Eötvös Loránd University
Faculty of Science

Master's Thesis

The chromatic polynomial

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Abstract

After introducing the concept of the chromatic polynomial of a graph, we describe its basic properties and present a few examples. We continue with observing how the coefficients and roots relate to the structure of the underlying graph, with emphasis on a theorem by Sokal bounding the complex roots based on the maximal degree. We also prove an improved version of this theorem. Finally we look at the Tutte polynomial, a generalization of the chromatic polynomial, and some of its applications.

Acknowledgements

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Contents

Chapter 0

Introduction

George David Birkhoff introduced the chromatic polynomial in 1912 as an attempt to prove the four color theorem [4]. He noticed that the number of ways a certain map can be painted with at most k colors exhibits polynomial dependence on k. This observation made it possible to find or preclude some roots using algebraic and analytic methods and draw the corresponding conclusion about k-colorability. Today we usually define the chromatic polynomial for arbitrary graphs as extended by H. Whitney in 1932 [22][23].

A fundamental property of the chromatic polynomial is that it can be reduced to that of two slightly smaller graphs, those resulting from the deletion and the contraction of and edge e respectively. This operation, defined in section 1.3, gives us an algorithm to recursively calculate the chromatic polynomial for any graph, and explicit formulae for some special classes of graphs.

There are several well-known results about the roots, coefficients and substitutions of the chromatic polynomial being linked to some graph theoretic properties of G. We'll explore a few of these relations in chapter 2.

The study of chromatic polynomials, as a nearly hundred years old area of algebraic graph theory, is sustained by continuous development. For example, while it has been long known that integer roots of the chromatic polynomial are bounded by the maximal degree of G, a recent result of Sokal [18], motivated by statistical mechanics, has proved an analogous claim for all complex roots. Chapter 3 is devoted to presenting an improved version of Sokal's theorem.

Finally we'll look at an extension of the chromatic polynomial defined by W. T. Tutte in 1954. The dichromate, now called the Tutte polynomial of G, is the most general graph invariant that satisifes the aforementioned deletion-contraction recurrence, with notable applications in physics, topology, probability theory and of course combinatorics. Some of these applications are examined in chapter 4.

As with any sufficiently broad subject, it is impossible to do anything more than just scratching the surface. We will try, however, to introduce the reader into the basics of chromatic polynomials, providing pointers to more detailed descriptions where appropriate.

Chapter 1

Preliminaries

After briefly recalling the concept of vertex coloring we define the chromatic polynomial and introduce some of its basic properties. We also calculate the chromatic polynomial for some special classes of graphs as examples. Most, if not all, of the material in this chapter can be found in [3] or [9].

1.1 Graph coloring

Graph coloring, or more specifically *vertex coloring* means the assignment of colors to the vertices of a graph in such a way that no two adjacent vertices share the same color.

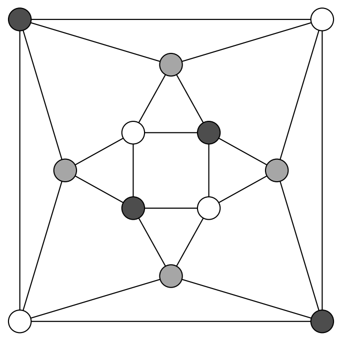

Figure 1: 3-coloring of the cuboctahedral graph

This definition allows us to use a separate color for each vertex. From a mathematical perspective graph coloring is only interesting if we restrict the permissible colors to a fixed finite set $\mathcal{S}$. It is easy to see that the choice of actual colors is irrelevant, and therefore any graph property related to coloring may only depend on the cardinality $|\mathcal{S}| = k$. We may as well label the nodes using the numbers 1, 2, ..., k.

Formally, a *k-coloring* of a graph G is a function $\sigma : V(G) \to \{1, 2, \ldots, k\}$ which satisfies $\sigma(i) \neq \sigma(j)$ for any edge $e = ij$. Note that it is not compulsory to use all the colors. The graph is said to be *k-colorable* if such a function exists. The *chromatic number* $\chi(G)$ is the minimal k for which the graph is k-colorable, and we say that G is *k-chromatic* if $\chi(G) = k$.

A graph containing a loop cannot be properly colored while multiple edges don't add any additional restriction on the coloring. Therefore we'll assume that the graphs being examined are simple until we return to multigraphs in chapter 4.

Vertex coloring is a central concept of graph theory having a large number of well-known facts and theorems. Let's recall a few of them:

- A graph is 2-colorable, also called *bipartite*, if and only if it contains no odd cycle. This property can be polynomially checked e.g. by using breadth-first search.
- Deciding 3-colorability (or k-colorability for any $k \geq 3$) is NP-complete and finding the chromatic number is #P-complete. In laymen's terms it means that that it is practically infeasible to calculate them in an efficient way.
- The four-color theorem states that every planar graph is 4-colorable.
- A bounded degree graph can be greedily colored using $D+1$ colors where D denotes the maximal degree. Brooks' theorem tells us that for all simple graphs except complete graphs and odd cycles D colors are also sufficient.
- A graph containing a clique of size k needs at least k colors. This statement cannot be reversed: Mycielski's construction reveals that there are even triangle-free graphs with an arbitrarily high chromatic number.

