Flows in networks - a probabilistic approach

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“Ad maiorem Dei gloriam“

In memoriam
P. Gustav Teres S.J.

Abstract

My aim is to study a transport equation on an ergodic network and related questions using a probabilistic approach. The process described by the transport equation can be regarded as a linear extension of a random process on the network. This enables me to use tools and results from probability theory (in particular Markov chain theory) to describe the asymptotic behaviour of the flow on the network. I show that the linear operator mapping the initial distribution to the asymptotic distribution is strongly linked to a suitable factorization of the underlying graph, thereby answering a question raised by Prof. R. Nagel. I also exploit the probabilistic approach to prove that controlling the system at a single vertex, there is only need for a finite time control. The general idea and the basic methods are first presented on a simple network, and are then adapted to allow the treatment of the general case.

CONTENTS

PREFACE

Consider a closed system of pipelines in which some material flows with constant speed. There are some nodes at which some of the pipelines meet and then split again. We suppose that there is no loss of material, neither along the pipelines, nor at the nodes. We would like to understand how this system behaves in the long run, and find out which parameters of the network determine its asymptotic behaviour.

This is the physical setting that has led to the study of flows in networks as a dynamic process rather than the traditional graph theoretical approach in which one looks for optimal solutions to a transportation problem with capacities and costs assigned to the edges and nodes of the graph. The case of flows as a dynamic process on finite networks has already been treated by E. Sikolya in [5], where the flow is viewed as a solution to a transport equation on the edges with a boundary condition at the nodes that reflects the material conservation during the process. This PDE setting allowed for a treatment of the problem with tools from semigroup theory, and led to a description of the asymptotic behaviour of the system. While answering questions about asymptotic periodicity, the question of how to determine the exact periodic or stationary state the system converges to with a given initial state remained open, and Prof. R. Nagel asked at the Hungarian-German Workshop on Evolution Equations on Dobogókő in March 2007 whether the mapping from the initial states to the corresponding asymptotic state — which was proven to induce a direct sum decomposition of the state space — could be linked to some appropriate factorisation of the network.

As mentioned, discrete or combinatorial processes in networks have been systematically studied for several decades, and this question suggested that an attempt should be made to exploit the results in those fields to bring new insight to the question of asymptotic behaviour of flow on networks. The discrete process of finite Markov chains was the one which to me seemed the most promising in this aspect, and I would in this thesis like to present on the special case of strongly connected networks how Markov chain theory and the corresponding probabilistic interpretation of the flow in networks can be used to reproduce many of the results obtained with semigroup methods, but also to answer the above question about the connection between asymptotic behaviour and network factorisation. This is also a good example of how very different, both continuous and discrete parts of mathematics — operator semigroup theory, probability theory and graph theory — come together in a physically motivated problem.

In Chapter 1, we first present the definitions and results from Markov chain theory needed later, with some notions being generalised to allow the handling of L^1-functions rather than only probability vectors. Then we formulate the probabilistic interpretation of flows in networks that will constitute the foundation on which will be based our treatment of the asymptotic behaviour of flows in networks. The process that in [5] is described by a transport equation will be reformulated as the linear extension of an appropriate stochastic process in the network motivated by the original physical process.

In Chapter 2, we show how simple ergodic networks with unit edge lengths can be transformed to allow the application of convergence results from Markov chain theory presented in Chapter 1 to reproduce the results on asymptotic behaviour obtained by semigroup theory (see [5]).

In Chapter 3, we exploit the methods previously presented to fully characterise the effects of one-vertex control of an ergodic network with unit edge lengths, and show that any asymptotically reachable state is in fact exactly reachable by bounded-time control.

In Chapter 4, we generalise the results of Chapter 2 to ergodic networks with rationally dependent cycle lengths. To this end we introduce a set-valued distance function on these networks that will help take the idea of network transformation one step further, allowing us to transform these networks into the simpler ones treated in Chapter 2 without changing their asymptotic behaviour. This distance function is also what will allow us to define the factor network whose flow semigroup captures the asymptotic behaviour of the flow semigroup of the original network and answer the question raised by Prof. R. Nagel. All this time we use probabilistic (and some graph theoretical) methods to produce semigroup theory results, but at the end, in connection with ergodic networks with rationally independent cycle lengths, we turn this around and use a semigroup theory result to formulate a limit theorem for the stochastic process.

1. DEFINITIONS AND PRELIMINARIES

1.1 Finite Markov Chains

In this section we start by summarising notions and results from finite Markov chain theory, mainly based on the monograph [4], but see also [3] and [2].

Definition 1.1.1. A *finite Markov chain* (MC) is a discrete-time ($T = \mathbb{N}_0$) stochastic process $\{X_t\}$ with finite state space $V = \{v_1, \dots, v_n\}$ for which the probability of entering a certain state only depends on the last state occupied, i.e.

$$\mathbf{P}(X_t = v_j | X_0 = v_{i_0}, X_1 = v_{i_1}, \dots, X_{t-1} = v_{i_{t-1}}) = \mathbf{P}(X_t = v_j | X_{t-1} = v_{i_{t-1}}).$$

The probabilities $p_{ij,t} := \mathbf{P}(X_t = v_j | X_{t-1} = v_i)$ $(i, j \in \overline{1, n})$ are called the *transition probabilities at time* $t \in \mathbb{N}^+$.

We are here only going to use MC-s with time-independent transition probabilities. Let $\{X_t\}$ be an MC and let P denote its *transition matrix*, i.e. the matrix $(p_{ij})_{i,j} \in \mathbb{R}^{n \times n}$

Remark. *In this section, in accordance with the usual probability theory notation, we use P to denote the transition matrix. However, for the sake of simplicity, P will in subsequent sections denote the transposed transition matrix.*

The probabilities $p_{ij}^{(t)} := \mathbf{P}(X_t = v_j | X_0 = v_i)$ are called the *t-step transition probabilities*. The state v_j is said to be *reachable* from state v_i if there exists a time $t \in T$ for which the t-step transition probability $p_{ij}^{(t)}$ is non-zero.

A Markov chain may be represented as a weighted directed graph $D(V, \overrightarrow{E})$ where $\overrightarrow{E}$ is the set of directed edges $\{\overrightarrow{v_i v_j} : i, j \in \overline{1, n}, p_{ij} > 0\}$, the edge $e_\alpha = \overrightarrow{v_i v_j}$ having weight $w_\alpha := p_{ij}$. We shall later see that the similarity of this graph representation to a network is not without reason.

We introduce a partial ordering $\mathscr{R} \subset V \times V$ on the set of states defined by $(v_i, v_j) \in \mathscr{R} \Leftrightarrow \exists t \in T : p_{ij}^{(t)} > 0$. It may easily be verified that $(v_i, v_i) \in \mathscr{R}$ and $((v_i, v_j) \in \mathscr{R}) \wedge ((v_j, v_k) \in \mathscr{R}) \Rightarrow (v_i, v_k) \in \mathscr{R}$. This partial ordering induces an equivalence relation $\sim$:

$$v_i \sim v_j \Leftrightarrow ((v_i, v_j) \in \mathscr{R}) \wedge ((v_j, v_i) \in \mathscr{R})$$

on the set V, partitioning it into the set of equivalence classes $\{V^1, \ldots, V^m\}$. This set inherits an induced partial ordering, and we write $V^a \leq V^b$ if there are elements $v_{\mathbf{a}} \in V^a$ and $v_{\mathbf{b}} \in V^b$ such that $(v_{\mathbf{a}}, v_{\mathbf{b}}) \in \mathscr{R}$.

Definition 1.1.2.
• The *irreducible sets* of the chain are the maximal classes with respect to $\mathscr{R}$, (i.e. the maximal elements of the partition with respect to $\leq$).
• States that do not belong to an irreducible set are called *transient*. The set of transient states is called the *transient part* of the MC.
• $v_i \in V$ is an *absorbing state* if $\{v_i\}$ is a maximal class.

Definition 1.1.3. • An *absorbing chain* is a Markov chain in which each irreducible set consists of a single absorbing state.
• An *irreducible/ergodic* chain is a Markov chain for which V is an irreducible set.

Remark. *It can be shown that a Markov chain is irreducible iff the corresponding transition matrix is irreducible, and this is also equivalent to the underlying graph being strongly connected (i.e. for any ordered pair of vertices there is a directed path connecting the first to the second).*

We shall often need the following theorem about the transition matrix of ergodic MC-s (see e.g. Theorem 11.10 in [2]).

Theorem 1.1.1. *For an ergodic MC with transition matrix P, there is a unique probability vector π_P such that $\pi_P P = \pi_P$. π_P is strictly positive, and any P-invariant row vector is a multiple of π_P.*

For a state v_i of an irreducible set, we may consider the number

$$d_i := gcd\{k : \mathbf{P}(X_k = v_i | X_0 = v_i) > 0\},$$

called the *period of the state*. It is known that this period is in fact independent of the state considered within the irreducible set, and the common period d is called the *period of the set*.

Definition 1.1.4. There are two types of irreducible sets with respect to their structure:
-*cyclic* sets, for which $d > 1$
-*regular* sets, for which $d = 1$.

For cyclic MC-s we have a partition of the set $\overline{1,n}$ — and through it of the set of states — into d sets defined by:

$$I_m := \left\{ j \in \overline{1,n} : \mathbf{P}(X_{m+k\cdot d} = v_j | X_0 = v_1) > 0 \text{ for a sufficiently large } k \in \mathbb{N} \right\}$$

and $V_m := \{v_i : i \in I_m\}$ ($m \in \overline{0, d-1}$). These sets of states are all invariant under the d-th iterate of the original MC (i.e. under the stochastic process

$\left\{X_t^d\right\} := \{X_{td}\}$, $t \in T$), and as such constitute isolated regular MC-s with transposed transition matrix $P_m := (P^d)_{I^m \times I^m}$. To each of these matrices there belongs a stationary probability row vector π_{P_m}.

We shall also need the following theorem about the asymptotic behaviour of ergodic MC-s (for the first part see e.g. Corollary 4.1.5 in [4], for the second apply the first part using the above mentioned).

Theorem 1.1.2. *For an ergodic MC with transition matrix P the following holds:*

- If the MC is regular, then $(P^a)_{a=1}^{\infty}$ converges exponentially to a matrix A which has identical rows equal to π_P, i.e.

$$\|P^a - A\| \leq b \cdot r^a$$

with suitable constants $b \in \mathbb{R}^+$, $r \in (0,1)$, and thus for any probability row vector $\mathbf{p}$ we have

$$\forall a \in \mathbb{N}_0 : \ \|\mathbf{p}P^a - \pi_P\|_1 \leq b \cdot r^a$$

- If the MC has period $d > 1$, then for any probability row vector $\mathbf{p}$ there exists a unique P^d-invariant probability row vector $\widetilde{\mathbf{p}}$ that satisfies

$$\forall a \in \mathbb{N}_0 : \ \|\mathbf{p}P^a - \widetilde{\mathbf{p}}P^a\|_1 \leq b \cdot r^a$$

with suitable constants $b \in \mathbb{R}^+$, $r \in (0,1)$. In addition we have $(\widetilde{\mathbf{p}})_{I^m} = (\sum_{j \in I^m} \mathbf{p}_j)\pi_{P_m}$ for all $m \in \overline{0, d-1}$.

By the linearity of P, these results may be extended to arbitrary complex vectors:

Corollary 1.1.3. *For an ergodic MC with transition matrix P the following holds:*

-If the MC is regular, then for any vector $\mathbf{v} \in \mathbb{C}^n$ we have

$$\forall a \in \mathbb{N}_0 : \ \|\mathbf{v}P^a - (\sum_{j=1}^{n} \mathbf{v}_j)\pi_P\|_1 \leq (\sum_{j=1}^{n} |\mathbf{v}_j|) \cdot b \cdot r^a$$

with suitable constants $b \in \mathbb{R}^+$, $r \in (0,1)$

- If the MC has period $d > 1$, then for any vector $\mathbf{v} \in \mathbb{C}^n$ there exists a unique P^d-invariant vector $\widetilde{\mathbf{v}} \in \mathbb{C}^n$ that satisfies

$$\forall a \in \mathbb{N}_0 : \ \|\mathbf{v}P^a - \widetilde{\mathbf{v}}P^a\|_1 \leq (\sum_{j=1}^{n} |\mathbf{v}_j|) \cdot b \cdot r^a$$

with suitable constants $b \in \mathbb{R}^+$, $r \in (0,1)$. In addition we have $(\widetilde{\mathbf{v}})_{I^m} = (\sum_{j \in I^m} \mathbf{v}_j)\pi_{P_m}$ for all $m \in \overline{0, d-1}$.

