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On representation related problems in algebraic logic

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Contents

1 Introduction

1.1 Prelude

We start by surveying the subject and summing up the results presented in this work. In order to do so in an efficient way, we assume that the reader is familiar with mathematical logic. However, for completeness, in Section 1.2 below we will systematically recall all the notions and concepts we need in later sections.

Model theory and computability theory are traditional sub-disciplines of mathematical logic. In model theory one investigates the relationships between formalized statements (formulas) and stuctures (or models). One of the main aims is to describe all structures in which a given theory (i.e. set of first order formulas) is true. At that level of generality this ambitious aim seems to be untractible. Hence, instead of it, model theorists are trying to characterize those theories which have a structure theorem, that is, whose models can be described in a comprehensive way. Recenty, related investigations are very active. Along the results of Morley, Shelah, Hrushovski, Cherlin, Pillay and others, it turned out, that theories have a "structure theoretic" hierarchy of complexity: in some cases the possible models are relatively easy to describe, in some other cases this is much more difficult, while in some other cases such a complete "comprehensive" description of all models is impossible for theoretical reasons. This hierarchy is related to different degrees of stability, i.e. to the size of the Stone–spaces of the theory. Somewhat roughly, but more concretely, categorical theories (which are the simplest ones from structure theoretic point of view) have small (finite or countable) Stone–spaces, while the Stone–spaces of unstable theories are of large (uncountable) cardinality.

Computability theory studies a different kind of complexity. Here, two of the traditional aims are:

- to study the structure of the partially ordered sets of complexity classes, and
- to develop methods showing that particular subsets of natural numbers have a large computational complexity (i.e. they are not recursive, not recursively enumerable, etc.).

By classical results of Rosser, Gödel and Church, it turned out, that there are finite sets of formulas whose set of logical consequences is not recursive. In addition, the first order theories of certain natural structures are much more complicated in the computational sense. This also gives a natural hierarchy of complexity of structures: a structure is more complicated than another iff its theory (as a subset of finite sequences over a finite alphabet) has a larger computational complexity.

Beside these two traditional research directions, there is a third one called "recursive model theory". In recursive model theory one studies countable structures in which all the basic relations and operations are computable (i.e. recursive) subsets; such structures are called computable. Computable structures are interesting from purely theoretical point of view as well as practical purposes: countably infinite, but computable structures may be represented and manipulated by computers. By striking results of Ershov, Arslanov and others, there are many natural examples known for countable structures which are not isomorphic to any computable structures. For example, there are countable orderings, Boolean algebras and fields which do not have a computable isomorphic copy. Instead of dealing with particular structures, it would be interesting to obtain results in a higher level of generality: which theories have "complicated" models (i.e. models without a computable isomorphic copy).

In the present work we are trying to establish connections between structure theoretic complexity and computational complexity. Our main new results are as follows.

> In Theorem 3.1 we show that if a theory T is complicated in the model theoretic sense, that is, at least one of it's Stone spaces is uncountable, then T has a "complicated" countable model. More precisely, T has a countable model which is not isomorphic to any computable structure.
>
> In Theorem 3.11 we show, that there exists a theory T which is as simple as possible from the model theoretic sense (namely, T is $\aleph_0$-categorical, hence all of its Stone-spaces are finite), but at the same time the unique countable model of T does not have a computable isomorphic copy.

As we mentioned, Ershov and Arslanov established natural examples for countable structures which do not have computable isomorphic copies. It is also natural to ask what can be proven if isomorphism is replaced by elementary equivalence. By theorem 3.11 it follows that there exists a consistent theory T in a finite language which does not have computable models at all. A countable model of T is an example for a countable structure which is not elementarily equivalent with any computable structure.

Our methods are more general: we obtain similar results for models whose basic relations and operations belong to other, higher complexity classes. For more details we refer to [9].

In section 1.3 we give a short overview on the concept of complexity that we will use here. In section 2.1 we define the complexity of a structure and prove a technical lemma. As a corollary we gain a known result about the existence of a non-computable ordering on ω. We take a look on infinite languages as far as our subject concerned in section 2.2. In section 3.1 and 3.2 we prove our main results on the non-existence of computable presentation of certain theories. We sketch a further way of research on this subject in section 4.

1.2 A survey on the basic concepts of model theory

A *similarity type* or *signature* is a set of symbols with non-negative integers assigned to them. A *first order language* is a set of well-formed terms and a set of well-formed formulas built up from the usual logical symbols, namely the connectives, variables and miscellaneous symbols, of first order logic and non-logical symbols given by a similarity type. We say that the language is finite iff the corresponding similarity type is finite. The underlying sets of the absolutely free algebras are referred to as *word algebras*. The set of *terms* is the word algebra generated by the variables and constant symbols considered as the basis and by function symbols as the operations with the arity given by the similarity type. We deal with languages with a countable infinite set of variables which are usually referred to as v_i. The set of *atomic formulas* consists of all substitutions of terms into relation symbols according to their arity. Thus an atomic formula is built up from an n-ary relation symbol and an ordered n-tuple of terms. If equality is in the set of logical symbols then we allow atomic formulas constructed by equality as a relation symbol. The set of *formulas* is the word algebra generated by the atomic formulas and the connectives of the language.

We use the following set of connectives $\{\neg, \vee\} \cup \{\exists v_i : i \in \omega\}$. The other usual logical symbols are abbreviations of sequences built up by the use of these.