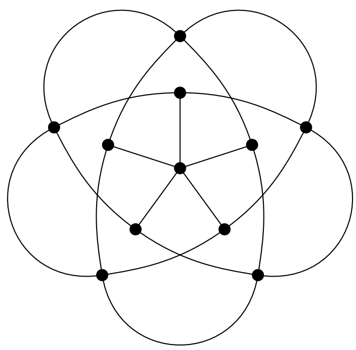

Figure 2: the Grötzsch graph (contains no triangle but requires four colors)

- Graphs whose induced subgraphs have an equal chromatic and clique number are called *perfect graphs*. Bipartite graphs and their line graphs, as well as the complements of perfect graphs are perfect. In general, they can be recognized by the absence of odd cycles of length ≥ 5 as induced subgraphs in G or its complement. Perfectness can be checked in polynomial time and the chromatic number of perfect graphs can also be efficiently determined.

1.2 The chromatic polynomial

Consider the number of different k-colorings of a given graph G as a function of k, and denote it by $\mathsf{chr}(G, k)$.

Theorem 1.1. *$\mathsf{chr}(G, k)$ is a polynomial of k.*

Proof. For any coloring of G the nonempty color classes constitute a partition of $V(G)$ where each part is a stable vertex set. We may count those colorings that give a certain partition and add them up for all such partitions to find the total number of colorings. Since $V(G)$ is a finite set, it has a finite number of partitions, so it is sufficient to show that the number of colorings for a single partition is a polynomial of k.

Fix a partition with p parts, each of them being a stable set. By assigning a different color to each part, we get all the colorings belonging to the partition. We may pick the first color in k possible ways, the second in $k-1$ ways, etc. so there are $k(k-1)\dots(k-p+1)$ colorings, which is obviously a polynomial. Note that this also works when $k < p$. $\square$

Corollary 1.2. *$\mathsf{chr}(G, k)$ has a degree of $n = |V(G)|$.*

Proof. There is no partition with more than n parts and only a single partition with exactly n parts. For this partition, the number of colorings is a polynomial of degree n while for all other partitions it has a degree $< n$. The sum of such polynomials is one of degree n. $\square$

Having established this fact, we may call $\mathsf{chr}(G, k)$ the *chromatic polynomial* of G.

Despite being an elementary structure in algebra, polynomicity implies several nice properties. First, we are no longer bound to evaluate it at positive integers: it is possible to substitute any $q \in \mathbb{C}$. They no longer carry the meaning of the number of possible q-colorings, but for certain values they still do have some meaning as we'll see later. Another advantage is that we can examine the polynomial's coefficients and roots and connect them to graph properties and invariants.

1.3 Deletion-contraction property

How do we determine the chromatic polynomial of a given graph?

One way would be to iterate through all hypothetic k-colorings, count the valid ones and use interpolation to reconstruct the polynomial. This is quite cumbersome, however. Note that we can't expect miracles: if 3-colorability was NP-complete, calculating the chromatic polynomial of a general graph won't be easier. But we might find a way to work directly with the polynomial so that we can build on our previous results and also have claims about some graph families.

The idea is simple. Select two vertices i and j from $V(G)$ with no edge between them. We may classify colorings into the following two classes:

- those with i and j colored differently, and
- those with i and j having the same color.

The first class corresponds to the colorings of the graph with an edge added between i and j, that is, $G + ij$. This edge ensures that the colors assigned to the two vertices are indeed different. The second class can be similarly mapped to the colorings of the graph where i and j are unified into a single vertex, thus being forced to have the same color. This latter method is also called the contraction of $\{i, j\}$ and denoted by G/ij.

Expressing this observation with a formula yields

$$\mathsf{chr}(G, k) = \mathsf{chr}(G + ij, k) + \mathsf{chr}(G/ij, k) \ . \tag{1.1}$$

Usually we apply this identity in reverse. For an edge $e = ij$, substitute $G \backslash e$ into G and rearrange the equation so that we obtain

$$\mathsf{chr}(G, k) = \mathsf{chr}(G \backslash e, k) - \mathsf{chr}(G/e, k) \ . \tag{1.2}$$

Note that in the current form we have a relation between these chromatic polynomials evaluated at some positive integer k. However, since two degree n polynomials agreeing on $n + 1$ points are identical, the same expression also holds for the polynomial itself. We'll omit this kind of reasoning in the future.

Since both $G \backslash e$ and G/e have fewer edges than G, we may apply this observation to facilitate induction or recursion for statements about the chromatic polynomial. This method will prove quite powerful to be used several times, so we'll refer to it as the *deletion-contraction argument*.