1.2 L^1 Markov Chains

In order to be able to link the behaviour of networks to that of Markov chains, we have to generalise the convergence theorem for ergodic MC-s to L^1-classes of ergodic MC-s with common transition matrix.

Note that from now on, P denotes the *transposed* transition matrix of a given MC.

Let $(X, \mathscr{A}, \mu)$ be a measure space, and for each $x \in X$ let us take a regular MC with given transposed transition matrix P_X and an initial (column) vector $\mathbf{v}_x \in \mathbb{C}^n$ such that $(x \mapsto (\mathbf{v}_x)_j) \in L^1(X, \mathscr{A}, \mu)$ for all $j \in \overline{1,n}$. We shall be interested in the joint behaviour of these Markov chains, i.e. in the functions $(x \mapsto (P^a \mathbf{v}_x))$ where $a \in \mathbb{N}_0$. P_X being a linear operator, the coordinate functions will remain in $L^1(X, \mathscr{A}, \mu)$, and it thus makes sense to speak of L^1-convergence.

We know by Corollary 1.1.3 that for all $x \in X$ there exists a unique vector $\widetilde{\mathbf{v}_x} \in \mathbb{C}^n$ such that

$$\|P_X^a \mathbf{v}_x - \widetilde{\mathbf{v}_x}\|_1 \leq (\sum_{j=1}^{n} |(\mathbf{v}_x)_j|) \cdot b \cdot r^a$$

where $\widetilde{\mathbf{v}_x} = (\sum_{j=1}^{n} (\mathbf{v}_x)_j) \pi_{P_X}^T$. It is then clear that $(x \mapsto (\widetilde{\mathbf{v}_x})_j) \in L^1(X)$ for all j, and thus using this (a.e.) pointwise estimate we obtain that for all $a \in \mathbb{N}_0$:

$$\begin{aligned}
\|(x \mapsto P_X^a \mathbf{v}_x) - (x \mapsto \widetilde{\mathbf{v}_x})\|_1 &= \int_X \|P_X^a \mathbf{v}_x - \widetilde{\mathbf{v}_x}\|_1 d\mu \leq \int_X b \cdot r^a \sum_{j=1}^{n} |(\mathbf{v}_x)_j| d\mu \\
&= b \cdot r^a \sum_{j=1}^{n} \|(x \mapsto (\mathbf{v}_x)_j)\|_1 \\
&= b \cdot r^a \cdot \|(x \mapsto \mathbf{v}_x)\|_1.
\end{aligned}$$

Reformulating the above, looking at this L^1-class of Markov chains as a single $L^1(X, \mathscr{A}, \mu)$-valued MC, we obtain the following:

Proposition 1.2.1. *If P is the transposed transition matrix of a regular MC, then for all $\mathbf{v} \in (L^1(X))^n$ there exists a unique vector $\widetilde{\mathbf{v}} \in (L^1(X))^n$ such that*

$$\|P^a \mathbf{v} - \widetilde{\mathbf{v}}\|_1 \leq b \cdot r^a \|\mathbf{v}\|_1 \; \forall a \in \mathbb{N}_0$$

with suitable constants $b \in \mathbb{R}^+$ and $r \in (0,1)$. In addition we have that $\widetilde{\mathbf{v}} = (\sum_{j=1}^{n} \mathbf{v}_j) \pi_P^T$, and thus $\|\widetilde{\mathbf{v}}\|_1 \leq \|\mathbf{v}\|_1$ since $\|\pi_P^T\|_1 = 1$.

Similarly we can prove the following:

Proposition 1.2.2. *If P is the transposed transition matrix of a cyclic MC with period d, then for all $\mathbf{v} \in (L^1(X))^n$ there exists a unique vector $\widetilde{\mathbf{v}} \in (L^1(X))^n$ such that*

$$P^d \widetilde{\mathbf{v}} = \widetilde{\mathbf{v}}$$
$$\text{and} \quad \|P^a \mathbf{v} - P^a \widetilde{\mathbf{v}}\|_1 \leq b \cdot r^a \|\mathbf{v}\|_1 \quad \forall a \in \mathbb{N}_0$$

with suitable constants $b \in \mathbb{R}^+$ and $r \in (0,1)$. In addition we have $(\widetilde{\mathbf{v}})_{I^m} = (\sum_{j \in I^m} \mathbf{v}_j)\pi^T_{P_m}$ for all $m \in \overline{0, d-1}$, and in consequence $\|\widetilde{\mathbf{v}}\|_1 \leq \|\mathbf{v}\|_1$.

1.3 Networks and flows

Let us first recall a definition from graph theory.

Definition 1.3.1. A *directed walk* on a directed graph $G(V, \overrightarrow{E})$ is a sequence $v_{i_0}, e_{i_1}, v_{i_1}, \ldots, e_{i_m}, v_{i_m}$ $(m \in \mathbb{N}^+)$ where the v_i-s are vertices, the e_i-s are directed edges such that e_{i_j} has tail in $v_{i_{j-1}}$ and head in v_{i_j} for all $j \in \overline{1, m}$. A *cycle* on a directed graph $G(V, \overrightarrow{E})$ is a directed walk in which no two vertices coincide apart from $v_{i_0} = v_{i_m}$.

Definition 1.3.2. A weighted directed graph $D = (V, \overrightarrow{E})$ in which the edges have nonnegative lengths, there are no cycles containing exclusively zero-length edges, and which satisfies the Kirchhoff Law, i.e. that for any vertex $v_i \in V$ the sum of the weights on the edges having tail in v_i is equal to 1 will be called a *network*.

Let $\mathcal{N}$ be a network with

$$n := |V|,\ k := |\overrightarrow{E}|,$$

and denote by w_α the weight on the edge e_α, by l_α the length of the same edge, and by e^t_α and e^h_α its tail and head, respectively. An edge e_α with length l_α can then be identified with the interval $[0, l_\alpha] \subset \mathbb{R}$, and the edges thus inherit the standard Lebesgue measure λ. Let $\mathcal{P}_\mathcal{N} \subset L^1\left(\coprod_{\alpha=1}^k [0, l_\alpha]\right)$ denote the subset of positive functions with unit norm.

We would now like to reformulate the process described by the transport equation used in [5] to the linear extension of a suitable stochastic process. Let us consider the following random process on $\mathcal{N}$: a particle is moving with unit speed along the directed edges of the network, and when arriving at a vertex v_i, it continues its journey along the edge e_α with probability w_α if e_α has its tail in v_i. If the particle is sent to a zero-length edge, then it passes immediately to its endpoint. The absence of cycles of zero-length edges together with the Kirchhoff Law condition on the weights given in the definition guarantee the well-definedness of this process. This can easily be translated into a stochastic process where the random variables $\mathcal{F}_t$ $(t \in [0, \infty))$ represent the position of the particle at time t (we omit the exact details of the stochastic process).

If we suppose that each random variable $\mathcal{F}_t$ has a probability distribution on $\coprod_{j=1}^k [0, l_\alpha]$ that is absolute continuous with respect to the Lebesgue measure (it is in fact enough to suppose the absolute continuity of the distribution of $\mathcal{F}_0$), we can consider the derivate functions $f^t \in \mathcal{P}_\mathcal{N} \subset L^1\left(\coprod_{\alpha=1}^k [0, l_\alpha]\right) \cong \prod_{\alpha=1}^k L^1[0, l_\alpha]$ $(t \in [0, \infty))$. This yields a family of mappings $\widetilde{T}(t) : \mathcal{P}_\mathcal{N} \to \mathcal{P}_\mathcal{N} \subset \prod_{\alpha=1}^k L^1[0, l_\alpha]$ defined by

$$\widetilde{T}(t) f^0 := f^t$$

for nonnegative values of t. It is clear from the nature of the process that

$$\widetilde{T}(t+s) = \widetilde{T}(t) \circ \widetilde{T}(s) \ \forall\, t, s \in [0, \infty).$$

By extending these mappings linearly to the space of (not necessarily probability) distributions $\prod_{\alpha=1}^k L^1[0, l_\alpha]$ on $\mathcal{N}$ we obtain a semigroup of operators $(T(t))_{t \geq 0}$,

$$T(t) : \prod_{\alpha=1}^k L^1[0, l_\alpha] \to \prod_{\alpha=1}^k L^1[0, l_\alpha].$$

Definition 1.3.3. We call this semigroup of operators a *flow* on the network $\mathcal{N}$. $T(t)$ is then called the *flow at time t*.

Definition 1.3.4. An initial distribution $f \in \prod_{\alpha=1}^k L^1[0, l_\alpha]$ on a network $\mathcal{N}$ is called $(\mathcal{N}, d)$-*invariant* if $T(d)f = f$.

If $f \in \prod_{\alpha=1}^k L^1[0, l_\alpha]$ is an initial distribution on a network $\mathcal{N}$, then f^t denotes the distribution after time t, i.e. $f^t := T(t)f$ where $(T(t))_{t \geq 0}$ is the flow on $\mathcal{N}$. Let further F_i^t $(i \in \overline{1, n})$ and F^t be defined by

$$F_i^t := \sum_{\{\beta : e_\beta^t = v_i\}} f_\beta^t$$

and

$$F^t := (F_1^t, \ldots, F_n^t)^T$$

for all $t \geq 0$. Notice that for all $t \geq 0$ we have $\|\widetilde{T}(t)\| = 1$, and thus also $\|T(t)\| = 1$, i.e.

$$\|f^t\|_1 \leq \|f\|_1.$$

Finally let us introduce the class of networks we are going to treat in the rest of this paper.

Definition 1.3.5. We call a network *ergodic* if the underlying graph is strongly connected.

2. ERGODIC NETWORKS WITH UNIT EDGE LENGTHS

We start our study with networks having unit edge lengths, and then pass on to general ergodic networks in Chapter 4.

Using the notations of the previous Chapter, let $\mathcal{N}$ be an ergodic network with all edges having unit length, and let us further suppose that there are no multiple directed edges (loops are allowed). Then denote by P the transposed transition matrix defined by the weights of the corresponding graph. Let us fix an initial distribution $f \in (L^1[0,1])^n$. Then due to the Kirchhoff Law, for any time $t \geq 1$ the coordinate functions of the distributions corresponding to edges having tail in a same given vertex will be equal up to a constant factor, namely the weight of the edges: if e_α has its tail in v_i, then

$$f^t_\alpha = w_\alpha \cdot F^t_i,\ \alpha \in \overline{1,k}, t \geq 1$$

(for notations, see end of previous Chapter). Let $Q \in \mathbb{C}^{k\times n}$ be the transposed weighted outgoing incidence matrix of the underlying graph, i.e.

$$Q_{\alpha,i} = \begin{cases} w_\alpha, & \text{if } e^t_\alpha = v_i; \\ 0, & \text{otherwise.} \end{cases}$$

Then for $t \geq 1$, we have $f^t = QF^t$.