A *structure* or *model* with a given similarity type has the following two ingredients. A set, which is called the *underlying set* or *universe* of the structure, and an *interpretation* of the similarity type on the underlying set, in other words, a map which assigns functions and relations on the universe to each function and relation symbol in the type respectively according to the prescribed arity. We denote a structure with a fraktur letter and the universe of it with a capital latin.

A *valuation* is a map from the set of variables to the universe of a structure. The value of a term by means of a valuation is an element of the universe pointed out by the operations using the values of the valuation map instead of the variables and the term as a counting plan. In other words, by defining the valuation map on the constant symbols according to the interpretation the valuation map extends to the free algebra of terms in a unique way, which gives a value to each term. An atomic formula is true in a structure according to a valuation if the relation which is the interpretation of the relation symbol used in the atomic formula is held for the ordered values of terms of the formula. In symbols, $\mathfrak{A} \vDash \phi[e]$.

The truth of a formula according to a valuation is defined inductively. $\mathfrak{A} \vDash (\phi \vee \psi)[e]$ iff $\mathfrak{A} \vDash \phi[e]$ or $\mathfrak{A} \vDash \psi[e]$, $\mathfrak{A} \vDash \neg\phi[e]$ iff $\mathfrak{A} \nvDash \phi[e]$ and finally $\mathfrak{A} \vDash \exists v_i \phi[e]$ iff there exists an element in the universe of $\mathfrak{A}$, say a, and an evaluation e' such that $e(v_j) = e'(v_j)$ for $j \neq i$ and $e'(v_i) = a$ so as $\mathfrak{A} \vDash \phi[e']$.

A formula is true in a structure, symbolised as $\mathfrak{A} \vDash \phi$, iff $\mathfrak{A} \vDash \phi[e]$ for every possible valuation e.

An occurrence of a variable v_i is called *bounded* if it is in the scope of act of a connective $\exists v_i$, othewise it is called a *free occurrence*. A formula is said to be *closed* (or *sentence*) iff there are no variables with free occurance in it. It is easy to verify that the truth of a closed formula over a structure is independent of valuations. The set of formulas with free variables only from the set $\{v_1, \dots, v_n\}$ is denoted by F_n.

We write $\mathfrak{A} \vDash \phi[a_{i_1}, \dots, a_{i_n}]$ instead of $\mathfrak{A} \vDash \phi[e]$ if the only free variables of ϕ are $v_{i_1}, \dots, v_{i_n}$ and $e(v_{i_j}) = a_{i_j}$ $(j = 1 \dots n)$.

Let T be a set of formulas. We call it a *first order theory* if it is satisfiable or consistent. These two conditions are equivalent due to the completeness theorem of Gödel. By $\mathfrak{A} \vDash T$, we symbolize the fact that all the formulas of the theory T are true over $\mathfrak{A}$. In this case the structure $\mathfrak{A}$ is called a model of T. The *theory of a structure* is the set of all true formulas over it, $\mathrm{Th}(\mathfrak{A}) = \{\phi : \mathfrak{A} \vDash \phi\}$. The notation $T \vDash \Gamma$ is an abbreviation for $\forall \mathfrak{A} : \mathfrak{A} \vDash T \Rightarrow \mathfrak{A} \vDash \Gamma$.

Two structures of the same similarity type are *elementary equivalent*, $\mathfrak{A} \equiv \mathfrak{B}$, iff $\mathfrak{A} \vDash \phi \Leftrightarrow \mathfrak{B} \vDash \phi$ for every formula ϕ, that is $\mathrm{Th}(\mathfrak{A}) = \mathrm{Th}(\mathfrak{B})$ in short. Two structures of the same similarity type are *isomorphic*, $\mathfrak{A} \cong \mathfrak{B}$, iff there is a bijection

$g : \mathfrak{A} \to \mathfrak{B}$ such that $g(f^{\mathfrak{A}}(a_1, \ldots, a_n)) = f^{\mathfrak{B}}(g(a_1), \ldots, g(a_n))$ and $\langle a_1, \ldots, a_n \rangle \in R^{\mathfrak{A}}$ iff $\langle g(a_1), \ldots, g(a_n) \rangle \in R^{\mathfrak{B}}$ is held for all $a_i \in A$ and for each function and relation symbol in the signature. This means that $\mathfrak{A} \vDash \phi[e] \Leftrightarrow \mathfrak{B} \vDash \phi[g \circ e]$, for every formula ϕ and every valuation e. We say that $\mathfrak{B}$ is an *elementary extension* of $\mathfrak{A}$, (or that $\mathfrak{A}$ is an *elementary substructure* of $\mathfrak{B}$), $\mathfrak{A} \prec \mathfrak{B}$, iff $\mathfrak{A}$ is a subalgebra in $\mathfrak{B}$ furthermore for every valuation e whose range is in the universe of $\mathfrak{A}$ (i.e. for every valuation e over $\mathfrak{A}$) and for for every formula ϕ, we have $\mathfrak{A} \vDash \phi[e] \Leftrightarrow \mathfrak{B} \vDash \phi[e]$. Let us notice the following consequences of these definitions: if $\mathfrak{A} \cong \mathfrak{B}$ then $\mathfrak{A} \equiv \mathfrak{B}$, and if $\mathfrak{A} \prec \mathfrak{B}$ then $\mathfrak{A} \equiv \mathfrak{B}$.

The Löwenheim–Skolem theorem states that if a theory has a model of an infinite cardinality then it has a model of arbitrary infinite cardinality either. Moreover, the larger structure can be chosen to be an elementary extension of the smaller.