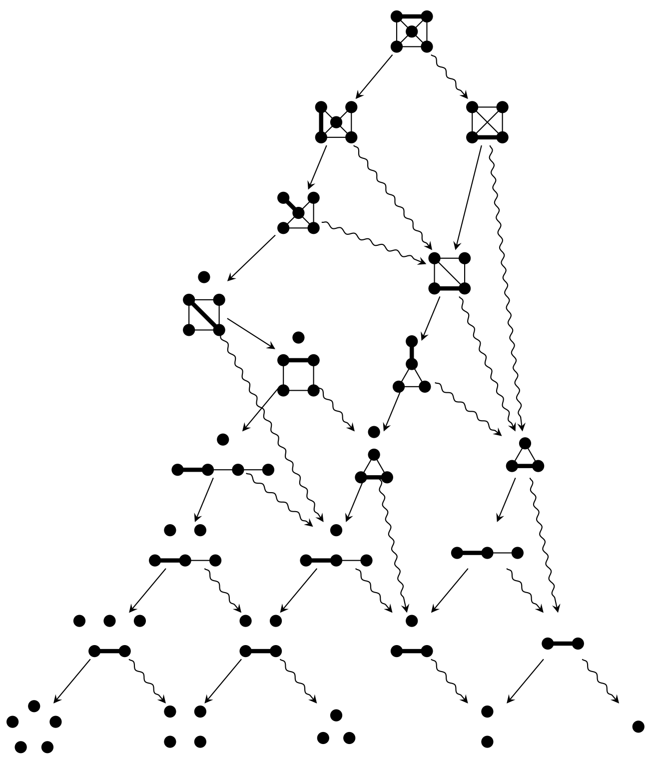

Figure 3: recursion by edge deletion and contraction

For example, it gives an alternate proof for $\mathsf{chr}(G, k)$'s polynomicity. Perform induction by the number of edges. For an edgeless graph all the k^n colorings are permissible and thus the claim is true. Otherwise $\mathsf{chr}(G, k)$ can be written as the difference of two terms which by induction are polynomials. Therefore $\mathsf{chr}(G, k)$ is also a polynomial.

In practice the chromatic polynomial of a general graph is usually calculated by this recursion. It's rather slow, having an asymptotic runtime of $O(\varphi^{n+m})$ where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, and n and m denote the number of nodes and edges respectively. So a number of heuristics are used to speed it up. These include the separate handling of sparse and dense graphs, removing edges in the former case and adding them in the latter, and isomorphism rejection techniques to avoid handling duplicate cases more than once.

Remark 1.1. Some recent advances made it possible to calculate the chromatic polynomial in vertex-exponential time. For more details, see [6] and [5]. Also note that there exist polynomial time algorithms for some special classes of graphs such as chordal graphs and graphs having bounded clique-width.

1.4 Examples

Let's calculate the chromatic polynomial for some specific graph families.

Claim 1.3. *The chromatic polynomial of the empty graph on n vertices is*

$$\mathsf{chr}(\overline{K_n}, k) = k^n \ . \tag{1.3}$$

Proof. It is enough to verify the claim for $k \in \mathbb{N}$. Each of the n vertices can be independently colored using any of the k colors, which gives a total of k^n possibilities. □

Claim 1.4. *The chromatic polynomial of the complete graph is*

$$\mathsf{chr}(K_n, k) = k(k-1)\dots(k-n+1) \ . \tag{1.4}$$

Proof. No two vertices can share the same color. Ordering them arbitrarily the first one can be assigned any of the k colors, the second one can be assigned $k-1$ colors regardless of our choice for the first, the third one can get $k-2$ colors, etc. This is essentially the same argument as in section 1.2 for partitions. □

Claim 1.5. *Any tree T_n on n vertices has a chromatic polynomial of*

$$\mathsf{chr}(T_n, k) = k(k-1)^{n-1} \ . \tag{1.5}$$

Proof. Apply induction by n. For a single vertex, the statement is true. In the general case, select a leaf, detach it and use induction for the remnant of the tree. It can be colored in $k(k-1)^{n-2}$ distinct ways, while the selected leaf can be assigned any color except that of its neighbor's, resulting in $k-1$ possibilities. □

Remark 1.2. Graphs sharing the same chromatic polynomial are called *chromatic equivalents*. We've just established that all trees on n vertices are chromatically equivalent.