We are now going to construct a new network $\widehat{\mathcal{N}}_\tau$ that for $t \geq 1$ behaves in the same way as our original network. First we pick a value $\tau \in [0,1]$. Then we call the vertices V_i and V'_i, the edges E'_i and E_α ($1 \leq i \leq n$, $1 \leq \alpha \leq k$). For every i, the edge E'_i is a directed edge of length τ from V_i to V'_i with weight 1. For every α, if e_α was a directed edge such that $e^t_\alpha = v_i$ and $e^h_\alpha = v_j$, then E_α is a directed edge such that $E^t_\alpha = V'_i$ and $E^h_\alpha = V_j$ of length $1-\tau$ and weight w_α. We then define the initial distributions along the edges as follows:

- on E_α, the initial distribution $p_\alpha \in L^1[0,1-\tau]$ is $p_\alpha := f^1_\alpha|_{[\tau,1]}$
- on E'_i, the initial distribution $q_i \in L^1[0,\tau]$ is $q_i := F^1_i|_{[0,\tau]}$

In the random process setting, this corresponds to delaying the decision-making at the original vertices with τ. The behaviour of the two networks is strongly linked, as can be seen from the following equalities:

$$p^{t-1}_\alpha = f^t_\alpha|_{[\tau,1]},\quad q^{t-1}_i = F^t_i|_{[0,\tau]} \qquad 1 \leq t, 1 \leq i \leq n, 1 \leq \alpha \leq k.$$

If we stretch this to the limit case $\tau = 1$, the edges E_α will have length 0 (notice though that no cycle with only such edges arise). As such, these edges allow a transition from the original Kirchhoff-type boundary condition with weights on the outgoing edges to a new Kirchhoff-type boundary condition with weights on the inflow: the flow on this network $\widehat{\mathcal{N}} := \widehat{\mathcal{N}}_1$ can be reinterpreted as a system of functions $r \in (L^1[0,1])^n$ with rightshift (yielded by the E_i') satisfying the boundary condition $r(0) = Pr(1)$ (yielded by the E_α)! Notice that after time $t = 1$ the distribution r^1 is therefore going to be exactly Pr. Thus, seen at unit time intervals, this network behaves just like an L^1 Markov chain as defined in the first Chapter. This observation, as mentioned in the Introduction, is our main motivation to try and use MC methods to obtain results about the asymptotic behaviour of flows in networks .

We are going to explicitly describe the behaviour of $\widehat{\mathcal{N}}$, and through it, we shall obtain an explicit description of the behaviour of the original network beyond $t = 1$. Our initial distribution on $\widehat{\mathcal{N}}$ was $q = F^1$. For any $t \geq 1$ the combination of the rightshift and the unit time behaviour yields:

$$q^{t-1}(s) = \begin{cases} (P^{\lfloor t \rfloor} q)(1 + s - \{t\}) & \text{for } s \in [0, \{t\}] \\ (P^{\lfloor t \rfloor - 1} q)(s - \{t\}) & \text{for } s \in [\{t\}, 1], \end{cases}$$

where $\{t\} := t - \lfloor t \rfloor$. From the previous Chapter we know that if our MC has period d, then for any $r \in (L^1[0,1])^n$ there exists a unique vector $\widetilde{r} \in (L^1[0,1])^n$ for which

$$\begin{array}{cl} P^d \widetilde{r} = \widetilde{r} & \text{and} \\ \|P^m r - P^m \widetilde{r}\|_1 \leq c\rho^m \|r\|_1 & \text{for some } c \in \mathbb{R}^+, \rho \in (0,1) \text{ and all } m \in \mathbb{N}_0. \end{array}$$

Since for (nonnegative) integers m we have $P^m \widetilde{r} = \widetilde{r}^m$ these equations yield:

$$\begin{array}{cl} \widetilde{r}^d = \widetilde{r} & \text{and} \\ \|r^m - \widetilde{r}^m\|_1 \leq c\rho^m \|r\|_1 & \text{for some } c \in \mathbb{R}^+, \rho \in (0,1) \text{ and all } m \in \mathbb{N}_0. \end{array}$$

Let us convert this back to the original network. Let $\widetilde{F} := \widetilde{q}^{d-1}$, and $\widetilde{f} := Q\widetilde{F}$. Since all the weights w are positive, we then for $t \geq 1$ have:

$$\begin{aligned}
\left\|f^t - \widetilde{f}^t\right\|_1 &= \left\| \begin{pmatrix} f_1^t \\ f_2^t \\ \vdots \\ f_k^t \end{pmatrix} - \begin{pmatrix} \widetilde{f}_1^t \\ \widetilde{f}_2^t \\ \vdots \\ \widetilde{f}_k^t \end{pmatrix} \right\|_1 = \left\| \begin{pmatrix} F_1^t \\ F_2^t \\ \vdots \\ F_n^t \end{pmatrix} - \begin{pmatrix} \widetilde{F}_1^t \\ \widetilde{F}_2^t \\ \vdots \\ \widetilde{F}_n^t \end{pmatrix} \right\|_1 \\
&= \left\| \begin{pmatrix} q_1^{t-1} \\ q_2^{t-1} \\ \vdots \\ q_n^{t-1} \end{pmatrix} - \begin{pmatrix} \widetilde{q}_1^{t+d-1} \\ \widetilde{q}_2^{t+d-1} \\ \vdots \\ \widetilde{q}_n^{t+d-1} \end{pmatrix} \right\|_1 \\
&= \left\| q^{t-1}\big|_{[0,\{t\}]} - \widetilde{q}^{t-1}\big|_{[0,\{t\}]} \right\|_1 + \left\| q^{t-1}\big|_{[\{t\},1]} - \widetilde{q}^{t-1}\big|_{[\{t\},1]} \right\|_1 \\
&= \left\| \left(P^{\lfloor t \rfloor} q\right)\Big|_{[1-\{t\},1]} - \left(P^{\lfloor t \rfloor} \widetilde{q}\right)\Big|_{[1-\{t\},1]} \right\|_1 + \\
&\quad \left\| \left(P^{\lfloor t \rfloor -1} q\right)\Big|_{[0,1-\{t\}]} - \left(P^{\lfloor t \rfloor -1} \widetilde{q}\right)\Big|_{[0,1-\{t\}]} \right\|_1 \\
&\leq c\rho^{\lfloor t \rfloor} \left\| q|_{[1-\{t\},1]} \right\|_1 + c\rho^{\lfloor t \rfloor -1} \left\| q|_{[0,1-\{t\}]} \right\|_1 \leq c\rho^{\lfloor t \rfloor -1} \|q\|_1 \\
&\leq b\rho^t \|q\|_1 = b\rho^t \left\|F^1\right\|_1 = b\rho^t \left\|f^1\right\|_1 \leq b\rho^t \|f\|_1 ,
\end{aligned}$$

where $b = \frac{c}{\rho^2}$.

But also

$$\|\widetilde{f}\|_1 = \|\widetilde{F}\|_1 = \|\widetilde{q}^{d-1}\|_1 = \|\widetilde{q}\|_1 \leq \|q\|_1 = \|F^1\|_1 = \|f^1\|_1 \leq \|f\|_1$$

and for $t < 1$ we have

$$\begin{aligned}
\left\|f^t - \widetilde{f}^t\right\|_1 &= \|(f - \widetilde{f})^t\|_1 \leq \|f - \widetilde{f}\|_1 \\
&\leq \|f\|_1 + \|\widetilde{f}\|_1 \leq 2\|f\|_1 \leq \frac{2}{\rho}\rho^t \|f\|_1.
\end{aligned}$$

Putting together the above estimates, we obtain the following

Theorem 2.0.1. *For any initial distribution $f \in (L^1[0,1])^k$ on the network $\mathcal{N}$, there exists a unique distribution $\widetilde{f} \in (L^1[0,1])^k$ which satisfies the following:*

- $\widetilde{f}^d = \widetilde{f}$
- $\left\|f^t - \widetilde{f}^t\right\|_1 \leq a\rho^t \|f\|_1$ *with suitable constants $a \in \mathbb{R}^+$ and $\rho \in (0,1)$.*

In addition,

$$\left\|\widetilde{f}\right\|_1 \leq \|f\|_1 .$$

Let us now consider the mapping

$$\pi : (L^1[0,1])^k \to (L^1[0,1])^k, f \mapsto \widetilde{f}.$$

This mapping is clearly linear, due to the linearity of the flow itself. Obviously all elements of the range are distributions that are $(\mathcal{N}, d)$-invariant. But it can be seen that all distributions that are $(\mathcal{N}, d)$-invariant are actually mapped to themselves, and therefore the range of the mapping is the sublattice $X_d^{\mathcal{N}}$ of $(\mathcal{N}, d)$-invariant vectors in $X := (L^1[0,1])^k$. Since the mapping is the identity on $X_d^{\mathcal{N}}$, it yields a decomposition of X into the direct sum $Ran(\pi) \oplus Ker(\pi)$. Thus π is in fact a projection onto $X_d^{\mathcal{N}}$. $Ran(\pi)$ is invariant under the flow, since for any $t > 0$ and $g \in X_d^{\mathcal{N}}$ we have $(g^t)^d = g^{t+d} = (g^d)^t = g^t$. As it can easily be verified, $\pi(f^t) = (\pi(f))^t$ for any $t \geq 0$ and $f \in X$, and so $Ker(\pi)$ is also invariant under the flow. We therefore obtain the following for the flow-semigroup $(T(t))_{t\geq 0}$

Corollary 2.0.2. *Let $\mathcal{N}$ be a strongly connected network with unit edge lengths. Let d denote its period. For the decomposition $X = X_d^{\mathcal{N}} \oplus Ker(\pi)$ we have*

- *$X_d^{\mathcal{N}}$ and $Ker(\pi)$ are T(t)-invariant subspaces*
- *the operators $S(t) := T(t)|_{X_d^{\mathcal{N}}}$ form a bounded C_0-group with period d on $X_d^{\mathcal{N}}$ ($\|S(t)\|_{X_d^{\mathcal{N}}} = 1$)*
- *the semigroup $T(t)|_{Ker(\pi)}$ is uniformly exponentially stable, and*

$$\|T(t) - S(t) \circ \pi\|_X \leq a e^{t \log \rho},$$

where $a \in \mathbb{R}^+$ and $\rho \in (0,1)$

This corresponds to a special case of Proposition 2.4.5. in [5]. We are going to prove the general case in Chapter 4.

Definition 2.0.6. The mapping

$$\pi : (L^1[0,1])^k \to (L^1[0,1])^k, f \mapsto \widetilde{f}$$

is called the *asymptotic mapping* on $\mathcal{N}$.

3. VERTEX CONTROL

Until now, we have considered the network as an isolated system. However, it is also interesting to know how this system reacts to external influence. In this chapter, we are going to study how the system can be influenced through control at one or more vertices.

Let $\mathcal{N}$ be a strongly connected network.

Definition 3.0.7. *Controlling* $\mathcal{N}$ in a given vertex v_i is allowing to tap or fill up the network through that vertex, i.e. we change the original Kirchhoff Law boundary condition at v_i

$$\sum_{\{\alpha:e_\alpha^t=v_i\}} w_\alpha \cdot f_\alpha^t(0) = \sum_{\{\beta:e_\beta^h=v_i\}} f_\beta^t(l_\beta)$$

to

$$\sum_{\{\alpha:e_\alpha^t=v_i\}} w_\alpha \cdot f_\alpha^t(0) = \sum_{\{\beta:e_\beta^h=v_i\}} f_\beta^t(l_\beta) + g_i(t)$$

for $t \geq 0$, where g_i is the *control function*, taken from $L^1_{loc}[0,\infty)$.

Notice that simultaneous control at several vertices can also easily be defined: we choose a subset $V^I = \{v_i | i \in I\}$ of the vertices ($I \subset \overline{1,n}$), and we change the boundary condition at each of them through the control functions $g_i \in L^1_{loc}[0,\infty)$ ($i \in I$).

Several interesting questions arise concerning control. One possibility is to ask for the space of states that can be reached at some specific time $t_0 \geq 0$ from the constant zero initial state if the control function can be arbitrarily chosen. It is clear that this space grows as t_0 increases. Also one may ask for the space of states that can be reached without specifying the time. In connection with this, an interesting question is whether the system is controllable in finite time, that is if there exists a time t_{max} for which any reachable state is reachable at t_{max}. A further question is whether — and in what sense — a network is more controllable than an other. This is, as can be expected, dependent on the structure of the graph.

Let us introduce a few notations and define the key notions needed in this chapter. Notice that due to the linearity of the flow, the control itself is also linear, hence the spaces defined below are truly linear subspaces.