A theory is *κ-categorical* iff, up to isomorphism, it has a unique model of cardinality κ. We say that a structure $\mathfrak{A}$ is κ-categorical iff its theory $\mathrm{Th}(\mathfrak{A})$ is κ-categorical.

A set of formulas with an upper bound of free variables, say Γ, is *finitely satisfiable* over $\mathfrak{A}$ iff $\forall \Gamma_0 \in [\Gamma]^{<\omega}\ \exists \vec{a} \in A\, \forall \phi \in \Gamma_0 : \mathfrak{A} \vDash \phi[\vec{a}]$. The word – type – is usually used in a second meaning too, from now on we will consider this second meaning.

Definition 1.1 *An n-*type *of a structure $\mathfrak{A}$ is a maximal set of formulas from F_n that is finitely satisfiable over $\mathfrak{A}$.*

The set of n-types of $\mathfrak{A}$ is denoted by $S_n(\mathfrak{A})$.

Definition 1.2 *A type $p \in S_n(\mathfrak{A})$ is* realised *by an n-tuple of elements $\vec{a} \in A$ iff $\forall \phi \in p : \mathfrak{A} \vDash \phi[\vec{a}]$.*

Definition 1.3 *The* type of tuple *$\vec{a}$ over $\mathfrak{A}$ is $\mathrm{tp}_{\mathfrak{A}}(\vec{a}) = \{\phi \in F_n : \mathfrak{A} \vDash \phi[\vec{a}]\}$. This is a type of the structure $\mathfrak{A}$ indeed.*

The type of an element collects all the information that is expressible about it via the first order language of a structure.

Definition 1.4 *An* n-type *of a theory, say T, is a set of formulas, say p, with the following properties.*

- *Every formula is from F_n,*

- *every finite subset of it is realizable that is:* $\forall p_0 \in [p]^{<\omega}\ \exists \mathfrak{A}_{p_0} : \mathfrak{A}_{p_0} \models T,\ \exists \vec{a} \in A_{p_0}\, \forall \phi \in p_0 : \mathfrak{A} \models \phi[\vec{a}]$.

- p *is maximal concerning these properties.*

The set of n-types over T is denoted by $S_n(T)$.

Lemma 1.5 *For a type $p \in S_n(T)$ there is a structure $\mathfrak{A}$ such that $\mathfrak{A} \models T$ and p is finitely satisfiable over $\mathfrak{A}$. Hence $S_n(\mathrm{Th}(\mathfrak{A})) = S_n(\mathfrak{A})$. Moreover there is a model of T over which p is realizable.*

Proof. It is enough to prove the last statement. Let p be as in the statement. Add n pieces of new constant symbols to the signature of T, $t' = t \cup \{\vec{c}\}$. Consider the theory $T' = T \cup \{\bigwedge p_0(\vec{c}) : p_0 \in [p]^{<\omega}\}$. By the definition of $S_n(T)$ every finite subset of this theory has a model so owing to the compactness theorem there is a structure $\mathfrak{A} \models T'$. The interpretations of $\vec{c} \in A$ realise p over $\mathfrak{A}$. ■

Definition 1.6 *A topological space X is a* Stone–space *iff it is compact, Hausdorff and has a basis consisting of clopen (i.e. closed and open at the same time) sets.*

Let B be a Boolean algebra. B^* denotes the set of ultrafilters of B. A topology can be defined on B^* by taking a subbasis $\{N_b = \{\mathcal{U} \in \mathcal{B}^* : b \in \mathcal{U}\}\}_{b \in B}$. In this way we associate a topological space to a Boolean algebra. For a topological space X the subsets $[X]$ constitute a Boolean algebra. X^* denotes the subalgebra of the clopen sets.

Theorem 1.7 (Stone) *The space B^* is a Stone–space and the subbasis defined above is a basis. $B \cong B^{**}$ (in fact, the mapping $b \mapsto N_b$ is an isomorphism) and $X \cong X^{**}$ iff X is a Stone–space.*

Lemma 1.8 *The n-types form a Stone–space with the basis:*
$\{N_\phi = \{p \in S_n(T) : \phi \in p\}\}_{\phi \in F_n}$

For more details we refer e.g. to theorem 6.2.3. of Hodges [5].

Definition 1.9 *Introduce an equivalence relation on F_n relative to a theory T as follows: for $\phi, \psi \in F_n$ stipulate $\phi \equiv \psi$ iff $T \models \forall v_1 \dots \forall v_n (\phi \leftrightarrow \psi)$. The equivalence classes form a Boolean algebra $B_n(T)$. The operations are given by*

$$
\begin{aligned}
&(\phi/\!\equiv) \wedge (\psi/\!\equiv) \ \doteq \ (\phi \wedge \psi)/\!\equiv \\
&\neg(\phi/\!\equiv) \ \doteq \ (\neg\phi)/\!\equiv \\
&1 \doteq (v_1 = v_1)/\!\equiv \\
&0 \doteq (v_1 \neq v_1)/\!\equiv
\end{aligned}
$$

These are the so called Lindenbaum–Tarski algebras.

