two non-isomorphic equinumerous finite models $\mathcal{A}_N, \mathcal{B}_N$ which are Φ_N-elementary substructures of $\mathcal{C}$. Let $\mathcal{F}$ be a countably incomplete, $\mathfrak{c}^+$-good ultrafilter on a suitable set I and let $f : I \to \omega$ be a function such that for every $n \in \omega$ we have $\{i \in I : f(i) \geq n\} \in \mathcal{F}$ (such an f may be easily constructed using that $\mathcal{F}$ is countably incomplete). Finally let $\mathcal{A} = \Pi_{i \in I} \mathcal{A}_{f(i)} / \mathcal{F}$, and $\mathcal{B} = \Pi_{i \in I} \mathcal{B}_{f(i)} / \mathcal{F}$. But then Theorem 4.2 implies, that $\{i \in I : \mathcal{A}_i \cong \mathcal{B}_i\} \in \mathcal{F}$; this contradicts to the choices of $\mathcal{A}_N, \mathcal{B}_N$. ∎

Theorem 4.4 *Let $T = \{\varphi_i : i \in \omega\}$ be an $\aleph_1$-categorical complete theory having a finite elimination base. Denote the i-th initial segment of T by T_i. Then there exists $N = N(T)$ such that $\forall n > N$, if $\mathcal{A}, \mathcal{B} \models T_n$, and $|A| = |B|$, then $\mathcal{A} \cong \mathcal{B}$.*

Proof. By way of contradiction, suppose for all $N \in \omega$ there exist (at least) two non-isomorphic equinumerous (finite) models $\mathcal{A}_N, \mathcal{B}_N \models T_N$. Let $\mathcal{A} = \Pi_{i\in\omega}\mathcal{A}_i/\mathcal{F}$, and $\mathcal{B} = \Pi_{i\in\omega}\mathcal{B}_i/\mathcal{F}$ (where $\mathcal{F}$ is a countably incomplete, good ultrafilter). Clearly $\mathcal{A}, \mathcal{B} \models T$, therefore $\mathcal{A} \equiv_e \mathcal{B}$, hence by categoricity $\mathcal{A} \cong \mathcal{B}$. Since the common theory of $\mathcal{A}$ and $\mathcal{B}$ have (the same) finite elimination base, there exists a finite distribution of formulas simultaneously witnessing UEDP for $\langle \mathcal{A}, \mathcal{B} \rangle$.

As an easy consequence of Theorem 3.8 and Corollary 3.14 it follows that there is a decomposable isomorphism between $\mathcal{A}$ and $\mathcal{B}$. Particularly $\{i \in I : \mathcal{A}_i \cong \mathcal{B}_i\} \in \mathcal{F}$ which leads to contradiction. Thus we can conclude that there exists $N \in \omega$ such that up to isomorphism there is at most one n-element model of T_i for all $N \leq i$ and $n \in \omega$. ∎

We note that Theorem 4.4 may also be derived (in a completely different way) from earlier results of Zilber and Cherlin.

4.2 On the Zilber-Cherlin-Harrington-Lachlan Theorem

In this section we investigate homogeneity of finite substructures of stable (and not necessary $\aleph_0$-stable) structures. Particularly, we give a simple proof related to the converse of a special case of the Zilber-Cherlin-Lachlan-Harrington theorem.

Definition 4.5 *For a structure $\mathcal{A}$, a set Φ of formulas, and $\bar{a} \in A$,*

$$E_{\bar{a}}^{\Phi,\mathcal{A}}(x,y) \overset{def}{\Longleftrightarrow} \mathrm{tp}_{\Phi}^{\mathcal{A}}(x/\bar{a}) = \mathrm{tp}_{\Phi}^{\mathcal{A}}(y/\bar{a}), \text{ that is,}$$

for all $\varphi \in \Phi$ we have $\mathcal{A} \models \varphi(x, \bar{a})$ iff $\mathcal{A} \models \varphi(y, \bar{a})$. When Φ or $\mathcal{A}$ is clear from the context, we omit them.