Definition 3.0.8. Let $I \subset \overline{1,n}$, and let $g \in (L^1_{loc}[0,\infty))^I$ be a control function. Denote by $f^{g,t} \in (L^1[0,1])^k$ the state of the network at time t when applying g_i at vertex v_i $(i \in I)$ with the initial state $f^{g,0} \equiv 0$. Then

- the linear subspace
$$\mathcal{R}^I_t := \bigcup_{g \in (L^1_{loc}[0,\infty))^I} \left\{f^{g,t}\right\} \subset (L^1[0,1])^k \cong L^1([0,1],\mathbb{C}^k)$$
will be called the *exact reachability space at time* t belonging to I,
- the *exact reachability space* belonging to I is the space of states
$$\mathcal{R}^I := \bigcup_{t \geq 0} \mathcal{R}^I_t \subset (L^1[0,1])^k,$$
- and the *approximate reachability space* belonging to I is the space of states
$$\overline{\mathcal{R}^I} := \overline{\bigcup_{t \geq 0} \mathcal{R}^I_t} \subset (L^1[0,1])^k.$$

Usually we can not expect any of these spaces to be equal to the space $L^1(\mathcal{N})$ due to restrictions caused by the boundary conditions at the vertices. It is therefore not a good approach to classify networks with respect to whether the exact (or approximate) reachability space is equal to the whole state space. The reachability spaces take the global structure into consideration, and since we have to — to some extent — factor in the structure of the network, a good.

If we want a good measure of how controllable a network is, we do have to take into account some of its structural aspects, as we can not expect the reachability spaces to equal the whole state space. We have to find some intermediate space which has some dependence on the network structure, but is big enough for not being reachable for all networks.

Let us now take a closer look at the effect of the Kirchhoff Law on control. Due to the outgoing flow being distributed along the outgoing edges according to fixed ratios, it is clear that whatever our control is, if $e^t_\alpha = e^t_\beta$, then the restrictions of the states to these edges differ only by a factor w_α/w_β for any time $t \geq 0$. Using the notations of Chapter 2, let us again consider the network $\widehat{\mathcal{N}}$, this time with initial state zero. Let us apply the same control to this network as to our original network $\mathcal{N}$, i.e. if we have a control function $g \in (L^1_{loc}[0,\infty))^I$ applied to $\mathcal{N}$, then we apply g_i at V_i in $\widehat{\mathcal{N}}$ for all $i \in I$, and denote by $F^{g,t} \in (L^1[0,1])^I$ the state thus obtained at time t. It then follows from the above that we have
$$f^{g,t} = QF^{g,t}$$

for all $t \geq 0$ (for the definition of Q, see the beginning of Chapter 2). This means that the states reachable by control are uniquely determined by the corresponding states on the network $\widehat{\mathcal{N}}$ *without* time delay. Let us denote by

$$\mathcal{S}_t^I \subset L^1([0,1],\mathbb{C}^n)$$

the *exact reachability space at time t* belonging to I on $\widehat{\mathcal{N}}$, and by

$$\overline{\mathcal{S}^I} \in L^1([0,1],\mathbb{C}^n)$$

the *approximate reachability space* belonging to I defined by

$$\overline{\mathcal{S}^I} := \overline{\bigcup_{t \geq 0} \mathcal{S}_t^I} \subset L^1([0,1],\mathbb{C}^n).$$

We then have

$$\mathcal{R}_t^I = Q\mathcal{S}_t^I$$

and

$$\overline{\mathcal{R}^I} = Q\overline{\mathcal{S}^I},$$

and so obviously $\overline{\mathcal{R}^I} \subset QL^1([0,1],\mathbb{C}^n)$.

An obvious observation is that applying control at a subset of vertices V^I, we can not expect to reach a larger space of functions than if we simultaneously apply control at all vertices. This will turn out to be a useful reference for measuring controllability, and gives rise to the following definition.

Definition 3.0.9. The network $\mathcal{N}$ is called *maximally controllable at* V^I if all states approximately reachable through control applied at all vertices simultaneously can also be approximately reached through control applied only at V^I, i.e. if

$$\overline{\mathcal{R}^I} = \overline{\mathcal{R}^{\overline{1,n}}}.$$

However, it is easily seen that due to the simple structure of $\widehat{\mathcal{N}}$, we have

$$\overline{\mathcal{S}^{\overline{1,n}}} = L^1([0,1],\mathbb{C}^n),$$

and thus

$$\overline{\mathcal{R}^{\overline{1,n}}} = QL^1([0,1],\mathbb{C}^n),$$

leading to the following Corollary.

Corollary 3.0.3. *The network* $\mathcal{N}$ *is called* maximally controllable *at* V^I *if* $\mathcal{R}^I \subset QL^1([0,1],\mathbb{C}^n)$ *is satisfied with equality.*

Remark. *Actually this corresponds to the notion of* maximal reachability space *used in [1] (cf. Lemma* 4.1*).*

3.1 One-vertex control

We shall now restrict ourselves to control applied at a single vertex. Without loss of generality, we may assume that this vertex is v_1. In view of a less cumbersome notation, we shall omit the upper index $\{1\}$ from the different reachability spaces. As mentioned earlier, the flow on the network $\widehat{\mathcal{N}}$ is a rightshift combined with a Markov chain iteration as boundary condition, and so the effect of the control function g can be described precisely.

Let $\mathbf{e}_1$ denote the unit vector $(1, 0, \ldots, 0)^T \in \mathbb{C}^n$. It is then possible to see that

Proposition 3.1.1. *When applying the control function $g \in L^1_{loc}[0, \infty)$ at vertex v_1, the state $F^{g,t} \in L^1([0,1], \mathbb{C}^n)$ on the modified network $\widehat{\mathcal{N}}$ at time $t \geq 0$ satisfies*

$$F^{g,t}(\epsilon) = \begin{cases} \sum_{j=0}^{\lfloor t \rfloor}(g(j + \{t\} - \epsilon) \cdot P^{\lfloor t \rfloor - j}\mathbf{e}_1) & \textit{for a.e.} \quad \epsilon \in [0, \{t\}] \\ \sum_{j=1}^{\lfloor t \rfloor}(g(j + \{t\} - \epsilon) \cdot P^{\lfloor t \rfloor - j}\mathbf{e}_1) & \textit{for a.e.} \quad \epsilon \in [\{t\}, 1] \end{cases}$$

Proof. For $t \leq 1$, the combination of the modified boundary condition at v_1 together with right-shift yields

$$F^{g,t}(\epsilon) = \begin{cases} g(t - \epsilon) \cdot \mathbf{e}_1 & \text{for a.e.} \quad \epsilon \in [0, t] \\ 0 & \text{for a.e.} \quad \epsilon \in [t, 1] \end{cases}$$

Consider now an arbitrary $t \geq 0$. Let $g^0 := g|_{[0,\{t\}]}$, and $g^j := g|_{[j-1+\{t\}, j+\{t\}]}$ for all $1 \leq j \leq \lfloor t \rfloor$. The system being linear, we then have

$$F^{g,t} = \sum_{j=0}^{\lfloor t \rfloor} F^{g^j,t}.$$

But due to the above, and to the fact that $f^1 = Pf$ for any function $f \in L^1([0,1], \mathbb{C}^n)$, we also have

$$F^{g^0,t}(\epsilon) = (\chi_{[0,\{t\}]} g(\{t\} - \epsilon) \cdot \mathbf{e}_1)^{\lfloor t \rfloor} = \begin{cases} (g(\{t\} - \epsilon) \cdot P^{\lfloor t \rfloor}\mathbf{e}_1) & \text{for a.e.} \quad \epsilon \in [0, \{t\}] \\ 0 & \text{for a.e.} \quad \epsilon \in [\{t\}, 1] \end{cases}$$

where $\chi_{[0,\{t\}]}$ is the characteristic function of the interval $[0, \{t\}]$, and

$$F^{g^j,t}(\epsilon) = (g(j + \{t\} - \epsilon) \cdot \mathbf{e}_1)^{\lfloor t \rfloor - j} = (g(j + \{t\} - \epsilon) \cdot P^{\lfloor t \rfloor - j}\mathbf{e}_1) \quad \text{for a.e.} \quad \epsilon \in [0, 1]$$

for all $1 \leq j \leq \lfloor t \rfloor$. By summing these equations, we obtain just what we needed. □

Let

$$A_0 := \{\underline{0}\}$$

and

$$A_j := lin\left\{\mathbf{e}_1, P\mathbf{e}_1, \ldots, P^{j-1}\mathbf{e}_1\right\}$$

for $j \geq 1$. Then, with the identification

$$L^1([0,1],\mathbb{C}^n) \cong L^1([0,\{t\}],\mathbb{C}^n) \times L^1([\{t\},1],\mathbb{C}^n)$$

we obtain the following.

Corollary 3.1.2. $F^{g,t} \in L^1([0,\{t\}], A_{\lfloor t \rfloor+1}) \times L^1([\{t\},1], A_{\lfloor t \rfloor}) =: \mathcal{A}_t \ \ \forall t \geq 0.$

Since $g \in L^1_{loc}[0,\infty)$ is arbitrary, we actually have

$$\mathcal{A}_t = \mathcal{S}_t.$$

But $A_j \subset A_{j+1}$ for all $j \geq 0$, and $A_j = A_n$ for all $j \geq n$. In addition A_n is a closed subspace in $\mathbb{C}^n$. Therefore for any $t \in [0,\infty)$ we have $\mathcal{A}_t \subset L^1([0,1], A_n) = \mathcal{A}_n$ with $\mathcal{A}_n$ being a closed sublattice in $L^1([0,1],\mathbb{C}^n)$, and so

$$\overline{\mathcal{S}} = \overline{\bigcup_{t\geq 0} \mathcal{S}_t} = \mathcal{S}_n$$

Consequently we obtain the following

Theorem 3.1.3. *The approximate reachability space for the network $\widehat{\mathcal{N}}$ with control at vertex v_1 coincides with the exact reachability space at time n, i.e.*

$$\overline{\mathcal{S}} = \mathcal{S}_n = L^1([0,1], A_n),$$

and passing to the network $\mathcal{N}$:

$$\overline{\mathcal{R}} = \mathcal{R}_n = L^1([0,1], QA_n)$$

Thus $\mathcal{N}$ is maximally controllable at v_1 if and only if $A_n = \mathbb{C}^n$

In other words, control beyond time $t = n$ is superfluous, and any approximately reachable state is exactly reachable.

3.2 One-vertex control with non-zero initial state

If we are interested in the approximate reachability space when the initial state is nonzero, the situation gets slightly different. As the initial functions on outgoing edges at a given vertex are not necessarily equal up to a constant factor, transcribing the problem to the network $\widehat{\mathcal{N}}$ at $t = 0$ seems impossible. But, by linearity of the flow, the effect of the uncontrolled flow on the initial state f^0 adds up with the control, and the exact reachability space at time t thus turns into the Minkowski sum $\mathcal{R}_t^f = f^t + \mathcal{R}_t$. The approximate reachability space will then be

$$\overline{\mathcal{R}^f} = \overline{\bigcup_{t\geq 0} \mathcal{R}_t^f} = \overline{\bigcup_{t\geq 0} (f^t + \mathcal{R}_t)}$$

Let us first look at the effect of the closure. Take a convergent sequence $(h_i) \subset \bigcup_{t\geq 0} \mathcal{R}_t^f$, where $h_i \in \mathcal{R}_{t_j}$ $(t_i \geq 0)$. Let the limit be called $h \in L^1([0,1], \mathbb{C}^k)$.

(1) If the sequence (t_i) contains a convergent subsequence (t'_j) with limit t', let us take the corresponding subsequence (h'_j) of (h_i). Then

$$h'_j = f^{t'_j} + r_j,$$

where $r_j \in \mathcal{R}_{t'_j}$. Since the flow is strongly continuous, we have

$$\lim_{j\to\infty} f^{t'_j} = f^{t'},$$

and then the sequence (r_j) has to converge to $h - f^{t'}$. But it is clear from the above that

$$\mathcal{R}_{\tau_1} \subset \mathcal{R}_{\tau_2} \quad \text{if} \quad \tau_1 < \tau_2.$$

Thus for any $\epsilon > 0$ there exists a positive integer J such that for any $j > J$ we have $t_j < t' + \epsilon$, and so $r_j \in \mathcal{R}_{t'+\epsilon}$. The spaces $\mathcal{R}_\tau$ are closed for any $\tau \geq 0$, and so

$$h - f^{t'} \in \mathcal{R}_{t'+\epsilon}$$

for all $\epsilon > 0$. But we also have

$$\mathcal{R}_{t'} = \bigcap_{\epsilon>0} \mathcal{R}_{t'+\epsilon},$$

and therefore $h \in \mathcal{R}_{t'}^f$.