Lemma 1.10 *The Stone–space of types $S_n(T)$ can be identified with the Stone–dual of the Lindenbaum–Tarski algebra, $B_n^*(T)$.*

Proof. There is a natural pairing between the n-types over T and ultrafilters on $B_n(T)$. ■

1.3 Complexity

There are many different ways of approaching the concept of computability. We can raise the question as a membership problem (i.e. whether a given word is a member of a given language) or as a problem of computing a function with a given input. A possible way to define a computational method, for instance Turing-machine, RAM-machine or finite automaton. We can define a complexity class by choosing a method and take restrictions on certain sort of resources in terms of the length of the input (see Papadimitriou [7, Ch. 7.]). On the other hand we can point out directly a certain family of functions or languages as an etalon complexity. For instance, it is a common way to define the family of recurisve functions as a set of complete maps from finite power of ω to ω containing addition, multiplication, projection and closed under substitution and the so called μ-operation (see Csirmaz[4, Ch.10.]). By dropping the completeness we obtain the family of partial recursive functions. We can define new complexity classes by enabling the use of an answer to another problem in the course of the computing as a simple step. In the other case we can put a new function to our clone. In other words we can use another problem as an oracle for our purpose. But we have just distinguished problems up to this point. To be able to speak about complexity classes we need to compare different questions. The suitable tools for this are the different kind of reductions (see [7, Ch.8.]). Within recursive problems the polinomial reductions are very usefull. For our purpose, we do not need such a fine resolution so we distinguish classes only up to recursivity. (We refer to Simpson [1, Ch. C.4.])

Definition 1.11 *Consider two relations on ω, say $R_1 \subset \omega^n$, and $R_2 \subset \omega^m$. We say that R_1 has a* reduction *to R_2, in symbols $R_1 \prec R_2$, iff there exsists a recursive algorithm or map, say M, such that $w \in R_1 \Leftrightarrow M(w) \in R_2$. We say that R_1 is* recursive relative *to R_2.*

This definition means that R_2 is at least as hard problem as R_1. If we have a solution that solves the membership problem for R_2 then it may also be utilized to solve the membership problem for R_1 up to recursivity as well. It is easy to see that $\prec$ is a reflexive and transitive relation. Consequently – as it is well known – the relation $\prec$ determines an equivalence relation $\sim$ via the stipulation: $R_1 \sim R_2$ iff $R_1 \prec R_2$ and $R_2 \prec R_1$.

Definition 1.12 *By a* complexity class *we mean an equivalence class of relations of arbitrary arity on ω, under the equivalence relation $\sim$.*

These kind of complexity classes are called Turing degrees or degrees of unsolvability (see [1]). We denote them by calligraphic letters like $\mathcal{D}$. (If we had used a reduction with more restrictions we would have obtained a more detailed classification such as Karp–classes, for example.)

Lemma 1.13 *(1) A complexity class is always countable.*
(2) The set of all complexity classes has cardinality $2^{\aleph_0}$.
(3) Comlexity classes are partially ordered by the induced ordering of $\prec$.
(4) A set of complexity classes has an upper bound (under $\prec$) iff it is countable.

Proof. For more details we refer to [1]. ∎

2 Preliminaries

2.1 Presentation of a structure

Let us consider structures of the form $\mathfrak{B} = \langle \omega, R_i^{\mathfrak{B}} \rangle_{i \in L}$ where L is a finite first order language. Such a structure is defined to be *computable* if $R_i^{\mathfrak{B}}$ is a recursive subset of a direct power of ω for all $i \in L$. It is equivalent to say that the sets defined by atomic formulas of L in $\mathfrak{B}$ are recursive.

We obtain other notion by changing some words in the above definition. Instead of atomic formulas we can consider a specified set of formulas, say Φ. A structure $\mathfrak{B}$ is Φ-*decidable* if all of the sets defined by formulas from Φ are recursive. We simply say *decidable* if it is true for all formulas of L. On the other hand we can change the scope of permitted sets defined by the formulas allowing any other complexity class instead of recursive relations. For instance, recursively enumerable or arithmetical sets may also be considered. Recall that a subset R of a direct power of ω is defined to be *arithmetical* iff there is a first order formula in the language of arithmetic defining R. It is well known that recursive and recursively enumerable sets are arithmetical, but arithmetical sets may be much more complicated than recursively enumerable sets.

Definition 2.1 *Let L be a finite first order language and $\mathcal{D}$ be a complexity class. We say, that an L-structure $\mathfrak{A}$ has a $\mathcal{D}$-*computable presentation *if there is a structure $\mathfrak{B} = \langle \omega, R_i^{\mathfrak{B}} \rangle_{i \in L}$ such that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic and the relations $R_i^{\mathfrak{B}}$ are in $\mathcal{D}$ for every $i \in L$.*

We can similarly define $(\mathcal{D}, \Phi)$-decidable presentations of a structure. It is clear that such a presentation is $\mathcal{D}$-computable. Since we prove negative results in this paper we use this strongest form, and for short we omit the word "computable". In this situation we also say, that $\mathfrak{B}$ is $\mathcal{D}$-*presented.*

Lemma 2.2 *Let $\mathcal{D}$ be a complexity class. Suppose $\mathcal{H}$ is an uncountable family of pairwise non-isomorphic countable structures. Then there exists $\mathfrak{A} \in \mathcal{H}$ such that $\mathfrak{A}$ does not have a $\mathcal{D}$-presentation.*