Definition 4.6 *Let $\mathcal{A}$ be a structure, Φ a set of formulas, and $l \in \omega$. Then $\mathcal{A}$ is said to have the (Φ, l)-equivalence property iff the following holds: if $\bar{a}, \bar{b} \in {}^{l}A$ and $\mathrm{tp}_{\Phi}^{\mathcal{A}}(\bar{a}/\emptyset) = \mathrm{tp}_{\Phi}^{\mathcal{A}}(\bar{b}/\emptyset)$ then $\langle A, \bar{a}, E_{\bar{a}}^{\Phi} \rangle \cong^{g} \langle A, \bar{b}, E_{\bar{b}}^{\Phi} \rangle$ such that for every $x \in A$ we have $g[\mathrm{tp}_{\Phi}(x/\bar{a})] = \mathrm{tp}_{\Phi}(g(x)/\bar{b})$.*

Lemma 4.7 *Let $\mathcal{A} = \Pi_{i\in I}\mathcal{A}_i/\mathcal{F}$ be an ultraproduct of finite structures such that $\langle\mathcal{A},\mathcal{A}\rangle$ has* UEDP, *with the corresponding distribution $\langle\Phi_i : i \in I\rangle$, where $\mathcal{F}$ is a countably incomplete, $\mathfrak{c}^+$-good ultrafilter. Suppose for every $n \in \omega$ we have*

$$\{i \in I : \mathcal{A}_i \textit{ has the } (\Phi_i, n) - \textit{equivalence property}\} \in \mathcal{F}.$$

Then $\langle\mathcal{A},\mathcal{A}\rangle$ is relatively $\mathfrak{c}^+$-homogeneous.

Proof. Let $f : A \to A$ be a decomposable elementary mapping, $a \notin \mathrm{dom}(f)$ and $C \in \big[\mathrm{dom}(f)\big]^{\leq\mathfrak{c}}$. We have to prove, that $f\big[\,\mathrm{tp}^{\mathcal{A}}(a/C)\big]$ can be realized by a suitable element $b \in A \smallsetminus \mathrm{ran}(f)$.

Let $p = f\big[\,\mathrm{tp}^{\mathcal{A}}(a/C)\big] = \{\varphi(v, f(\bar{c})) : \varphi(v,\bar{c}) \in \mathrm{tp}^{\mathcal{A}}(a/C)\}$. Since $|C| \leq \mathfrak{c}$ by saturatedness there exists $b \in A$ which realizes p. We show, that b can be chosen outside of $\mathrm{ran}(f)$.

Let f_i and D_i $(i \in I)$ be a decomposition of f and $\mathrm{dom}(f)$, respectively. Because $a \notin \mathrm{dom}(f)$ it follows, that $J_0 = \{i : a_i \notin D_i\} \in \mathcal{F}$. Adjoin a new relation symbol R to the language of $\mathcal{A}$ and interpret it as $\mathrm{ran}(f)$. f is decomposable, hence R is also decomposable, say $R = \langle R_i : i \in I\rangle/\mathcal{F}$. Since $\mathcal{F}$ is $\mathfrak{c}^+$-good, it follows that $\Pi_{i\in I}\langle\mathcal{A}, R_i\rangle/\mathcal{F}$ is $\mathfrak{c}^+$-saturated as well. So, because of p is closed under conjunctions, it is enough to show that $\neg R(v) \wedge \varphi(v,\bar{d})$ can be satisfied in $\mathcal{A}$, for every $\varphi \in p$.

To do so, let $\varphi(v, f(\bar{c})) \in p$, with $\bar{c} = \langle\bar{c}_i : i \in I\rangle/\mathcal{F}$. Since f is an elementary mapping, $f_0 = \{\langle c, f(c)\rangle : c \in \bar{c}\} \subseteq f$ is also elementary. Note, that f_0 is finite, hence by Theorem 3.13 [12] it is decomposable. Therefore UEDP implies $J_1 = \{i \in I : \mathrm{tp}^{\mathcal{A}_i}_{\Phi_i}(\bar{c}_i/\emptyset) = \mathrm{tp}^{\mathcal{A}_i}_{\Phi_i}(f_i(\bar{c}_i)/\emptyset)\} \in \mathcal{F}$. Particularly, f_i preserves all $\psi \in \Phi_i$, for every $i \in J_1$.