(2) If the sequence (t_i) does not contain a convergent subsequence, it must tend to infinity. Also, it has to contain a convergent subsequence *mod d* (i.e. on $\mathbb{R}/d\mathbb{Z}$), where d is the period of the underlying MC. Let (t'_j) be such a subsequence, and let $\tau \in [0,d) \cong \mathbb{R}/d\mathbb{Z}$ be its limit. Then we have both

$$\lim_{j\to\infty} \left\| f^{t'_j} - \widetilde{f}^{t'_j} \right\| = 0 \quad \text{and} \quad \lim_{j\to\infty} \left\| \widetilde{f}^{t'_j} - \widetilde{f}^{\tau} \right\| = 0,$$

which yield $\lim_{j\to\infty} f^{t'_j} = \widetilde{f}^\tau$. At the same time $h'_j - f^{t'_j} \in \mathcal{R}_{t'_j} \subset \mathcal{R}_n$ for all j, and by closedness of $\mathcal{R}_n$ we then have $h - \widetilde{f}^\tau \in \mathcal{R}_n$.

We can thus write

$$\mathcal{R}^f = \left(\bigcup_{t\geq 0} (f^t + \mathcal{R}_t) \right) \cup \left(\bigcup_{\tau\in[0,d)} \left(\widetilde{f}^\tau + \mathcal{R}_n\right) \right).$$

If we denote by Γ^f the asymptotic orbit

$$\Gamma^f := \bigcup_{\tau \in [0,d)} \{\widetilde{f}^\tau\}$$

of the initial state f, we obtain

$$\mathcal{R}^f = \left(\bigcup_{t \geq 0} (f^t + \mathcal{R}_t) \right) \cup \left(\Gamma^f + \mathcal{R}_n \right).$$

But as we shall later see in Chapter 4, $\Gamma^f \subset \mathcal{R}_n$, and we may thus state the following

Proposition 3.2.1. *When applying control at vertex v_1 to the network $\mathcal{N}$ with initial state $f \in L^1([0,1], \mathbb{C}^k)$, the approximate reachability space is*

$$\mathcal{R}^f = \left(\bigcup_{t \geq 0} (f^t + \mathcal{R}_t) \right) \cup \mathcal{R}_n.$$

3.3 Control at V^I

We now shortly outline the situation in the case when control is allowed in more than one vertex, say in the set of vertices $V^I = \{v_i : i \in I\}$ where $I \subset \overline{1, n}$. When controlling simultaneously at several vertices, the effect of the control at each vertex adds up, and the reachability space will be the linear hull of the individual reachability spaces. Therefore, when following the reasoning of Section 3.1, instead of taking the subspaces $A_j \subset \mathbb{C}^n$ generated by the iterates of the first unit vector, we have to take the subspaces

$$A_j^I := lin \left\{ P^a \mathbf{e}_i : a \in \overline{0, j-1}, i \in I \right\}$$

jointly generated by the iterates of the unit vectors

$$\mathbf{e}_i := (0, \ldots, \overset{i}{\widehat{1}}, \ldots, 0)^T \in \mathbb{C}^n$$

corresponding to each of the vertices v_i in which control is applied. The exact reachability space will then be

$$\mathcal{R}^I = \mathcal{R}_n^I = L^1([0,1], QA_n^I),$$

and so $\mathcal{N}$ is maximally controllable in $\{v_i : i \in I\}$ iff $A_n^I = \mathbb{C}^n$.

4. GENERAL ERGODIC NETWORKS

We are now going to study general ergodic networks with arbitrary strictly positive edge lengths (we shall later see that this restriction can be removed). Let $\mathcal{M}$ be an ergodic network with edge lengths $l_\alpha > 0$ ($\alpha \in \overline{1,k}$). We shall try to transform this network into a network with unit edge lengths without changing the asymptotic behaviour of the flow. To this end we present several types of transformations related to the underlying graph. As we shall see, only a special class of networks will be transformable in such ways, and the remaining networks will exhibit a significantly different asymptotic behaviour.

At first notice that if a network has integer edge lengths, then by inserting flow-through vertices (i.e. vertices that have a single incoming edge, and a single outgoing edge with weight 1) at unit intervals along edges longer than 1, the flow is not modified. Therefore networks with integer edge lengths do not differ from networks with unit edge lengths. A more general class of networks is that of networks with rational edge length ratios. For these we could rescale the network to obtain integer lengths, and add flow-through vertices as necessary. But to keep the period of the periodic states unchanged, we would then have to rescale the speed of the flow. Alternatively, we may instead rescale time. This gives rise to the following definition.

Definition 4.0.1. We call two networks $\mathcal{G}$ and $\mathcal{H}$ *time-scale equivalent* if there exist a weighted directed graph isomorphism between their underlying graphs $D_\mathcal{G}$ and $D_\mathcal{H}$ and a constant $c \in \mathbb{R}^+$ such that the image of any edge in $\mathcal{G}$ with length l_α is an edge in $\mathcal{H}$ with length $c \cdot l_\alpha$.

If two networks $\mathcal{G}$ and $\mathcal{H}$ are time-scale equivalent, then by rescaling time with the constant factor c we transform the flow on the first network to the flow on the second (the states are also stretched with factor c): $f(\cdot) \in L^1(\mathcal{G})$ is transformed into $1/cf(\cdot/c) \in L^1(\mathcal{H})$. Through this norm-preserving transformation we may say that the flow on $\mathcal{G}$ converges to a periodic one of period d iff the flow on $\mathcal{H}$ converges to a periodic flow of period $c \cdot d$. In the previous chapter, we did not allow for multiple directed edges. But if a network with rational edge lengths contains multiple directed edges (MDEs), then adding a flow-through vertex at the midpoint of each such edge leads us back to an equivalent network with rational edge lengths without MDEs. Since any network with rational edge lengths without MDEs is time-scale equivalent to a network with integer edge lengths without MDEs, the asymptotic behaviour of all networks with rational edge lengths can be fully described with the help of Theorem 2.0.1. To formulate the corresponding theorem about the asymptotic flow, we first need the following definition.

Definition 4.0.2. (1) The length of a directed walk

$$W = v_{i_0}, e_{i_1}, v_{i_1}, \ldots, e_{i_m}, v_{i_m}$$

on a network is

$$l_W := \sum_{j=1}^{m} l_{i_j}$$

(2) The *period* of a network $\mathcal{M}$ with rational edge length ratios is defined by

$$d := \gcd\{l_{\mathscr{C}} : \mathscr{C} \text{ is a directed cycle in } \mathcal{M}\}$$

As we shall later on in Section 4.2, this definition is in accordance with the definition of the period of MC-s given in 1.1 and used for networks with unit edge lengths. Thus trough rescaling time and applying Theorem 2.0.1 we obtain the following.

Theorem 4.0.1. *Let $\mathcal{M}$ be a network with rational edge length ratios. Then for any initial state $f \in L^1(\mathcal{M})$ on the network $\mathcal{M}$, there exists a unique state $\widetilde{f} \in L^1(\mathcal{M})$ which satisfies the following:*

- $\widetilde{f}^d = \widetilde{f}$
- $\left\| f^t - \widetilde{f}^t \right\|_1 \le a\rho^t \|f\|_1$ *with suitable constants $a \in \mathbb{R}^+$ and $\rho \in (0,1)$,*

where d is the period of the network. In addition,

$$\left\| \widetilde{f} \right\|_1 \le \|f\|_1 .$$

This resolves the case of networks with rational edge length ratios. But what with networks which have edge lengths with irrational ratio? As can be guessed from the previous theorem, the edge lengths in themselves are not relevant for the behaviour of the flow. This behaviour is in fact determined by the ratios of the *cycle lengths*. To be able to simplify our investigation, we will have to introduce a new type of equivalence between networks.

4.1 On Perturbations of Networks

Consider an arbitrary network $\mathcal{H}$ (with *nonnegative* edge lengths), and let v_i be a node which has no incoming zero-length edge (such a node has to exist by the definition of a network). Let further $\epsilon \in \mathbb{R}^+$ be such that $l_\alpha > \epsilon$ whenever the edge e_α has its head in v_i. We then modify the network as follows:

- We increase the length of all edges with tail in v_i with ϵ
- We reduce the length of all edges with head in v_i with ϵ

Remark that if e_α is a loop, then its length remains unaltered. We then obtain a new network $\mathcal{H}^*$. In order to compare the behaviour of the flow in the networks $\mathcal{H}$ and $\mathcal{H}^*$, we also have to map states of the first to states of the second. Let therefore $f \in \prod_{\alpha=1}^{k} L^1([0,l_\alpha],\mathbb{C})$ be a state of $\mathcal{H}$. We then define the corresponding state of $\mathcal{H}^*$ as follows:

- $f^*_\alpha \equiv f_\alpha$ whenever both endpoints of e_α differ from v_i
- $f^*_\alpha \equiv f_\alpha|_{[0,l_\alpha-\epsilon]}$ whenever e_α has head in v_i and tail in a different vertex
- $f^*_\alpha|_{[\epsilon,l_\alpha+\epsilon]} \equiv f_\alpha$ and

$$f^*_\alpha|_{[0,\epsilon]} := w_\alpha \cdot \sum_{\{\beta:\, e^h_\beta = v_i\}} f_\beta|_{[l_\beta-\epsilon,l_\beta]}$$

whenever e_α has tail in v_i and head in a different vertex

- $f^*_\alpha|_{[\epsilon,l_\alpha]} := f_\alpha|_{[0,l_\alpha-\epsilon]}$ and

$$f^*_\alpha|_{[0,\epsilon]} := w_\alpha \cdot \sum_{\{\beta:\, e^h_\beta = v_i\}} f_\beta|_{[l_\beta-\epsilon,l_\beta]}$$

whenever e_α is a loop in v_i.

Even though it is not possible to recover f itself from f^*, the state f^ϵ (i.e. the state on $\mathcal{H}$ at time ϵ with the initial state being f) can be obtained as:

- $f^\epsilon_\alpha = (f^*)^\epsilon_\alpha$, i.e. $f^\epsilon_\alpha|_{[\epsilon,l_\alpha]} = f^*_\alpha|_{[0,l_\alpha-\epsilon]}$ and

$$f^\epsilon_\alpha|_{[0,\epsilon]} = w_\alpha \cdot \sum_{\{\beta:\, e^h_\beta = v_j\}} f^*_\beta|_{[l^*_\beta-\epsilon,l^*_\beta]}$$

whenever both endpoints of e_α differ from v_i, its tail being in v_j

-$f^\epsilon_\alpha|_{[\epsilon,l_\alpha]} = f^*_\alpha$ and

$$f^\epsilon_\alpha|_{[0,\epsilon]} = w_\alpha \cdot \sum_{\{\beta:\, e^h_\beta = v_j\}} f_\beta|_{[l^*_\beta-\epsilon,l^*_\beta]}$$

whenever e_α has head in v_i and tail in a different vertex v_j

- $f^\epsilon = f^*_\alpha|_{[0,l_\alpha]}$ whenever e_α has tail in v_i and head in a different vertex
- $f^\epsilon_\alpha = f^*_\alpha$ whenever e_α is a loop in v_i.

Notice that this mapping $* : L^1(\mathcal{H}) \to L^1(\mathcal{H}^*)$ also satisfies $(f^t)^* = (f^*)^t$. Hence through this transformation of the $(network, state)$ pair the behaviour of the original network completely determines the behaviour of the new network, while the behaviour of the new network completely determines that of the original with an ϵ-delay. Thus the flows on the two networks exhibit the same asymptotic behaviour.

Definition 4.1.1. If the above relations hold, we call $\mathcal{H}^*$ an *ϵ-perturbation* (in v_i) of $\mathcal{H}$, and $\mathcal{H}$ a *$(-\epsilon)$-perturbation* of $\mathcal{H}^*$.

Definition 4.1.2. Two networks $\mathcal{G}$ and $\mathcal{H}$ with nonnegative edge lengths are called *perturbation equivalent* if there is a finite sequence of networks $(\mathcal{G}_a)_{a=0}^b$ ($b \in \mathbb{N}^+$), with $\mathcal{G}_0 := \mathcal{G}$ and $\mathcal{G}_b := \mathcal{H}$, such that $\mathcal{G}_a$ is a δ_a-perturbation of $\mathcal{G}_{a-1}$ with suitable $\delta_a \in \mathbb{R}$ for all $a \in \overline{1,b}$.