Proof. Let $\mathcal{H}_0 = \{\mathfrak{A} \in \mathcal{H} : \mathfrak{A}$ has a $\mathcal{D}$-presentation $\}$ and for every $\mathfrak{A} \in \mathcal{H}_0$ let $\mathcal{D}(\mathfrak{A})$ be a $\mathcal{D}$-presented structure such that $f_{\mathfrak{A}} : A \to \mathcal{D}(A)$ is an isomorphism between $\mathfrak{A}$ and $\mathcal{D}(\mathfrak{A})$. Observe, that for every distinct $\mathfrak{A}, \mathfrak{B}$ we have $\mathcal{D}(\mathfrak{A}) \neq \mathcal{D}(\mathfrak{B})$, otherwise $(f_{\mathfrak{B}})^{-1} \circ f_{\mathfrak{A}}$ would be an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$. Hence, the function $\alpha : \mathfrak{A} \mapsto \mathcal{D}(\mathfrak{A})$ is injective. In addition, there are only countably many $\mathcal{D}$-presented structures, so the range of α is countable. It follows, that $\mathcal{H}_0$ (which is the domain of α) is also countable. Consequently, there exists $\mathfrak{A} \in \mathcal{H} \setminus \mathcal{H}_0$; this $\mathfrak{A}$ does not have a $\mathcal{D}$-presentation. ■

Corollary 2.3 *(1) There is an ordering on ω which does not have a computable presentation.*
(2) There is a well-ordering on ω which does not have an arithmetical presentation.

Proof. Since (2) implies (1), it is enough to show (2). Let $\mathcal{H}$ be the set of (isomorphism types of) countable well-orderings and let $\mathcal{D}$ be the set of arithmetical relations on ω. Since $|\mathcal{H}| = \aleph_1$, the statement follows from Lemma 2.2. ■

2.2 Remarks on infinite languages

Under infinite language we mean that there are infinitely many relation and function symbols in the corresponding similarity type. In this case two different concepts of representation emerge. Consider the structure $\mathfrak{B} = \langle \omega, R_i^{\mathfrak{B}} \rangle_{i \in \omega}$. We say that this is *$\mathcal{D}$-computable* iff the following membership problem is in $\mathcal{D}$: We have the input of the form: $\langle i, \{a_1, \ldots, a_{n_i}\} \rangle$. The question is, whether the n_i-tuple of elements of B is in the relation $R_i^{\mathfrak{B}}$ or not (n_i is the arity of R_i). By this way of extending the definition of $\mathcal{D}$-computablility for infinite languages, with a slight modification of the conditions, the statement of Lemma 2.2 still holds. We say that the structure $\mathfrak{B}$ is *weak-$\mathcal{D}$-computable* iff the membership problem for a tuple and for an arbitrary fixed relation is in $\mathcal{D}$.

It is easy to construct an uncountable set of pairwise non-isomorphic structures on ω using an infinite language. The family is parametrized by the subsets of ω. For a fixed $S \subset \omega$ the structure $\mathfrak{M}_S$ is the following:

$$\mathfrak{M}_S = \langle \omega,\ R_i \rangle_{i \in \omega} \quad \text{where } R_i = \begin{cases} \{1\} & \text{if } \ i \in S \\ \emptyset & \text{if } \ i \notin S \end{cases}$$

These structures are obviously weak-$\mathcal{D}$-computable for any complexity class $\mathcal{D}$. On the other hand, Lemma 2.2 implies that there is a non-$\mathcal{D}$-computable structure in the above defined familiy. In addition, as it is easy to see, each $\mathfrak{M}_S$ is $\aleph_0$-categorical. So there exist continuum many, pairwise non-isomorphic $\aleph_0$-categorical structures. A natural adaptation of Lemma 2.2 implies, that at least one of them is not isomorphic to any $\mathcal{D}$-presented structure (in strong sense). On one hand this result is best possible: $\aleph_0$-categorical structures are the simpliest ones from structure theoretic point of view, however, the computational complexity of such a simple structure may be arbitrary large.

On the other hand, the above result can be considered as a kind of cheating, since we permitted to use infinitely many relational symbols in the similarity type, constructing continuum many pairwise non-isomorphic structures had become rather easy. Therefore the restriction to finite languages put sublety into the subject.

3 Results on finite languages

3.1 Case of large Stone–space

As we mentioned in the introduction, a theory is used to consider "complicated" from the model theoretic point of view iff at least one of its Stone spaces is of uncountable cardinality. Among other things, in the present subsection we show that if a theory T has at least one uncountable Stone–space then T has a countable model which does not have a computable presentation.

Theorem 3.1 *Let $\mathcal{D}$ be a complexity class and let T be a first order theory in a finite language such that there is an $n \in \omega$ with $|S_n(T)| \geq \aleph_1$. Then T has a countable model which is not isomorphic to any $\mathcal{D}$-presented structure.*

Proof. We proove the statement by transfinite recursion. Let us suppose that we have a countable set of countable structures $\mathfrak{A}_\alpha\,(\alpha < \lambda < \aleph_1)$ such that they are pairwise nonisomorphic and are models of T. Each structure can realize only countably many types from $S_n(T)$, since a single element realizes a unique type. Hence, these countably many structres realize countably many types alltogether. Let us choose a type $p \in S_n(T)$ which has not been realized yet. By Lemma 1.5 we obtain that there is a structure $\mathfrak{B}$ which realizes p and is a model of T. By using the downward Löwenheim–Skolem theorem we obtain that there is a countable structure $\mathfrak{B}'$ such that $\mathfrak{B}' \prec \mathfrak{B}$ and contains a prescribed countable subset of B. We require only to contain an element which realizes p. Since $\mathfrak{B}'$ is an elementary part of $\mathfrak{B}$ containing this element it is still realize p. Thus $\mathfrak{B}'$ has the property: $\mathfrak{B}' \models T$, $\mathfrak{B}' \ncong \mathfrak{A}_\alpha\,(\alpha < \lambda)$. In this way we construct $\aleph_1$ many pairwise nonisomorphic models of T. From Lemma 2.2 the result follows. ■

Remark 3.2 *It is well-known (see, e.g. theorem 6.3.4. of Hodges [5]) that $|S_n(T)| \geq \aleph_1$ implies $|S_n(T)| = 2^{\aleph_0}$.*

3.2 Case of small Stone–space

As we already mentioned, from structure theoretic point of view, a theory T is as simple as possible, iff it is $\aleph_0$-categorical, that is iff T has a unique countable model. The well known results of Svenonius, Ryll–Nardzewski and others show a connection between categoricity and the size of the Stone–spaces.