So by assumption

$$J_2 = J_0 \cap J_1 \cap \{i \in I : \langle A_i, E^{\Phi_i}_{\bar{c}_i}\rangle \cong^{g_i} \langle A_i, E^{\Phi_i}_{f_i(\bar{c}_i)}\rangle\} \in \mathcal{F}.$$

For every $i \in J_2$ we have $a_i \in a_i/E^{\Phi_i}_{\bar{c}_i} \smallsetminus D_i$, particularly, since A_i is finite,

$$\big|a_i/E^{\Phi_i}_{\bar{c}_i}\big| > \big|(a_i/E^{\Phi_i}_{\bar{c}_i}) \cap D_i\big|.$$

Since g_i is an isomorphism, there is an $e_i \in A_i$ with $g_i\big[a_i/E^{\Phi_i}_{\bar{c}_i}\big] = e_i/E^{\Phi_i}_{f_i(\bar{c}_i)}$. It follows, that

$$\big|e_i/E^{\Phi_i}_{f_i(\bar{c}_i)}\big| > \big|f_i\big[a_i/E^{\Phi_i}_{\bar{c}_i} \cap D_i\big]\big| = \big|e_i/E^{\Phi_i}_{f_i(\bar{c}_i)} \cap R_i\big|.$$

Hence, there exists $b_i' \in e_i/E^{\Phi_i}_{f_i(\bar{c}_i)} \setminus R_i$. Then clearly

$$f_i\big[\,\mathrm{tp}^{\mathcal{A}_i}_{\Phi_i}(a_i/\bar{c}_i)\big] = \mathrm{tp}^{\mathcal{A}_i}_{\Phi_i}(b_i'/f_i(\bar{c}_i)).$$

Since $i \in J_2$ was arbitrary, there exists $b \in A$ such that $\{i : b_i = b_i'\} = J_2 \in \mathcal{F}$, and obviously $b \notin \mathrm{ran}(f)$.

Now by assumption, φ is equivalent to a boolean combination of suitable members of Φ_i in a big set of indices. As we have seen, $f_i\big[\,\mathrm{tp}^{\mathcal{A}_i}_{\Phi_i}(a_i/\bar{c}_i)\big] = \mathrm{tp}^{\mathcal{A}_i}_{\Phi_i}(b_i/f_i(\bar{c}_i))$, consequently b satisfies φ outside R, as desired. ∎

Theorem 4.8 *Let $\mathcal{A} = \Pi_{i\in I}\mathcal{A}_i/\mathcal{F}$ be a stable structure such that $\langle \mathcal{A}, \mathcal{A}\rangle$ has* UEDP*, where $|A_i| < \aleph_0$ for all $i \in I$, and $\mathcal{F}$ is a countably incomplete, good ultrafilter. Then the following are equivalent:*

(i) $(\forall n \in \omega)$ $\{i \in I : \mathcal{A}_i$ has the (Φ_i, n)-equivalence property$\} \in \mathcal{F}$.

(ii) Any decomposable elementary mapping $g : A \to A$ extends to a decomposable automorphism of $\mathcal{A}$.

Proof. We start by showing (ii)⇒(i). Suppose, seeking a contradiction, there exists $n \in \omega$ such that, in a big set of indices $\mathrm{tp}^{\mathcal{A}_i}_{\Phi}(\bar{a}_i/\emptyset) = \mathrm{tp}^{\mathcal{A}_i}_{\Phi}(\bar{b}_i/\emptyset)$ but $\langle A_i, \bar{a}_i, E^{\Phi_i}_{\bar{a}_i}\rangle \not\cong \langle A_i, \bar{b}_i, E^{\Phi_i}_{\bar{b}_i}\rangle$ for some $\bar{a}_i, \bar{b}_i \in A_i$. Let $\bar{a} = \langle \bar{a}_i : i \in I\rangle/\mathcal{F}$ and $\bar{b} = \langle \bar{b}_i : i \in I\rangle/\mathcal{F}$. Then $\mathrm{tp}^{\mathcal{A}}(\bar{a}/\emptyset) = \mathrm{tp}^{\mathcal{A}}(\bar{b}/\emptyset)$, thus the function f mapping a onto b is elementary. By Theorem 3.13 (b) of [12], f is decomposable, and by (ii), f extends to a decomposable automorphism $f_{a,b} \in \mathrm{Aut}(\mathcal{A})$. But then $(f_{a,b})_i$ is an isomorphism between $\langle A_i, \bar{a}_i, E^{\Phi_i}_{\bar{a}_i}\rangle$ and $\langle A_i, \bar{b}_i, E^{\Phi_i}_{\bar{b}_i}\rangle$ in a big set of indices, which is a contradiction.