Claim 1.6. *The cycle of length n has the chromatic polynomial*

$$\mathsf{chr}(C_n, k) = (k-1)^n + (-1)^n(k-1) \ . \tag{1.6}$$

Proof. Apply induction based on the deletion-contraction argument. For $n = 3$ we already know the claim since $C_3 = K_3$. Deleting an edge from a cycle results in a path, which is actually a tree and therefore its chromatic polynomial is $k(k-1)^{n-1}$. Contracting an edge yields a cycle of length $n-1$ which by the inductive hypothesis has a chromatic polynomial of $(k-1)^{n-1}+(-1)^{n-1}(k-1)$. The difference is $k(k-1)^{n-1}-(k-1)^{n-1}-(-1)^{n-1}(k-1) = (k-1)^n + (-1)^n(k-1)$. □

Remark 1.3. Cycles are the only graphs having the chromatic polynomial in (1.6). Such graphs that are unambiguously characterized by their chromatic polynomials are called *chromatically unique.* Thus the cycle graph, as well as the aforementioned empty and complete graphs are chromatically unique. The question of chromatic equivalence and uniqueness are often studied together as chromaticity.

Claim 1.7. *The wheel graph on $n+1$ vertices has the chromatic polynomial*

$$\mathsf{chr}(W_{n+1}, k) = k(k-2)^n + (-1)^n k(k-2) \ . \tag{1.7}$$

Proof. If we assign an arbitrary color to the central vertex, the outer cycle has to be colored using the remaining $k-1$ ones. The previous claim tells us that it can be accomplished in $(k-2)^n + (-1)^n(k-2)$ different ways. Multiplication by k gives us the desired result. □

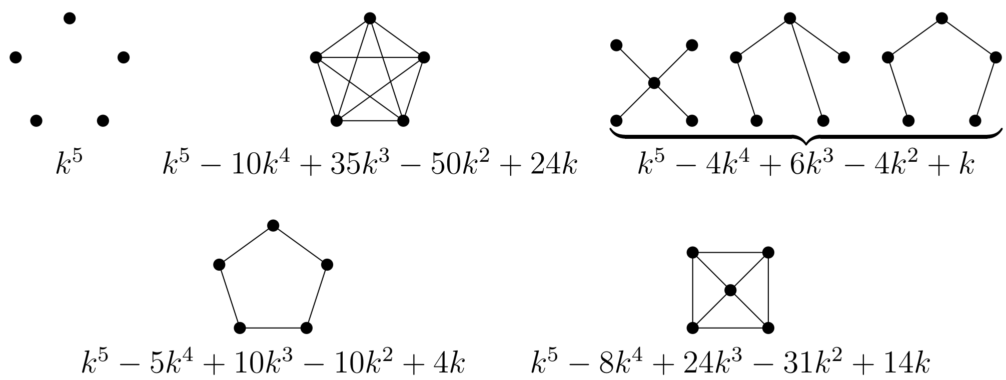

Figure 4: the chromatic polynomial for some special graphs on five vertices

And so on. Similar reasoning can be used to calculate the chromatic polynomial for some other classes such as ladders, prisms, interval graphs or complete bipartite graphs. For more examples see [15].

1.5 Constructions

Claim 1.8. *Suppose G and H are two disjoint graphs, i.e. $V(G) \cap V(H) = \varnothing$. Then*

$$\mathsf{chr}(G \cup H, k) = \mathsf{chr}(G, k)\mathsf{chr}(H, k) \ . \tag{1.8}$$

Proof. We may color the vertices of G and H independently of each other. $\square$

Claim 1.9. *If G and H share a single vertex, i.e. $V(G) \cap V(H) = \{v\}$, then*

$$\mathsf{chr}(G \cup H, k) = \frac{\mathsf{chr}(G, k)\mathsf{chr}(H, k)}{k} \; . \tag{1.9}$$

Proof. The number of colorings of H where v has a certain color cannot depend on the choice of this color because of formal symmetry (permutation of colors). So this number has to be $\frac{\mathsf{chr}(H,k)}{k}$. We may extend each of the $\mathsf{chr}(G, k)$ colorings of G in that many ways, resulting in $\frac{\mathsf{chr}(G,k)\mathsf{chr}(H,k)}{k}$ colorings for $G \cup H$. $\square$

Claim 1.10. *If the intersection of G and H is a complete graph K_m then*

$$\mathsf{chr}(G \cup H, k) = \frac{\mathsf{chr}(G, k)\mathsf{chr}(H, k)}{\mathsf{chr}(G \cap H, k)} \; . \tag{1.10}$$

Proof. Since $V(G) \cap V(H)$ spans a complete subgraph in H, its vertices must be always colored differently. Permuting the colors shows that any such assignment can be completed in the same number of ways, namely $\frac{\mathsf{chr}(H,k)}{k(k-1)\ldots(k-m+1)} = \frac{\mathsf{chr}(H,k)}{\mathsf{chr}(G\cap H,k)}$. The total number of colorings is therefore $\mathsf{chr}(G \cup H, k) = \frac{\mathsf{chr}(G,k)\mathsf{chr}(H,k)}{\mathsf{chr}(G\cap H,k)}$. $\square$

Remark 1.4. Graphs that can be decomposed into such G and H are called *quasi-separable* [3].