Theorem 3.3 *For a theory T the following two conditions are equivalent:*
(1) T is $\aleph_0$-categorical,
(2) $|S_n(T)| < \omega$ $(\forall n \in \omega)$.

Proof. The proof can be found in many work from the listed references. ■

In this subsection we show, that there exists an $\aleph_0$-categorical theory in a finite language whose unique countable model does not have a computable isomorphic copy (that is, altough T is simple from structure theoretic point of view, its unique countable model is still complicated from computatinal theoretic point of view).

In addition, we also show that there is a countable structure which is not elementarily equivalent to any computable (or any $\mathcal{D}$-presented) structures, where $\mathcal{D}$ is a given complexity class.

To prove these results first we need to recall and establish some connections between $\aleph_0$-categorical structures and certain permutation groups on ω.

Definition 3.4 *A permutation group $\mathcal{G} = \langle G, \circ, ^{-1}, 1\rangle$ is defined to be* closed *iff for every permutation $f \in {}^{\omega}\omega$ the following holds:*
$(\star)$ if for every finite $s \subseteq \omega$ there is $g_s \in G$ such that $f|_s = g_s|_s$, then $f \in G$.

The situation in $(\star)$ is denoted by $g_s \to f$ for a series of subsets $s \in [\omega]^{<\omega}$ oredered by containing.

Equip ω with the discrete topology. Then $\mathcal{G}$ is a closed permutation group iff it is a closed subset of ${}^{\omega}\omega$ in the corresponding product topology. For more details we refer to [5].

Clearly, the automorphism group of a first order structure is closed: if $g_n \in \mathrm{Aut}(\mathfrak{A})$ (for each n) and $g_n \to f$ then for every tuple $\vec{a}$ there exists an $n \in \omega$ such that $g_n(\vec{a}) = f(\vec{a})$. Since g_n is an automorphism of $\mathfrak{A}$, $\vec{a}$ satisfies a relation iff $f(\vec{a})$ does. Thus $f \in \mathrm{Aut}(\mathfrak{A})$.

Definition 3.5 *A permutation group $\mathcal{G}$ on X is said to be* oligomorphic *iff for every $k \in \omega$ the group acts on the k-tuples in a way that the number of orbits is finite.*

If $\mathcal{G}$ is an oligomorphic permutation group on X and $n \in \omega$ then $o_n^{\mathcal{G}}$ denotes the number of orbits of $\mathcal{G}$ on the set of n-tuples of X.

Lemma 3.6 *If $\mathcal{G}$ is a closed oligomorphic permutation group on ω then there exists an $\aleph_0$-categorical structure $\mathfrak{A}$ on ω with* $\mathrm{Aut}(\mathfrak{A}) = \mathcal{G}$.

Altough this theorem is well known, for completeness we include here a proof.

Proof. Firstly, we construct the so called canonical model for $\mathcal{G}$ in the form of $\mathfrak{A} = \langle \omega, R_i \rangle_{i \in \omega}$. For every $n \in \omega$, we introduce a relation symbol R_n with arity equals to $(n \cdot o_n^{\mathcal{G}})$. Let the interpretation as follows: $R_n(\vec{a_1}, \ldots, \vec{a_{o_n}}) \Leftrightarrow \vec{a_i}$ is in the ith orbit of n-tuples for all i. Consider an arbitrary n-tuple $\vec{b}$. Then there is a sequence of elements a_i^j such that $b_j = a_i^j$ for a unique i so that $R_n(\vec{a})$ is true in $\mathfrak{A}$. So we can write the elements of $\vec{b}$ into certain slots of R_n. The inclusion $\mathcal{G} \subseteq \mathrm{Aut}(\mathfrak{A})$ is trivial by our construction. Let $f \in \mathrm{Aut}(\mathfrak{A})$. Obviously, by changing the values of these slots to the elements of $f(\vec{a})$ then the relation remains true. So $\vec{a}$ and $f(\vec{a})$ are in the same $\mathcal{G}$ orbit too. So there is a $g_{\vec{a}} \in \mathcal{G}$ such that $g_{\vec{a}}(\vec{a}) = f(\vec{a})$. With $n = \{0, \ldots, (n-1)\} \subset \omega$ the above way defined elements of $\mathcal{G}$ converge, $g_n \to f$. Since $\mathcal{G}$ is closed, $f \in \mathcal{G}$. We obtained the fact that $\mathcal{G} = \mathrm{Aut}(\mathfrak{A})$.