Let us now return to our networks with strictly positive edge lengths. A perturbation doesn't change the length of the cycles, and thus the cycle lengths in any network that is perturbation equivalent to a network with rational edge length ratios are rationally dependent. Thus the following definition extends the notion of period also to these networks.

Definition 4.1.3. For an arbitrary network $\mathcal{M}$ with rationally dependent cycle lengths, let

$$d := gcd\{\text{length of } \mathscr{C} : \mathscr{C} \text{ is a directed cycle on } \mathcal{M}\}$$

be called the *period* of the network.

We may then formulate the equivalent of Theorem 4.0.1.

Theorem 4.1.1. *Let $\mathcal{M}$ be a network that is perturbation equivalent to a network with rationally dependent edge lengths. Then for any initial state $f \in L^1(\mathcal{M})$ on the network $\mathcal{M}$, there exists a unique state $\widetilde{f} \in L^1(\mathcal{M})$ which satisfies the following:*

- $\widetilde{f}^d = \widetilde{f}$
- $\left\| f^t - \widetilde{f}^t \right\|_1 \leq a\rho^t \left\| f \right\|_1$ *with suitable constants $a \in \mathbb{R}^+$ and $\rho \in (0,1)$,*

where d is the period of the network. In addition,

$$\left\| \widetilde{f} \right\|_1 \leq \left\| f \right\|_1 .$$

Let us now suppose that $\mathcal{H}$ (and thus also $\mathcal{H}^*$) is perturbation equivalent to a network with rationally dependent edge lengths and period d, and let us take a closer look at the mapping $* : L^1(\mathcal{H}) \to L^1(\mathcal{H}^*)$. It is, as earlier mentioned, not necessarily bijective, since we have an ϵ-delay when passing from $\mathcal{H}^*$ to $\mathcal{H}$. But let us look at the restriction $*|_{X_d^{\mathcal{H}}}$ to the space of $(\mathcal{H}, d)$-invariant states. Let $f \in X_d^{\mathcal{H}}$. Then

$$(f^*)^d = (f^d)^* = f^*,$$

i.e. $*$ maps $X_d^{\mathcal{H}}$ to $X_d^{\mathcal{H}^*}$. Let us now take an initial state $g \in X_d^{\mathcal{H}^*}$. We know that this — through the perturbation — uniquely determines a state on $\mathcal{H}$ with an

ϵ-delay. There exists an integer $a \in \mathbb{N}^+$ such that $a \cdot d > \epsilon$, and so g determines a unique state $g' \in L^1(\mathcal{H})$ at time $a \cdot d$ on $\mathcal{H}$. This state satisfies

$$(g')^* = g^{a \cdot d} = g.$$

Thus the restricted mapping

$$* : X_d^{\mathcal{H}} \to X_d^{\mathcal{H}^*}$$

is in fact a bijection.

In the next section we are going to determine the necessary and sufficient conditions for a network to be perturbation equivalent to a network with rationally dependent edge lengths.

4.2 A Distance Notion on Strongly Connected Graphs

To be able to study perturbation equivalency of strongly connected networks, we shall need some distance notion on networks. The underlying structure of a network being a directed graph, this distance notion should as a start be defined on the set of vertices (nodes). Defining a distance notion on the vertices of a directed graph is made difficult by the number of different walks leading from one vertex to the other. One often used possibility is to take the length of the shortest directed walk. But this would fail to capture the relevant structure of the graph (namely the cycles). We therefore propose a set-valued distance notion based on the lengths of all possible walks.

Definition 4.2.1. An *undirected walk* on a graph G is a sequence

$$v_{i_0}, e_{i_1}, v_{i_1}, \ldots, e_{i_m}, v_{i_m} \quad m \in \mathbb{N}^+$$

where the v_i-s are vertices, the e_i-s are (undirected) edges such that e_{i_j} is an edge between $v_{i_{j-1}}$ and v_{i_j} for all $j \in \overline{1,m}$. We say that v_{i_0} is the starting point of the walk, while v_{i_m} is its ending point.

Since networks are *directed* graphs, the direction of the edges renders the specification of the vertices superfluous. But the walk being undirected, we have to allow traveling along a directed edge in opposite direction, and we therefore introduce the notation e_α^{-1} to indicate that we travel along the edge e_α against its orientation.

Definition 4.2.2. An *undirected walk* on a network $\mathcal{M}$ is a sequence $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$ where each ε_j ($j \in \overline{1,m}$) is equal to e_α or e_α^{-1} for some $\alpha \in \overline{1,k}$.

To define the length of an undirected walk, we also have to assign a length to the "inverse" of an edge: we consider e_α^{-1} as having length $-l_\alpha$ for all $\alpha \in \overline{1,k}$.

Definition 4.2.3. The length of an undirected walk $W = \varepsilon_1, \dots, \varepsilon_m$ is

$$l_W := \sum_{j=1}^{m} l_{\varepsilon_j},$$

where $l_{\varepsilon_j} := \begin{cases} l_\alpha, & \text{if } \varepsilon_j = e_\alpha; \\ -l_\alpha, & \text{if } \varepsilon_j = e_\alpha^{-1}. \end{cases}$

Notice that if W_1 is a walk ending at v_i and W_2 is a walk starting at v_i, then $W := W_1 W_2$ is also a walk, and has length $l_W = l_{W_1} + l_{W_2}$. In addition, if W is a walk starting at v_i, then WW^{-1} is a closed walk from v_i to itself with zero length.

Defining a distance notion on the vertices of a graph is made difficult by the number of different walks leading from one vertex to the other. One often used possibility is to take the length of the shortest *directed* walk. But this would fail to capture the relevant structure of the graph (namely the cycles). We therefore propose a set-valued distance notion based on the lengths of all possible walks.

Definition 4.2.4. Let $\mathcal{M}$ be a strongly connected network. We define a set-valued distance $\mathtt{d} : V \times V \to \mathcal{P}(\mathbb{R})$ on the set of ordered pairs of vertices as follows:

$$\mathtt{d}(v_i, v_j) := \{l_W : W \text{ is an undirected walk starting at } v_i \text{ and ending at } v_j\}.$$

The network being strongly connected, all distance sets are nonempty. The following properties of $\mathtt{d}$ can easily be verified: for all $i, j, l \in \overline{1, n}$ we have

- $\mathtt{d}(v_i, v_j) = -\mathtt{d}(v_j, v_i)$,
- $\mathtt{d}(v_i, v_l) = \mathtt{d}(v_i, v_j) + \mathtt{d}(v_j, v_l)$,
- $\mathtt{d}(v_i, v_i) = \mathtt{d}(v_j, v_j)$,
- $\mathtt{d}(v_i, v_i)$ is closed under addition and substraction.

We now proceed to show the following proposition linking this distance notion to the period of a network.

Proposition 4.2.1. *Let $\mathcal{M}$ be a strongly connected network with rationally dependent cycle lengths and period d. Then*

$$\mathtt{d}(v_i, v_i) = d\mathbb{Z} \quad \forall i \in I$$

Proof. Due to the above, it is enough to show the equality for a single vertex, say v_1. Denote by C the set of cycles in the network, and denote by $\mathcal{C} \subset \mathbb{R}$ the set of real numbers that can be written as a linear combination with integer coefficients of cycle lengths, i.e.

$$\mathcal{C} := \{\sum_{j=1}^{m} a_j \cdot l_{\mathscr{C}_j} \in \mathbb{R} : \mathscr{C}_1, \ldots, \mathscr{C}_m \in C, m \in \mathbb{N}_0, a_1, \ldots, a_m \in \mathbb{Z}\}.$$

It is clear that $\mathcal{C}$ is in fact equal to $d\mathbb{Z}$.

First we show that the length of any undirected walk from v_1 to v_1 is an element in $\mathcal{C}$.

Let W be such a walk. We decompose this closed walk into a sequence of walks $W_1 W_2 \ldots W_m$ such that each of the walks W_j ($j \in \overline{1,m}$) consists only of edges of the network or only of inverses (i.e. is a directed walk or the inverse of a directed walk), while no two successive walks have the same orientation. Let v'_j denote the end of the walk W_j ($j \in \overline{1,m}$), and let $v'_0 := v_1$. If W_j is a directed walk, then let $W_j^* := W_j$. If W_j is the inverse of a directed walk, then let W_j^* be a directed walk from v'_{j-1} to v'_j. The closed walk $(W_j^*)^{-1} W_j^*$ from v'_j to itself has length zero, and thus

$$\begin{aligned} l_W &= l_{W_1 W_2 \ldots W_M} = l_{W_1[(W_1^*)^{-1}W_1^*]W_2[(W_2^*)^{-1}W_2^*]\ldots W_m[(W_m)^{-1}W_m^*]} \\ &= l_{[W_1(W_1^*)^{-1}]W_1^*[W_2(W_2^*)^{-1}]W_2^*\ldots[W_m(W_m)^{-1}]W_m^*} \end{aligned}$$

But notice that $W_1^* W_2^* W_3^* \ldots W_m^*$ is a closed directed walk from v_1 to v_1, while for all $j \in \overline{1,m}$ the walk $W_j(W_j^*)^{-1}$ is either the inverse of a closed directed walk from v'_j to itself, or a walk followed by its inverse. It is known from graph theory that any closed directed walk can be cut and reassembled into cycles through the following algorithm.

- Start the walk, and stop the first time you return to a vertex already visited.
- Cut out the resulting cycle between the two visits of the vertex.
- Repeat the first step with the reduced walk, until you reach the empty walk.

Thus the length of any closed directed walk is in $\mathcal{C}$, and therefore also the length of any closed undirected walk, since

$$\begin{aligned} l_W &= l_{[W_1(W_1^*)^{-1}]W_1^*[W_2(W_2^*)^{-1}]W_2^*\ldots[W_m(W_m)^{-1}]W_m^*} \\ &= l_{W_1^* W_2^* W_3^* \ldots W_m^*} + \sum_{j=1}^{m} l_{W_j(W_j^*)^{-1}}. \end{aligned}$$

In other words, we have $\mathtt{d}(v_1, v_1) \subset d\mathbb{Z}$. As $\mathtt{d}(v_1, v_1)$ is closed under addition and substraction, it has the form $c\mathbb{Z}$ for some multiple $c \in \mathbb{R}$ of d. It is now enough to show that $d \in \mathtt{d}(v_1, v_1)$. By the definition of d, there exist cycles

$\mathscr{C}_1, \ldots, \mathscr{C}_m \in C$ and integers $a_1, \ldots, a_m \in \mathbb{Z}$ such that $d = \sum_{j=1}^{m} a_j \cdot l_{\mathscr{C}_j}$. Denote by v_j^* the starting point of the cycle $\mathscr{C}_j$, and let Y_j be a walk from v_1 to v_j^* ($j \in \overline{1,m}$). Then

$$W := Y_1(\mathscr{C}_1)^{a_1}(Y_1)^{-1}Y_2(\mathscr{C}_2)^{a_2}(Y_2)^{-1}\ldots Y_m(\mathscr{C}_m)^{a_m}(Y_m)^{-1}$$

is a closed walk from v_1 to itself with length d, i.e. $d \in \mathtt{d}(v_1, v_1)$. □

Now let us investigate the behaviour of this distance function under perturbations of the network. Let $\mathcal{H}$ be a strongly connected network, and let its ϵ-perturbation in v_i be $\mathcal{H}^*$, for some $\epsilon \in \mathbb{R}$. Denote by $\mathtt{d}$ the distance function on the former, and by $\mathtt{d}^*$ the one on the latter. Then we have

- $\mathtt{d}^*(v_i, v_j) = \mathtt{d}(v_i, v_j) + \epsilon$ for all $j \neq i$.
- $\mathtt{d}^*(v_a, v_b) = \mathtt{d}(v_a, v_b)$ for all a, b different from i.

We are now ready to prove the following theorem.

Theorem 4.2.2. *Let $\mathcal{M}$ be a strongly connected network with rationally dependent cycle lengths. Then it is perturbation equivalent to a network with rationally dependent edge lengths.*

Proof. The proof is algorithmic, i.e. we provide an algorithm for transforming $\mathcal{M}$ to a network with rationally dependent edge lengths through a sequence of perturbations. Since $\mathcal{M}$ has rationally dependent cycle length, it has a period d.