Now we turn to the implication (i)⇒(ii). By (i) and Lemma 4.7, $\langle \mathcal{A}, \mathcal{A}\rangle$ is relatively $\mathfrak{c}^+$-homogeneous, hence by Theorem 3.14, (ii) follows. ∎

Our main goal is to prove the following theorem.

Theorem 4.9 *If an ultraproduct of finite structures is relatively $\mathfrak{c}^+$-homogeneous, $\aleph_0$-stable and tight then it has the* UEDP*.*

We cut the proof of Theorem 4.9 into the following two lemmas.

Lemma 4.10 *Let $\mathcal{A} = \Pi_{i\in I}\mathcal{A}_i/\mathcal{F}$ and let $f = \langle f_i : i \in I\rangle/\mathcal{F}$ be a decomposable elementary mapping on $\mathcal{A}$ such that* $\mathrm{dom}(f)$ *is an elementary substructure of $\mathcal{A}$. Then there exists $J \subseteq I$ such that f_i is an elementary mapping for every $i \in J$.*

Proof. By Lemma 3.2 for every relation symbol R there exists $J(R)$ such that f_i preserves R for every $i \in J(R)$. Let $J = \cap_{R\in L}J(R)$. Then f_i preserves every atomic formula for every $i \in J$. Taking into consideration that $\mathrm{dom}(f)$ is an elementary substructure of $\mathcal{A}$, the statement follows from a straightforward induction on the complexity of formulas. ∎

Lemma 4.11 *Suppose f is a decomposable elementary mapping on $\mathcal{A} = \Pi_{i\in I}\mathcal{A}_i/\mathcal{F}$ where each $\mathcal{A}_i$ is finite. Suppose in addition, that $\mathcal{A}$ is $\aleph_0$-stable, tight and relatively $\mathfrak{c}^+$-homogeneous. Then f can be extended to a decomposable elementary mapping f' such that* $\mathrm{dom}(f')$ *is an elementary substructure of $\mathcal{A}$.*

Proof. Let $\langle \varphi_n : n \in \omega\rangle$ be an enumeration of all first order formulas and let $\Phi_n = \{\varphi_k : k < n\}$. By tightness, there exist $\mathrm{dom}(f) = D_0 \subseteq D_1 \subseteq D_2 \subseteq \cdots$ such that $D_{i+1} \setminus D_i$ is finite and D_n is a Φ_n-elementary substructure of $\mathcal{A}$. It is easy to see that $\mathrm{dom}(f) \cup \cup_{n\in\omega} D_n$ is an elementary substructure of $\mathcal{A}$ hence by $\aleph_0$-stability it contains a prime model D^+ over $\mathrm{dom}(f)$. Clearly, $D^+ \setminus \mathrm{dom}(f)$ is countable. In addition, there is a prime model R^+ over $\mathrm{ran}(f)$. Since f is elementary, it follows from uniqueness of prime models in $\aleph_0$-stable theories that f can be extended to an elementary mapping $f^+ : D^+ \to R^+$. Let $f_n = f^+|_{\mathrm{dom}(f)\cup D_n}$. Since each $D_n \setminus \mathrm{dom}(f)$ is finite, each f_n is decomposable and elementary. Hence, by Lemma 3.2 for every $n \in \omega$ there exists $J_n \in \mathcal{F}$ such that for every $i \in J_n$, the i-th coordinate of f_n is Φ_n-elementary in $\mathcal{A}_i$. Now the statement follows from a diagonalization argument. ∎

Now Theorem 4.9 can be obtained by combining the above lemmas.

Originally, Zilber in [15], and independently Cherlin-Lachlan-Harrington in [3] obtained their non-finite axiomatizability theorem for totally categorical theories by showing that models $\mathcal{A}$ of such theories are n-approximable for all $n \in \omega$; this means that for every $n \in \omega$, every finite substructure $\mathcal{A}_0$ of $\mathcal{A}$ can be extended to a finite $\mathcal{B} \subseteq \mathcal{A}$ such that n-tuples of $\mathcal{B}$ lying in the same orbit of $\mathrm{Aut}(\mathcal{A})$ also lie in the same orbit of $\mathrm{Aut}(\mathcal{B})$. The next theorem verifies, that this approach was necessary, at least in the following special case.

Theorem 4.12 *Suppose $\mathcal{A}$ is stable and it has a finite elimination base Φ. Then the following are equivalent.*

(1) $\mathcal{A}$ is pseudo-finite (that is, elementarily equivalent with an ultraproduct of its finite substructures) such that almost every factor has the (Φ, n)-equivalence property for all $n \in \omega$;

(2) $\mathcal{A}$ is n-approximable for all $n \in \omega$.