Secondly, we have to show that this structure $\mathfrak{A}$ is $\aleph_0$-categorical. We have $\mathfrak{A}$ whose automorphism group is oligomorphic by assumption. Then for every $n \in \omega$ there are finitely many orbits on n-tuples. If two n-tuples, $\vec{a}$ and $\vec{b}$, are on the same orbit then $\mathrm{tp}_{\mathfrak{A}}(\vec{a}) = \mathrm{tp}_{\mathfrak{B}}(\vec{b})$. Therefore, the structure $\mathfrak{A}$ realizes only finitely many types, namely: $p_1, \ldots, p_{o_n} \in S_n(\mathfrak{A})$. There is a formula for every $i \neq j$ such that $\phi_{ij} \in p_i \backslash p_j$. Hence the formula, $\phi_i = \bigwedge_{i \neq j} \phi_{ij}$ is only in p_i. Thus $\mathfrak{A} \vDash \forall \vec{x} \bigvee_{i=1}^{o_n} \phi_i(\vec{x})$, and for every $\psi \in p_i$: $\mathfrak{A} \vDash \forall \vec{x}(\phi_i(\vec{x}) \to \psi(\vec{x}))$. So for an arbitrary formula $\theta \in F_n$ it is true that $\mathfrak{A} \vDash \forall \vec{x} \forall \vec{y}\, (\phi_i(\vec{x}) \wedge \phi_i(\vec{y}) \longrightarrow (\theta(\vec{x}) \leftrightarrow \theta(\vec{y})))$. Which implies that there are no other n-types than $p_1, \ldots, p_{o_n}$ in $S_n(\mathfrak{A})$. This means that all the Stone–spaces are finite. Theorem 3.3 implies that $\mathfrak{A}$ is $\aleph_0$-categorical, which completes the proof. ■

Lemma 3.7 *For any sequence $\{a_n \in \omega\}_{n \in \omega}$ there is an oligomorphic group $\mathcal{G}$ for which $o_n^{\mathcal{G}} > a_n$ for all $n \in \omega$.*

Proof. The proof can be found in Cameron [2]. ■

We say that two oligomorphic permutation groups $\mathcal{F}, \mathcal{G}$ have same orbit structures iff for every $n \in \omega$ we have $o_n^{\mathcal{F}} = o_n^{\mathcal{G}}$.

Lemma 3.8 *If $\mathcal{G}$ is an oligomorphic permutation group on an infinite set then there exists an oligomorphic permutation group on ω with the same orbit structure.*

Proof. First of all we encode the group and the action in a structure. The universe will contain the elements of $\mathcal{G}$ and the elements of the set the group is act on. The similarity type consists of the group operations, $\{\circ, ^{-1}, 1\}$, an operation symbol for the action, $\{\cdot\}$, and two unary relation symbols, $\{g, s\}$ group and set respectively, in order to distinguish to two sort of elements. In a theory we collect the axioms of a group and those of the action. The sentence:
$\exists x_1, \ldots, x_k \forall y (s(x_1) \wedge \cdots \wedge s(x_k) \wedge g(y) \rightarrow (g \cdot x_1 \neq x_2) \wedge \cdots \wedge (g \cdot x_{k-1} \neq x_k))$ means that there are at least k orbits of the group action. Likewise we can express that there are exactly k orbits via first order formulas. In a similar way we can express the same statement about the orbits of n-tuples.

Let us construct this structure from $\mathcal{G}$. By using the downward Lövenheim–Skolem theorem we gain a countable model. The orbit structure is fixed by the theory. After the enumeration of the universe the statement of the Lemma follows. ■

Lemma 3.9 *Let $\mathcal{G}$ be an oligomorphic permutation group on ω and let $\bar{\mathcal{G}}$ be its closure (in the topological sense). Then*

(1) $\bar{\mathcal{G}}$ is an oligomorphic permutation group;

(2) The orbit structures of $\mathcal{G}$ and $\bar{\mathcal{G}}$ are the same.

Proof. (1) is easy; (2) is straightforward. ■

Theorem 3.10 *There is a finite similarity type in which there are $2^{\aleph_0}$ many pairwise non-isomorphic $\aleph_0$-categorical structures on ω.*

Proof. First we show by transfinite recursion that there are $\aleph_1$ many oligomorphic permutation groups with pairwise different orbit structures. To do so, we apply transfinite recursion. Let us suppose we have $\{\mathcal{G}_\alpha : \alpha < \beta\}$ where β is a countable ordinal, and the $\mathcal{G}_\alpha$'s are oligomorphic permutation groups with pairwise different orbit structures. So we have β many sequences o^α_n which describes pairwise different orbit structures. For simplicity we ommit the letter $\mathcal{G}$ from the notation. Let $\iota : \omega \to \alpha$ be a surjection. Consider the sequence $\{1 + o^{\iota(n)}_n\}_{n\in\omega}$ as an input for Lemma 3.7. This lemma produce a new oligomorphic group with at least $1 + o^{\iota(n)}_n$ many orbits on n-tuples.