The *join* $G + H$ of two distinct graphs G and H is defined by connecting each vertex in G to each vertex in H. In other words, the complement of $G + H$ is the disjoint union of the complements of G and H. The chromatic polynomial of $G + H$ can be determined from those of G and H, but this time we'll need a more complicated operation.

Recall our partitioning argument from section 1.2. We have shown that the number of colorings belonging to a partition with p parts is determined by the *falling factorial*

$$(k)_p = k(k-1)\ldots(k-p+1) \; . \tag{1.11}$$

We have also established that if the number of valid partitions having p parts is a_p then the chromatic polynomial equals

$$\sum_p a_p (k)_p \; . \tag{1.12}$$

Now observe that no two points from G and H can go into the same part in any partition since they are interconnected with an edge. Thus the partitions of $G + H$ with p parts can always be subdivided into one of G and another of H having a total of p parts.

Symbolically it means that if

$$\mathsf{chr}(G, k) = \sum_p a_p (k)_p \quad \text{and} \quad \mathsf{chr}(H, k) = \sum_p b_p (k)_p \tag{1.13}$$

then

$$\mathsf{chr}(G+H,k) = \sum_p \left(\sum_i a_i b_{p-i} \right) (k)_p \ . \tag{1.14}$$

This operation is called the *umbral product* of the two polynomials, denoted by $\mathsf{chr}(G,k) \circ \mathsf{chr}(H,k)$. We have just proved that:

Theorem 1.11. *$\mathsf{chr}(G+H,k) = \mathsf{chr}(G,k) \circ \mathsf{chr}(H,k)$.* □

Usually, however, the chromatic polynomial is not written using the falling factorials but the powers of k. To convert them into each other, we'll need some elementary combinatorics:

$$(x)_n = \sum_{k=0}^{n} s(n,k) x^k \quad \text{and} \quad x^n = \sum_{k=0}^{n} S(n,k)(x)_k \ , \tag{1.15}$$

where $s(n,k)$ and $S(n,k)$ are the Stirling numbers of the first and second kind respectively. Thus the umbral product can be calculated using the following formula:

$$\left(\sum_k a_k x^k \right) \circ \left(\sum_k b_k x^k \right) = \sum_k \left[\sum_l s(l,k) \sum_i \left(\sum_j S(j,i) a_j \right) \left(\sum_j S(j,l-i) b_j \right) \right] x^k \tag{1.16}$$

Note that we needed the both polynomials $\mathsf{chr}(G,k)$ and $\mathsf{chr}(H,k)$ in their entirety to calculate the result even for a single k.

Corollary 1.12. *The chromatic polynomial of the complete bipartite graph is*

$$\mathsf{chr}(K_{n,m},q) = \sum_k \left[\sum_l s(l,k) \sum_i S(n,i) S(m,l-i) \right] q^k \ , \tag{1.17}$$

or equivalently,

$$\mathsf{chr}(K_{n,m},q) = \sum_k S(m,k) q(q-1)\dots(q-k+1)(q-k)^n \ . \tag{1.18}$$

Proof. Use $K_{n,m} = \overline{K_n} + \overline{K_m}$ and substitute into (1.16). □

Chapter 2

Some algebraic properties of the chromatic polynomial

In this chapter we'll summarize some well-known facts about the chromatic polynomial's coefficients, roots and substitutions as well as their relations to some graph-theoretic properties of G. Further details on this subject are available in [9] and [15].

2.1 Coefficients

Claim 2.1. *The lead coefficient of* $\mathsf{chr}(G, k)$ *is always 1.*

Proof. Use the partitioning argument from section 1.2. The only partition that contributes to the lead coefficient is the one with n parts, giving an addend of $k(k-1)\ldots(k-n+1)$ where the coefficient of k^n is 1. $\square$

Alternate proof. For $k \in \mathbb{N}$, each k-coloring of K_n is also valid for G while all k-colorings of G are permissible for $\overline{K_n}$. Therefore $\mathsf{chr}(K_n, k) \leq \mathsf{chr}(G, k) \leq \mathsf{chr}(\overline{K_n}, k)$. We have calculated these bounds in section 1.4, so we know that both of them are $\Theta(k^n)$, thus $\mathsf{chr}(G, k)$ is also $\Theta(k^n)$. For a polynomial it means that the lead coefficient is 1. $\square$

Claim 2.2. *The coefficient of* k^{n-1} *in* $\mathsf{chr}(G, k)$ *is the negative of the number of edges.*

Proof. Apply the deletion-contraction argument. By contracting an edge we obtain a graph on $n-1$ vertices, whose chromatic polynomial has 1 as the coefficient of k^{n-1} according to our previous claim. Therefore deleting the edge should augment the coefficient of k^{n-1} by 1, finally reaching in zero as all edges are removed. So initially it had to be the negative of the number of edges. $\square$

Claim 2.3. *The constant term, i.e. the coefficient of 1 in* $\mathsf{chr}(G, k)$ *is always zero.*

Proof. Substituting $k = 0$ into the chromatic polynomial yields 0 since G cannot be colored using 0 colors. $\square$

Alternate proof. The statement is obviously true for a graph without edges and carries over to general graphs by applying the deletion-contraction argument. $\square$

Claim 2.4. *The coefficient of k in $\mathsf{chr}(G,k)$ is nonzero if and only if G is connected.*

Proof. We know from section 1.5 that the the chromatic polynomial of a disconnected graph is the product of that of its components. If we have at least two terms, each being divisible by k, then their product is divisible by k^2, thus its coefficient of k is zero.