Proof. As supposed $\mathcal{A}$ has a finite elimination base, thus by Corollary 3.7 it follows that the pair $\langle \mathcal{A}, \mathcal{A} \rangle$ has UEDP. Assume (1), then $\mathcal{A}$ can be elementarily embedded in an ultraproduct $\mathcal{B}$ of finite structures in which, by Theorem 4.8 each decomposable elementary mapping extends to a decomposable automorphism. Now fix $n \in \omega$ and let $\mathcal{A}_0$ be a finite substructure of $\mathcal{A}$. If two n-tuples $\bar{a}, \bar{b} \in A_0$ have same type in $\mathcal{A}$ then they determine a decomposable elementary mapping $f : \bar{a} \mapsto \bar{b}$ which may be extended to a decomposable automorphism $f_{\bar{a},\bar{b}} \in \mathrm{Aut}(\mathcal{B})$. Let

$$R^{\mathcal{B}} = \{\langle \bar{a}, \bar{b}, x, f_{\bar{a},\bar{b}}(x)\rangle : \mathrm{tp}^{\mathcal{B}}(\bar{a}/\emptyset) = \mathrm{tp}^{\mathcal{B}}(\bar{b}/\emptyset),\ x \in B\} \subseteq {}^{2n+2}B.$$

Since $\mathcal{A}$ has a finite elimination base, there exists a formula τ_n, for which

$$\mathcal{B} \models \tau_n(\bar{a}, \bar{b}) \Longleftrightarrow \big(\mathrm{tp}^{\mathcal{B}}(\bar{a}/\emptyset) = \mathrm{tp}^{\mathcal{B}}(\bar{b}/\emptyset)\big).$$

The assertion that "$R^{\mathcal{B}}$ is an automorphism" (as a graph of a function on the last two coordinates) is first order expressible. Extend our language L with a single $2n+2$-ary relation symbol R, and interpret it as $R^{\mathcal{B}}$. Let

$$\varphi = (\forall \bar{a}, \bar{b})(\tau_n(\bar{a}, \bar{b}) \longrightarrow \text{"}R(\bar{a}, \bar{b}, \cdot, \cdot) \text{ is an automorphism"}).$$

Now $\mathcal{B} \models \varphi$, which means that if the types of $\bar{a}$ and $\bar{b}$ are the same, then there exists an automorphism of $\mathcal{B}$, which moves $\bar{a}$ to $\bar{b}$. It is enough to prove that $R^{\mathcal{B}}$ is decomposable. If $\mathrm{tp}^{\mathcal{B}}(\bar{a}/\emptyset) = \mathrm{tp}^{\mathcal{B}}(\bar{b}/\emptyset)$, then $J_{\bar{a},\bar{b}} = \{i \in I : \mathcal{A}_i \models \tau_n(\bar{a}_i, \bar{b}_i)\} \in \mathcal{F}$, and as above, the corresponding automorphism is decomposable, $f_{\bar{a},\bar{b}} = \langle f^i_{\bar{a},\bar{b}} : i \in I\rangle/\mathcal{F}$.

By Łoś lemma, we may assume that $f^i_{\bar{a},\bar{b}}$ is an automorphism of $\mathcal{A}_i$ for every $i \in J_{\bar{a},\bar{b}}$. Let

$$R_i^{\bar{a},\bar{b}} = \begin{cases} \{\langle \bar{a}_i, \bar{b}_i, x_i, f^i_{\bar{a},\bar{b}}(x_i)\rangle : x_i \in A_i\} & \text{if } i \in J_{\bar{a},\bar{b}} \\ \emptyset & \text{otherwise} \end{cases},$$

and let $R^{\mathcal{A}_i} = \bigcup\{R_i^{\bar{a},\bar{b}} : \mathrm{tp}^{\mathcal{A}}(\bar{a}/\emptyset) = \mathrm{tp}^{\mathcal{A}}(\bar{b}/\emptyset)\}$. Then $R^{\mathcal{A}} = \Pi_{i\in I} R^{\mathcal{A}_i}/\mathcal{F}$. It follows that there exists a big set of indices where φ is true, that is, all the elementary functions of size $\leq n$ had been extended, specially there exists a finite substructure $\mathcal{B}$ of $\mathcal{A}$ such that $A_0 \leq B$, and if n-tuples of $\mathcal{A}_0$ lying in the same orbit of $\mathrm{Aut}(\mathcal{A})$ then they also lie in the same orbit of $\mathrm{Aut}(\mathcal{B})$, consequently $\mathcal{A}$ is n-approximable. Since $n \in \omega$ was arbitrary, (2) follows.