Next, we show that there is a set $\mathcal{H}'$ consisting of $2^{\aleph_0}$ many oligomorphic permutation groups with pairwise different orbit structures. We build a tree of orbit structure sequences. We already have $\aleph_1$ many groups and sequences o^α_n either. For arbitrary k there are countably many truncated sequences with length k. There is a truncated sequence then which is the starting sequence of $\aleph_1$ many orbit structure sequences. To continue our truncated sequence with one step we have $\aleph_0$ many possiblities. But we still have $\aleph_1$ many sequences. A one step continuation of the truncated sequence which is still the starting sequence of $\aleph_1$ many orbit structure sequences is called large branching, and which is only a starting sequence of countably many is called small branching. So there is at least one large branching at this step. Let us consider all the ω steps. Suppose there is only finitely many points where there are at least two large branching continuations. Then there are only finitely many complete sequences which pass a large branching. Then apart from these the remaining small branchings yield only countably many complete sequences. This is contradictory to the supposition that we have $\aleph_1$ many different orbit structure sequences alltogether. Thus there are $\aleph_0$ choice points where there are at least two large branchings. Now we obtain by these points (or large branchings) a tree whose height is ω which yields $2^{\aleph_0}$ different orbit structure sequences. Let S be the set of these sequences (i.e. the complete branches of the above tree). So we have $|S| = 2^{\aleph_0}$. By construction, for each sequence $\{s_i\}_{i\in\omega} \in S$ and for arbitrary k we have an oligomorphic permutation group $\mathcal{G}^s_k$ such that $o^{\mathcal{G}^s_k}_n = s_n$ iff $n < k$. Consider the ultraproduct $\mathcal{G}_s = \prod_{k\in\omega} \mathcal{G}^s_k/\mathcal{U}$. Since there are cofinitely many groups with exactly s_n many orbits on n-tuples, the well known Łos lemma implies that $o^{\mathcal{G}_s}_n = s_n$. Hence we obtain the set $\mathcal{H}' = \{\mathcal{G}_s, s \in S\}$. where the $\mathcal{G}_s$'s are oligomorphic permutation groups with pairwise different orbit structures.

By Lemma 3.8, for every $\mathcal{G}_s \in \mathcal{H}'$ there exists an oligomorphic permutation group $\mathcal{F}_s$ on ω with the same orbit structure, by Lemma 3.9 we may assume $\mathcal{F}_s$ is closed as well. Finally, by Lemma 3.6 there is a countable structure $\mathfrak{A}_s$ such that $\mathrm{Aut}(\mathfrak{A}_s) = \mathcal{F}_s$. Then $\mathcal{H}'' = \{\mathfrak{A}_s : s \in S\}$ is a set of pairwise non-isomorphic, countable, $\aleph_0$-

categorical structures because their automorphism groups are oligomorphic and have pairwise different orbit structures.

Suppose t is a similarity type containing a distinguished unary relation symbol P and let $\mathfrak{B}$ be a structure of similarity type t. We say, that a structure $\mathfrak{A}$ is an induced substructure of $\mathfrak{B}$ by P iff the universe of $\mathfrak{A}$ is $P^{\mathfrak{B}}$ and the definable relations of $\mathfrak{A}$ coincide with the definable relations of $\mathfrak{B}$ restricted to P. This determines $\mathfrak{A}$ up to definitional equivalence, only.

By theorem 7.4.8 of Hodges [5] there is a finite similarity type t containing a distinguished unary relation symbol P such that every $\aleph_0$-categorical structure $\mathfrak{A}$ (possibly having an infinite language) is an induced substructure of an $\aleph_0$-categorical structure $\mathfrak{A}_t$ by P, where the similariy type of $\mathfrak{A}_t$ is t. Let $\mathfrak{A}, \mathfrak{B} \in \mathcal{H}''$ be arbitrary, but different. Then they have different orbit structures, hence $\mathfrak{A}_t$ cannot be isomorphic to $\mathfrak{B}_t$. In other words, the function $\mathfrak{A} \mapsto \mathfrak{A}_t$ is injective on $\mathcal{H}''$. Let $\mathcal{H} = \{\mathfrak{A}_t : \mathfrak{A} \in \mathcal{H}''\}$; clearly $\mathcal{H}$ contains $2^{\aleph_0}$ many pairwise non-isomorphic $\aleph_0$-categorical structures of similarity type t, as desired.

■

We note that the proof of Theorem 3.11 would remain correct, if in Theorem 3.10 we would establish the existence of $\aleph_1$ many pairwise non-isomorphic $\aleph_0$-categorical strucures only. Actually, in the first paragraph of Theorem 3.10 we are doing exactly this. The purpose of the remaining part of the proof of Theorem 3.10 is to establish the existence of continuum many $\aleph_0$-categorical pairwise non-isomorphic structures. This stronger result may be useful for further model theoretical investigations.

Theorem 3.11 *Let $\mathcal{D}$ be a complexity class.*

(1) There exists an $\aleph_0$-categorical structure of finite similarity type which is not isomorphic to a $\mathcal{D}$-presented structure.

(2) There is a first order theory again, in a finite similarity type which does not have a $\mathcal{D}$-presented model.

Proof. By Theorem 3.10 there exists a set $\mathcal{H}$ of pairwise non isomorphic, countable $\aleph_0$-categorical structures such that $|\mathcal{H}| = 2^{\aleph_0} \geq \aleph_1$. Now (1) follows from Lemma 2.2.

To show (2), let $\mathfrak{A}$ be a structure satisfying (1) and let $T = \mathrm{Th}(\mathfrak{A})$. Since $\mathfrak{A}$ is $\aleph_0$-categorical, every countable model of T is isomorphic to $\mathfrak{A}$, hence such a model cannot be $\mathcal{D}$-presented. ■

4 Concluding Remarks

We conclude this work by mentioning a further research direction. After answering the question for theories with not smaller than $\aleph_1$ and with finite Stone–spaces, the case of theories with countably infinite Stone–spaces is still open. From structure theoretic point of view this case has "intermediate complexity". In general, Lemma 2.2 seems unapplicable for them, and at the same time, there do not exist structure theorems (like Theorem 3.3 and Lemma 3.6) for theories with countable Stone–spaces.

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