For connected graphs we'll prove the slightly stronger result that the coefficient of k is positive if n is odd and negative if n is even. This works by induction based on the deletion-contraction argument. We can always select an edge that is not a bridge unless the graph is a tree, in which case the claim results from our formula for the chromatic polynomial stated in section 1.4. Otherwise both deletion and contraction gives a connected graph so we may continue the induction. $\square$

Remark 2.1. The same proof also shows that all coefficients of $\mathsf{chr}(G,k)$ except the constant one are nonzero if G is connected. The coefficient of k is called the chromatic invariant of G and denoted by $\beta(G)$.

Corollary 2.5. *The lowest nonzero coefficient is that of k^c where c is the number of connected components.*

Proof. The chromatic polynomial can be calculated as a product over the components, each term being divisible by k but not by k^2. Therefore the result is divisible by k^c but not by k^{c+1}. $\square$

Lemma 2.6. *Let $c(F)$ denote the number of components in a spanning subgraph F. Then*

$$\mathsf{chr}(G,k) = \sum_{F \subseteq E(G)} (-1)^{|F|} k^{c(F)} \; . \tag{2.1}$$

Proof. The number of ways we may assign colors to $V(G)$ so that vertices connected by F-edges *do* share the same color is $k^{c(F)}$. This is the number of colorings that *violate* the vertex coloring condition for all edges in F. Since the chromatic polynomial counts the colorings that violate this condition for no edges in $E(G)$, the result follows from the principle of inclusion-exclusion. $\square$

Claim 2.7. *The chromatic invariant of G equals the signed difference between the number of spanning subgraphs of G having an even resp. an odd number of edges.*

Proof. The claim can be rewritten as $\beta(G) = \sum (-1)^{|E(G')|}$ where G' iterates through all spanning subgraphs of G. It follows from the lemma by considering the coefficient of k on both sides of the equation. $\square$

Claim 2.8. *The coefficients of the chromatic polynomial alternate in sign. That is, for the the coefficient a_m of k^m we have $a_m \geq 0$ if $n \equiv m(2)$ and $a_m \leq 0$ otherwise.*

Proof. The claim holds for the empty graph and is preserved during a deletion-contraction step. $\square$

Claim 2.9. *For a connected graph G the coefficients satisfy*

$$1 = |a_n| < |a_{n-1}| < \ldots < \left|a_{\lfloor \frac{n}{2} \rfloor + 1}\right| \ . \tag{2.2}$$

Proof. We would like to show $|a_{m+1}| < |a_m|$ for $m > \frac{n}{2}$.

For trees we have $\mathsf{chr}(T_n, k) = k(k-1)^{n-1}$ from section 1.4, so $a_m = (-1)^{n-m}\binom{n-1}{m-1}$ and thus the claim is $\binom{n-1}{m} < \binom{n-1}{m-1}$. Rearranging transforms this to $n - m < m$ which we have assumed.

Otherwise we may select a non-bridge edge as in the proof of claim 2.4 and apply deletion-contraction. Our previous claim tells us that the corresponding coefficients of $\mathsf{chr}(G\backslash e, k)$ and $\mathsf{chr}(G/e, k)$ have opposite signs, and therefore their absolute values add up. For the contracted graph we have

$$0 = |a'_n| < 1 = |a'_{n-1}| < |a'_{n-2}| < \ldots \left|a'_{\lfloor \frac{n-1}{2} \rfloor + 1}\right| \ , \tag{2.3}$$

where the last index is no more than $\lfloor \frac{n}{2} \rfloor + 1$, so both $G\backslash e$ and G/e satisfy the inequalities in the claim and the final addition also preserves them. □

This claim is suspected to be possibly strengthened:

Conjecture 2.10 (unimodal conjecture)**.** *There exists some k such that*

$$|a_n| \leq |a_{n-1}| \leq \ldots \leq |a_{k+1}| \leq |a_k| \geq |a_{k-1}| \geq \ldots \geq |a_2| \geq |a_1| \ . \tag{2.4}$$

This claim has been verified for a few classes of graphs, but remains generally unknown. For some related results see [9].