For every j from 2 to n perturb the current network at v_j such that the new distance function satisfies $\mathtt{d}'(v_1, v_j) \subset d\mathbb{Q}$. This can be done, since before the perturbation we had $\mathtt{d}(v_1, v_j) = a_j + d\mathbb{Z}$, where $a_j \in \mathbb{R}$ is the length of a walk from v_1 to v_j, and so the perturbation parameter ϵ_j has to be chosen such that $a_j - \epsilon_j \in d\mathbb{Q}$.

Since we each time perturb the network at a different vertex, the distance of v_j from v_1 changes only at the $(j-1)$-st step, and therefore the distance function $\mathtt{d}^*$ on the final network $\mathcal{M}^* = (V, \overrightarrow{E}^*)$ satisfies

$$\mathtt{d}^*(v_1, v_j) \subset d\mathbb{Q}$$

for all $j \in \overline{1,n}$. But then all edge lengths are rational multiples of d, since for any edge e_α^* we have

$$l_\alpha^* \in \mathtt{d}^*((e_\alpha^*)^t, (e_\alpha^*)^h) = \mathtt{d}^*((e_\alpha^*)^t, v_1) + \mathtt{d}^*(v_1, (e_\alpha^*)^h) \subset d\mathbb{Q},$$

and so $\mathcal{M}^*$ is a network with rationally dependent edge lengths. □

Thus Theorem 4.1.1 takes the following form.

Theorem 4.2.3. *Let $\mathcal{M}$ be a network with rationally dependent cycle lengths. Then for any initial state $f \in L^1(\mathcal{M})$ on the network $\mathcal{M}$, there exists a unique state $\widetilde{f} \in L^1(\mathcal{M})$ which satisfies the following:*

- $\widetilde{f}^d = \widetilde{f}$
- $\left\| f^t - \widetilde{f}^t \right\|_1 \leq a\rho^t \|f\|_1$ *with suitable constants $a \in \mathbb{R}^+$ and $\rho \in (0,1)$,*

where d is the period of the network. In addition,

$$\left\| \widetilde{f} \right\|_1 \leq \|f\|_1 .$$

To fill in the last gap in the proof of Theorem 4.2.3, let us show that the period d of a network with unit edge lengths as defined in this Chapter coincides with the period of the underlying MC.

The latter was defined by $d' = gcd\{k : \mathbf{P}(X_k = v_i | X_0 = v_i) > 0\}$, where the choice of $i \in \overline{1,n}$ is arbitrary. Since $\mathbf{P}(X_k = v_i | X_0 = v_i) > 0$ is equivalent to the existence of a closed directed walk from v_i to itself of length k (all edges have unit length), we obviously have $d' \in \mathcal{C} = d\mathbb{Z}$, i.e. $d|d'$. But for any cycle $\mathscr{C}$, its length c is an element in the set $\{k : \mathbf{P}(X_k = v_i | X_0 = v_i) > 0\}$ for all $i \in \overline{1,n}$ for which $\mathscr{C}$ passes through v_i. Therefore $d'|c$, and thus also $d'|d$. The obtained identity $d' = d$ then completes the proof of Theorem 4.2.3.

Remark. *We started this chapter by restricting ourselves to networks with edges of strictly positive length, but we made use of networks that do contain such edges (e.g. $\widehat{\mathcal{N}}$) in the previous Chapter to establish our base case. This suggests that this strict positivity condition may be somewhat relaxed. In fact it can be completely removed, as any network can be perturbed to get rid of these edges (this is left as an exercise to the reader).*

Let us again consider the asymptotic mapping

$$\pi : L^1(\mathcal{M}) \to L^1(\mathcal{M}),$$
$$f \mapsto \widetilde{f}.$$

As in Chapter 2, this mapping is a projection to the sublattice $X_d^{\mathcal{M}}$ of $(\mathcal{M}, d)$-invariant vectors in $X := L^1(\mathcal{M})$ which yields a decomposition of X into the direct sum $Ran(\pi) \oplus Ker(\pi)$ of $(T(t))_{t\geq 0}$-invariant sublattices, and we thus obtain the generalisation of Corollary 2.0.2, which corresponds to Proposition 2.4.5. in [5].

Corollary 4.2.4. *Let $\mathcal{M}$ be a network with period d. For the decomposition $X = X_d^{\mathcal{M}} \oplus Ker(\pi)$ we then have*

- *$X_d^{\mathcal{M}}$ and $Ker(\pi)$ are T(t)-invariant subspaces*

- *the operators* $S(t) := T(t)|_{X_d^{\mathcal{M}}}$ *form a bounded* C_0*-group with period* d *on* $X_d^{\mathcal{M}}$ *(*$\|S(t)\|_{X_d^{\mathcal{M}}} = 1$*)*
- *the semigroup* $T(t)|_{Ker(\pi)}$ *is uniformly exponentially stable, and*

$$\|T(t) - S(t) \circ \pi\|_X \leq ae^{t \log \rho}$$

with suitable constants $a \in \mathbb{R}^+$ *and* $\rho \in (0, 1)$.

Now that we know the asymptotic behaviour of these networks, we move on to see how the asymptotic projection π and the decomposition of the state space are linked to the above defined distance function.

4.3 Asymptotic Behaviour and Factor Network

Let $\mathcal{M}$ be a strongly connected network with period d not containing any MDEs, and denote by $\mathtt{d}$ the distance function on the set of vertices of $\mathcal{M}$. Let further v_1 be one of the vertices.

We would like to extend $\mathtt{d}$ to also include points along the edges, whilst conserving the properties listed after Definition 4.2.4. Let $p \in e_\alpha$ be a point on an edge with tail in v_i. The edge being identified with $[0, l_\alpha]$, the point p corresponds to a real number $\mathtt{p} \in [0, l_\alpha]$. Now define the distance between v_i and p as

$$\mathtt{d}(v_i, p) := \mathtt{p} + \mathtt{d}(v_i, v_i)$$

This allows us to extend $\mathtt{d}$ to a function $\mathcal{M} \times \mathcal{M} \to \mathcal{P}(\mathbb{R})$ satisfying

- $\mathtt{d}(p, q) = -\mathtt{d}(q, p)$ for all $p, q \in \mathcal{M}$,
- $\mathtt{d}(p, r) = \mathtt{d}(p, q) + \mathtt{d}(q, r)$ for all $p, q, r \in \mathcal{M}$,
- $\mathtt{d}(p, p) = d\mathbb{Z}$ for all $p \in \mathcal{M}$.

Let $\mathcal{D}$ be a network consisting of a single vertex v and a single loop e of length d (and obviously weight 1). Let $\mathtt{D}$ denote the distance function on $\mathcal{D}$. Consider the mapping $\mathfrak{f}_{\mathcal{M}} : \mathcal{M} \to \mathcal{D}$ defined by

$$\mathfrak{f}_{\mathcal{M}}(p) := \mathtt{p} \quad \text{if} \quad \mathtt{D}(v, \mathtt{p}) = \mathtt{d}(v_1, p)$$

This induces a mapping $\mathfrak{F}_{\mathcal{M}} : L^1(\mathcal{M}) \to L^1(\mathcal{D})$ between the state spaces of the two networks through

$$(\mathfrak{F}_{\mathcal{M}}(f))(\mathtt{p}) := \sum_{p \in \mathfrak{f}_{\mathcal{M}}^{-1}(\mathtt{p})} f(p).$$

This way $\mathcal{D}$ can in fact be considered as a factor network of $\mathcal{M}$ with respect to the equivalence relation induced by $\mathtt{d}$, and — as can be checked — we have

$$\mathfrak{F}_{\mathcal{M}}(f^t) = (\mathfrak{F}_{\mathcal{M}}(f))^t$$

for any state $f \in L^1(\mathcal{M})$. In particular, since all states on $\mathcal{D}$ are $(\mathcal{D}, d)$-invariant,

$$\mathfrak{F}_{\mathcal{M}}(f) = \mathfrak{F}_{\mathcal{M}}(f^d).$$

The map $\mathfrak{F}_{\mathcal{M}}$ is clearly linear and bounded, since $\|\mathfrak{F}_{\mathcal{M}}(f)\|_1 \leq \|f\|_1$ for all $f \in L^1(\mathcal{M})$ with equality for all positive functions, and as such is a continuous linear operator with norm 1. We know that for any state $f \in L^1(\mathcal{M})$ we have $\|f^t - (\pi(f))^t\|_1 \stackrel{t\to\infty}{\longrightarrow} 0$. Thus, taking the time sequence $t = m \cdot d$ ($m \in \mathbb{N}$), we obtain from the previous equality that $\pi(f)$ has to satisfy

$$\mathfrak{F}_{\mathcal{M}}(\pi(f)) = \mathfrak{F}_{\mathcal{M}}(f).$$

Definition 4.3.1. This bounded linear operator $\mathfrak{F}_M : L^1(\mathcal{M}) \to L^1(\mathcal{D})$ is called the *factor mapping* belonging to $\mathcal{M}$.

Our goal is to show that the restricted mapping $\mathfrak{F}_{\mathcal{M}}|_{X_d^{\mathcal{M}}}$ is in fact a bijection.

Let $\mathcal{H}$ be an arbitrary network with period d, $\mathcal{H}^*$ an $\epsilon > 0$ perturbation of this network at some vertex different from v_1, and let $\mathcal{G}$ and $\mathcal{G}^*$ be their respective factor networks. Let the state space mappings induced by the factor mappings be called $\mathfrak{F}_H$ and $\mathfrak{F}_{H^*}$, respectively. Both factor networks consist of a single vertex and a single loop of length d. Thus we have a natural identification $\mathfrak{I}$ of the state spaces $L^1(\mathcal{G})$ and $L^1(\mathcal{G}^*)$. Also, as seen at the end of Section 4.1, the perturbation induces a bijection $* : X_d^{\mathcal{H}} \to X_d^{\mathcal{H}^*}$. It can then easily be verified that the diagram

$$\begin{array}{ccc} X_d^{\mathcal{H}} & \xrightarrow{\;*\;} & X_d^{\mathcal{H}^*} \\ \downarrow{\scriptstyle \mathfrak{F}_H} & & \downarrow{\scriptstyle \mathfrak{F}_{H^*}} \\ L^1(\mathcal{G}) & \xrightarrow{\;\mathfrak{I}\;} & L^1(\mathcal{G}^*) \end{array} \qquad (4.1)$$

is commutative.

Now let us get back to our network $\mathcal{M}$. Since it is a network with well-defined period, it is perturbation equivalent to a network $\mathcal{N}'$ with rationally dependent non-zero edge lengths. This network can then be turned into a network $\mathcal{N}$ with equal edge lengths by adding finitely many flow-through vertices along its edges. We may in addition suppose that none of the perturbations were made at v_1 (cf. the proof of Theorem 4.2.2). The successive perturbations induce a bijection

$$* : X_d^{\mathcal{M}} \to X_d^{\mathcal{N}'}.$$

Denote by $\mathfrak{F}_{\mathcal{M}}$, $\mathfrak{F}_{\mathcal{N}'}$ and $\mathfrak{F}_{\mathcal{N}}$ the factor mappings of the state spaces of the respective networks. The factor networks of $\mathcal{M}$, $\mathcal{N}'$ and $\mathcal{N}$ have — as seen above

— a natural identification, and so have their state spaces. The state spaces $X_d^{\mathcal{N}'}$ and $X_d^{\mathcal{N}}$ also have a natural identification $\mathfrak{Id}$. The network $\mathcal{N}'$ can be obtained from $\mathcal{M}$ with successive perturbations, and thus the commutativity of Diagram (4.1) implies the commutativity of the next diagram.

$$\begin{array}{ccccc} X_d^{\mathcal{M}} & \xrightarrow{\;*\;} & X_d^{\mathcal{N}'} & \xrightarrow{\;\mathfrak{Id}\;} & X_d^{\mathcal{N}} \\ \Big\downarrow {\scriptstyle \mathfrak{F}_{\mathcal{M}}} & & \Big\downarrow {\scriptstyle \mathfrak{F}_{\mathcal{N}'}} & & \Big\downarrow {\scriptstyle \mathfrak{F}_{\mathcal{N}}} \\ L^1(\mathcal{D}) & \xrightarrow{\;Id\;} & L^1(\mathcal{D}) & \xrightarrow{\;Id\;} & L^1(\mathcal{D}) \end{array} \tag{4.2}$$