Conversely, assume (2). Since $\mathrm{Th}(\mathcal{A})$ has a finite elimination base, it is $\aleph_0$-categorical, hence it can be axiomatized by $\forall\exists$-formulas. To show (1), it is enough to prove that every finite set of $\forall\exists$-formulas true in $\mathcal{A}$ is also true in a finite substructure of $\mathcal{A}$. So let Φ be a finite set of $\forall\exists$-formulas true in $\mathcal{A}$, let n be an upper bound for the lengths of the $\forall$-blocks of elements of Φ and let k be an upper bound for the $\exists$-blocks. Since $\mathcal{A}$ is $\aleph_0$-categorical, there exists a finite set $X \subseteq A$ such that every n-type over $\emptyset$ of $\mathcal{A}$ may be realized in X and there exists a finite $Y \subseteq A$ such that $X \subseteq Y$ and every k-type over X of $\mathcal{A}$ can be realized in Y. Then, by (2), there exists an n-homogeneous finite substructure $\mathcal{B}$ of $\mathcal{A}$ containing Y. We claim, that $\mathcal{B} \models \Phi$. To see this, let $\varphi = \forall\bar{x}\exists\bar{y}\psi$ where ψ is quantifier-free and let $\bar{a} \in B$ be arbitrary. Then there exists $\bar{a}' \in X$ such that $\mathrm{tp}^{\mathcal{A}}(\bar{a}/\emptyset) = \mathrm{tp}^{\mathcal{A}}(\bar{a}'/\emptyset)$. But then there is an automorphism f of $\mathcal{B}$ mapping $\bar{a}'$ onto $\bar{a}$. In addition, since $\mathcal{A} \models \varphi$, there exists $\bar{b} \in Y$ with $\mathcal{B} \models \psi(\bar{a}', \bar{b})$. Applying f, we obtain $\mathcal{B} \models \psi(\bar{a}, f(\bar{b}))$. Finally observe that since Φ is an elimination base, if $\mathcal{B}$ is a finite n-homogeneous substructure of $\mathcal{A}$ then $\mathcal{B}$ has the (Φ, n)-equivalence property. ∎

4.3 Future plans: the isomorphism problem

Finally in this section we sketch up a model-theoretical approach based on the constructions of subsections 3.3 and 4.1. We hope that using our techniques we will be able to prove some interesting theorems related to the complexity of the isomorphism problem, at least in a special class of finite structures. We must emphasize that there is no evidence for the success, but after all the approach may be interesting for its own.

The "isomorphism problem" is to decide algorithmically whether given to finite relational structures (for example, graphs) are isomorphic. This problem has been intensively studied because of it arises in a variety of practical applications such that circuit designs and molecular biology, and because its theoretical importance, as well. The complexity of the isomorphism problem is known to be in **NP**, but it resisted all the attempts to be classified as belonging to **P** or to be **NP**-complete.

To obtain results about finite structures first we construct infinite ultraproducts of finite structures and study these. This approach is similar in spirit to that of [16]. The basic idea is the following. If there would exist a property p_n of pairs of structures for all $n \in \omega$, such that the following stipulations hold:

(1) if $\mathcal{A}_n$ and $\mathcal{B}_n$ are finite structures having the property p_n then $\emptyset$ extends to a decomposable isomorphism between $\Pi\mathcal{A}_n/\mathcal{F}$ and $\Pi\mathcal{B}_n/\mathcal{F}$ (for suitable ultraproducts);

(2) if $\mathcal{A}_n \cong \mathcal{B}_n$ then they also have property p_n;

(3) for all $n \in \omega$, p_n can be checked in polynomial time,

then (similarly to the proof of Theorem 4.3) there would exists $N \in \omega$ such that two finite structures are isomorphic if and only if they have the property p_N. The first stipulation formulated above suggests using our method described in the previous sections. In order to do this, we have to find a property which guarantees all the conditions we need to extend partial isomorphisms to decomposable isomorphisms.

It seems that completing this approach needs a considerable amount of further investigations which we are planning to carry out later.

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