2.2 Roots

A nonnegative integer root k of the chromatic polynomial means noncolorability with k colors by definition. It follows that $k \in \mathbb{N}$ is a root if and only if $k < \chi(G)$.

Claim 2.11. *The multiplicity of 0 as a root equals the number of connected components in G.*

Proof. It follows from corollary 2.5. □

Lemma 2.12. *The derivative of the chromatic polynomial satisfies $(-1)^n \mathsf{chr}'(G, 1) > 0$ for any biconnected graph and ≥ 0 for any connected graph G.*

Proof. For connected graphs that are not biconnected there exists a cut vertex v and we may write the chromatic polynomial $\mathsf{chr}(G, q)$ as a product $\frac{\mathsf{chr}(G_1,q)\mathsf{chr}(G_2,q)}{q}$ according to claim 1.9. Neither G_1 nor G_2 is empty, thus both terms have a root at 1, so 1 is at least a double root of $\mathsf{chr}(G, q)$ and therefore its derivative also has a root at 1. It follows that the claim is satisfied with an equality.

At this point it is enough to consider biconnected graphs. For K_2 we have $\mathsf{chr}'(K_2, 1) = 1$. Otherwise both $G\backslash e$ and G/e are connected for any edge e, so we can obtain the weaker claim by using the deletion-contraction argument. To prove the stronger one, we'll show that there exists an edge e for which G/e is also biconnected.

The only possible cut vertex of G/e is the contracted one, since any other vertex would also separate G. So we are looking for such an $e = ij$ that removing i and j from $V(G)$ doesn't cut G apart.

There exists a longest path in G: let i be one of its endpoints and j its neighbor. Suppose that the removal of i and j leaves a disconnected graph and pick two points from two different components. By Menger's theorem (and $G \neq K_2$) there exist two vertex-disjoint paths between the two, and they have to traverse i and j respectively. The one going through i can be used to extend the selected longest path, implying a contradiction.

Therefore if we pick e as the final segment of a longest path, G/e is also biconnected and thus the claim holds. □

Consequence 2.13. *The multiplicity of 1 as a root of the chromatic polynomial equals the number of blocks in G.*

Proof. Repeatedly using claim 1.9 we can write $\mathsf{chr}(G, q)$ as the product of the chromatic polynomials of the blocks of G devided by a power of q. Each of these polynomials have a single root at 1 by the previous lemma. □

Claim 2.14. *The chromatic polynomial has no real root greater than $n - 1$.*

Proof. We have shown in section 1.2 that the chromatic polynomial is a sum of terms having the form $q(q-1)\dots(q-p+1)$ where $1 \leq p \leq n$, each of them possibly occuring multiple times. It is easy to see that such a term increases strictly monotonically for $q > n-1 \geq p-1$, and so does their sum as well.

Since $\mathsf{chr}(G, q)$ is nonnegative for $q = n-1$ and strictly increasing afterwards, it can have no root $> n-1$. □

Claim 2.15. *The chromatic polynomial of a graph has no negative real roots.*

Proof. By claim 2.8 we have

$$\mathsf{chr}(G, q) = \sum_{m=1}^{n} a_m q^m \tag{2.5}$$

where $a_m \geq 0$ if $n \equiv m(2)$ and $a_m \leq 0$ otherwise. Thus $(-1)^n a_m q^m \geq 0$ for any $q < 0$. We also know that $a_n = 1 > 0$, and therefore $(-1)^n \mathsf{chr}(G, q) > 0$ which implies that q cannot be a root. □

Claim 2.16. *The chromatic polynomial has no real roots between 0 and 1.*

Proof. It suffices to deal with connected graphs according to section 1.5. We show that $(-1)^n \mathsf{chr}(G, q) < 0$ for any $0 < q < 1$. This statement can be easily checked for trees where $\mathsf{chr}(G, q) = q(q-1)^{n-1}$ and otherwise it follows from the deletion-contraction property. □

An extension of these claims has been proved by Jackson [13]:

Theorem 2.17 (Jackson)**.** *The chromatic polynomial has no real roots in the interval* $(1, \frac{32}{27}]$. $\square$

Consequence 2.18. *There are no real roots in* $(-\infty, \frac{32}{27}]$ *except for 0 and 1.*

However, this proposition cannot be extended any further. Thomassen [21] has shown that:

Theorem 2.19 (Thomassen)**.** *The real roots of all chromatic polynomials are dense in* $[\frac{32}{27}, \infty)$. $\square$

Remark 2.2. The set of reals that are indeed roots of a suitable chromatic polynomial are much narrower. Clearly countably many graphs may only have $\aleph_0$ chromatic roots, but we also have some particular exceptions. For example, Read and Tutte have shown that $\varphi + 1 = \frac{\sqrt{5}+3}{2}$ is never a chromatic root [16]. Irrational numbers whose squares are rational constitute another excluded class.