It may also be supposed that $\mathcal{N}$ has unit edge lengths, otherwise we just rescale time. Also, it has no MDEs, since $\mathcal{M}$ had none. Denote then by P the transposed transition matrix of the underlying MC, and by $\mathtt{d}'$ the distance function on $\mathcal{N}$. Take an arbitrary state $g \in X_d^{\mathcal{N}}$, and consider the transformed network $\widehat{\mathcal{N}}$ (see Chapter 2) with the corresponding state $G \in X_d^{\widehat{\mathcal{N}}}$. g being $(\mathcal{N}, d)$-invariant, it satisfies

$$w_\beta \cdot g_\alpha = w_\alpha \cdot g_\beta$$

whenever $e_\alpha^t = e_\beta^t$. If $Q_{\mathcal{N}}$ denotes the transposed weighted outgoing incidence matrix of the underlying graph, we then have

$$g = Q_{\mathcal{N}} G,$$

and so this transformation yields a bijection $\mathfrak{n} : X_d^{\mathcal{N}} \to X_d^{\widehat{\mathcal{N}}}$, $\mathfrak{n}(g) := G$. Let us now consider the partition of the set $\overline{1,n}$ into the sets

$$I_b := \{i : b \in \mathtt{d}'(v_1, v_i)\},$$

and the corresponding partition of the set of vertices V into the sets

$$V_b := \{v_i : i \in I_b\}$$

for $b \in \overline{0, d-1}$ (see Section 1.1, after Definition 1.1.4). We then have

$$\mathfrak{F}_{\mathcal{N}}(g)|_{[b,b+1]} = \sum_{\{\beta : e_\beta^t \in V_b\}} g_\beta = \sum_{i \in I_b} G_i.$$

But $G \in X_d^{\widehat{\mathcal{N}}} \subset L^1([0,1], \mathbb{C}^n)$ is $(\widehat{\mathcal{N}}, d)$-invariant, meaning that

$$P_b((G)_{I_b}) = (G)_{I_b},$$

where $P_b := (P^d)_{I_b \times I_b}$. According to Theorem 1.1.1 there is a unique probability vector $\pi_{P_b} \in \mathbb{C}^{|I_b|}$ such that $P\pi_{P_b} = \pi_{P_b}$, and any P_b-invariant column vector is a multiple of π_{P_b}. $(G)_{I_b}(x)$ being P_b-invariant for a.e. $x \in [0,1]$, we then obtain

$$(G)_{I_b} = h_b \pi_{P_b}$$

for some $h_b \in L^1([0,1])$. But then

$$h_b = \sum_{j \in I_b} h_b (\pi_{P_b})_j = \sum_{j \in I_b} G_j = \mathfrak{F}_{\mathcal{N}}(g)|_{[b,b+1]}.$$

We have thus obtained the following:

$$(G)_{I_b} = \mathfrak{F}_{\mathcal{N}}(g)|_{[b,b+1]} \pi_{P_b},$$

i.e. that G is uniquely determined by $\mathfrak{F}_{\mathcal{N}}(g)$. $\mathfrak{n}$ being a bijection, this means that g is also uniquely determined by $\mathfrak{F}_{\mathcal{N}}(g)$. $\mathfrak{F}_{\mathcal{N}} : X_d^{\mathcal{N}} \to L^1(\mathcal{D})$ is thus a bijection.

The flow semigroup $(R(t))_{t \geq 0}$ on the network $\mathcal{D}$ can be identified with the rotation semigroup

$$(r(t)g)(h) := g(e^{-2\pi i \frac{t}{d}} h)$$

on $L^1(\Gamma_d)$ where

$$\Gamma_d := \{\gamma \in \mathbb{C} : |\gamma| = \frac{d}{2\pi}\}$$

through the bijection

$$\mathcal{D} \to \Gamma_d$$
$$\mathtt{p} \mapsto \tfrac{d}{2\pi} e^{2\pi i \frac{\omega}{d}} \quad \text{iff} \quad \omega \in \mathtt{D}(v, \mathtt{p}) \ .$$

With the definition

$$R(-t) := R(t)^{-1}$$

for $t \geq 0$ we obtain the rotation group $(R(t))_{t \in \mathbb{R}}$.

We can now state our main theorem, in which the connection between the asymptotic mapping and the network factorisation is made clear, hereby answering a question raised by Prof. R. Nagel.

Theorem 4.3.1. *For the ergodic network $\mathcal{M}$ with period d, the flow semigroup $(T(t))_{t \geq 0}$ on $\mathcal{M}$, the decomposition $X = X_d^{\mathcal{M}} \oplus Ker(\pi)$, the factor mapping $\mathfrak{F}_{\mathcal{M}} : L^1(\mathcal{M}) \to L^1(\mathcal{D})$ and the rotation group $(R(t))_{t \in \mathbb{R}}$ satisfy*

- *$X_d^{\mathcal{M}}$ and $Ker(\pi)$ are $T(t)$-invariant subspaces*
- *the semigroup $T(t)|_{Ker(\pi)}$ is uniformly exponentially stable, and*

$$\|T(t) - S(t) \circ \pi\|_X \leq a e^{t \log \rho}$$

with suitable constants $a \in \mathbb{R}^+$ and $\rho \in (0,1)$

- *the operators* $S(t) := T(t)|_{X_d^{\mathcal{M}}}$ *form a bounded* C_0*-group with period* d *on* $X_d^{\mathcal{M}}$ *(*$\|S(t)\|_{X_d^{\mathcal{M}}} = 1$*)*
- *the factor mapping* $\mathfrak{F}_{\mathcal{M}}$ *is a bijection between the spaces* $X_d^{\mathcal{M}}$ *and* $L^1(\mathcal{D})$ *for which*

$$\begin{array}{ccc} X_d^{\mathcal{M}} & \xrightarrow{\mathfrak{F}_{\mathcal{M}}} & L_1(\mathcal{D}) \\ \Big\downarrow S(t) & & \Big\downarrow R(t) \\ X_d^{\mathcal{M}} & \xrightarrow{\mathfrak{F}_{\mathcal{M}}} & L_1(\mathcal{D}) \end{array} \tag{4.3}$$

is commutative, and so the group $(S(t))_{t\in\mathbb{R}}$ *is isomorphic to the rotation group* $(R(t))_{t\in\mathbb{R}}$ *through the bijection* $\mathfrak{F}_{\mathcal{M}}$*.*

Also, the asymptotic mapping π *and the factor mapping* $\mathfrak{F}_{\mathcal{M}}$ *satisfy*

$$\pi = (\mathfrak{F}_{\mathcal{M}}|_{X_d^{\mathcal{M}}})^{-1} \circ \mathfrak{F}_{\mathcal{M}}.$$

Proof. The first three are taken from Corollary 4.2.4. The bijectivity of $\mathfrak{F}_{\mathcal{M}}|_{X_d^{\mathcal{M}}}$ follows from the commutativity of Diagram (4.2) and the bijectivity of $\mathfrak{F}_{\mathcal{N}}|_{X_d^{\mathcal{N}}}$. The commutativity of Diagram (4.3) and the relation between π and $\mathfrak{F}_{\mathcal{M}}$ then follow from the property $\mathfrak{F}_{\mathcal{M}}(f^t) = (\mathfrak{F}_{\mathcal{M}}(f))^t$.

□

The only remaining case is that of networks in which the cycle lengths are not rationally dependent. Our preceding treatment of networks relied heavily on the existence of the period and through it of the distance function. But when the cycle lengths are not rationally dependent, application of finite MC theory seems impossible. Instead, we will now cite a result from [5] (Theorem 2.5.2), and then reformulate it to a limit theorem for probability distributions.

Theorem 4.3.2. *Let* $\mathcal{N}$ *be an ergodic network with rationally independent cycle lengths. Then* $T(t)$ *converges strongly, but not uniformly, to the projection on* $X = L^1(\mathcal{N})$ *to the one dimensional space* X_0 *of stationary states.*

Let us now determine what this space X_0 is. Notice that if $g \in L^1(\mathcal{N})$ is a fixed state, then it is constant on each of the edges e_α ($\alpha \in \overline{1,k}$), and these constants $g_\alpha \in \mathbb{C}$ satisfy the boundary conditions imposed by the Kirchhoff Law, i.e.

$$g_\beta = w_\beta \sum_{\{\alpha : e_\alpha^h = e_\beta^t\}} g_\alpha$$

for all $\beta \in \overline{1,k}$. Let

$$G_i := \sum_{\{\alpha : e_\alpha^t = v_i\}} g_\alpha.$$

Then the vector $(G_i)_{i\in\overline{1,n}} \in \mathbb{C}^n$ satisfies

$$g = QG$$

where Q is the transposed weighted outgoing incidence matrix of the underlying graph. Since g is a fixed state of the network, G is a fixed (column) vector of P, where P is the transposed transition matrix corresponding to the network, and is thus a multiple of $\pi_{\mathcal{P}}$. If we denote by $\mathbf{1}_{\mathcal{N}}$ the constant 1 function on $\mathcal{N}$, we have

$$G_i = \frac{\int_{\mathcal{N}} g}{\|\mathbf{1}_{\mathcal{N}}\|_1} \pi_i (\mathbf{1}_{\mathcal{N}})_i$$

for all $i \in \overline{1,n}$. Thus the one dimensional subspace X_0 of stationary states on $\mathcal{N}$ is generated by the unique stationary probability density function on $\mathcal{N}$

$$\sigma_{\mathcal{N}} := QS_{\mathcal{N}},$$

where

$$S_{\mathcal{N}} := \left(\frac{1}{\|\mathbf{1}_{\mathcal{N}}\|_1} \pi_i (\mathbf{1}_{\mathcal{N}})_i \right)_{i\in\overline{1,n}}.$$

Reformulated to the stochastic process $\{\mathcal{F}_t\}$ (see Section 1.3), the above theorem then yields:

Theorem 4.3.3. *Let $\mathcal{N}$ be an ergodic network with rationally independent cycle lengths. Suppose that the random variable $\mathcal{F}_0$ has an arbitrary absolute continuous distribution on $\mathcal{N}$. Then the density functions $(f^t)_{t\geq 0}$ converge to the unique stationary probability density function $\sigma_{\mathcal{N}}$.*

CONCLUSION

The probabilistic approach to flows allowed us to give a more explicit description of the flow semigroup, simplifying the study of its asymptotics. We have seen that the asymptotic behaviour of a flow on an ergodic network $\mathcal{N}$ is very different depending on whether the (physical, not graph theoretical) length of the cycles in the network are rationally dependent or not. We showed that if they are rationally dependent, the flow converges exponentially to a rotation semigroup with period d equal to the greatest common divisor of the cycle lengths. We found that the asymptotic mapping π that maps the initial states to the d-periodic states they converge to is in fact a projection to the space of d-periodic states. With the help of a set-valued distance function defined on the network we could create a factor network $\mathcal{D}$ that fully captures the asymptotic behaviour of the original network, as the asymptotic flow on $\mathcal{N}$ is isomorphic to the flow on $\mathcal{D}$ through the factor mapping $\mathfrak{F}_{\mathcal{N}}$ between the state spaces $L^1(\mathcal{N})$ and $L^1(\mathcal{D})$. This mapping is a bijection when restricted to the space of d-periodic states on $\mathcal{N}$, and it turns out that we have

$$\pi = (\mathfrak{F}_{\mathcal{M}}|_{X_d^{\mathcal{M}}})^{-1} \circ \mathfrak{F}_{\mathcal{M}},$$

meaning that the direct sum decomposition $L^1(\mathcal{N}) = X_d^{\mathcal{M}} \oplus Ker(\pi)$ induced by π can be viewed as stemming from the factorisation $\mathfrak{f}_{\mathcal{N}}$ of the network $\mathcal{N}$. The more explicit description of the flow semigroup also allowed us to treat the question of vertex control of networks with unit edge-lengths, showing that — provided we start with zero initial state — any asymptotically reachable state is in fact exactly reachable, and that within time n, where n is the number of vertices in our network.

The presented probabilistic approach can be further developed to allow the treatment of non-ergodic networks, but also some classes of infinite networks. An interesting additional question is whether the stationary asymptotic behaviour of ergodic networks with rationally independent cycle lengths can somehow be obtained from the rationally dependent case through approximation and limit-transition arguments.

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