Since Birkhoff's original motivation to define the chromatic polynomial was to prove the four-color theorem, roots for planar graphs are also of significant importance. The famous Birkhoff-Lewis conjecture states:

Conjecture 2.20. *Planar graphs have no real roots in* $[4, \infty)$.

They have already solved the weaker version that $[5, \infty)$ contains no real roots. Since then, Appel and Haken have proved the four-color theorem which states that 4 is neither a root [1][2]. For the remaining interval $(4, 5)$, however, the question is still wide open.

Tutte has shown that for planar graphs $\mathsf{chr}(G, \varphi + 2) > 0$ where $\varphi = \frac{\sqrt{5}+1}{2}$ is the golden ratio [15]. He hoped that it takes us closer to the four-color theorem since $\varphi + 2 \approx 3.618$ is close to 4, but unfortunately there are some planar graphs having a root between the two. In fact, Royle has shown that there are roots arbitrarily close to 4 from below [17].

We may also ask about the complex chromatic roots. A result of Sokal [19] tells

Theorem 2.21 (Sokal)**.** *The roots of all chromatic polynomials are dense in* $\mathbb{C}$. $\square$

Despite this claim for general graphs, there exist root-free zones if we make some restrictions. These are particularly important from the point of view of statistical mechanics. A related theorem [18] shows

Theorem 2.22 (Sokal)**.** *There exists a universal constant C such that if G has maximum degree D, then all complex roots of* $\mathsf{chr}(G, q)$ *satisfy* $|q| < CD$. $\square$

A similar bound $|q| < CD + 1$ exists for the second largest degree. We'll prove this theorem in chapter 3. For the third largest degree, no such claim can be made: there are arbitrarily large chromatic roots even when all except two vertices have degree 2.

Remark 2.3. In Sokal's proof, C is the smallest number for which

$$\inf_{\alpha>0} \alpha^{-1} \sum_{n=2}^{\infty} e^{\alpha n} C^{-(n-1)} \frac{n^{n-1}}{n!} \leq 1 \ . \tag{2.6}$$

This value satisfies $C \leq 7.963907$.

2.3 Substitutions

For $k \in \mathbb{N}$ $\mathsf{chr}(G, k)$ means the number of k-colorings of G by definition. But there are some further locations where the evaluation of the chromatic polynomial is interesting.

Claim 2.23. $|\mathsf{chr}(G, -1)|$ *gives the number of acyclic orientations of* G*.*

Proof. Denote the number of acyclic orientations of G by $f(G)$. We show that $f(G) = (-1)^n \mathsf{chr}(G, -1)$. For the empty graph we have $f(\overline{K_n}) = 1$ and thus the proposition holds.

Now consider a nonempty graph G with an edge e selected. Suppose we have an orientation $\overrightarrow{G}_e$ on all edges except e and we would like to find out how many ways (i.e. 0, 1 or 2) we can extend it to an acyclic orientation $\overrightarrow{G}$ of G.

Notice that if there exists such an acyclic extension at all, removing the edge e won't break it. On the other hand, if we can't add e in either direction because both would close a directed path in $\overrightarrow{G}_e$ into a cycle, then these two paths make up a cycle in $\overrightarrow{G}_e$ by themselves. Therefore $\overrightarrow{G}_e$ is an acyclic orientation of $G \backslash e$ if and only if e can be added in at least one direction.

$\overrightarrow{G}_e$ specifies an orientation on G/e too. If it contains a cycle passing through the contracted point, one of the two possible orientations of e will extend this cycle into a larger one. And a cycle avoiding the contracted point will be kept intact. Thus if $\overrightarrow{G}_e$ is cyclic, pointing e in least one of the two directions will create a cycle. If $\overrightarrow{G}_e$ is acyclic, however, both directions of e will result in an acyclic graph, since any cycle in $\overrightarrow{G}$ would have been preserved during the contraction.

Thus if $\overrightarrow{G}_e$ specifies an acyclic orientation on 0, 1 or 2 of the graphs $G \backslash e$ and G/e, then it can be extended to G in 0, 1 or 2 ways respectively. This argument shows that $f(G) = f(G \backslash e) + f(G/e)$.

Since the same recursion holds for $f(G)$ and $(-1)^n \mathsf{chr}(G, -1)$ and they are equal for empty graphs, they have to be always equal. ☐

Claim 2.24. $\mathsf{chr}'(G, 0)$ *returns the chromatic invariant* $\beta(G)$*.*

Proof. The derivative of a polynomial at zero equals the linear coefficient. So the claim follows and the properties proved for $\beta(G)$ in section 2.1 apply. ☐

The chromatic polynomial also exhibits interesting behaviour at the so-called *Beraha numbers* $B_n = 2 + 2\cos\left(\frac{2\pi}{n}\right)$, but they are outside the scope of this study.

Chapter 3

Bounding complex roots

In this chapter we'll have another look at Sokal's theorem mentioned in section 2.2 and prove a slightly improved version of it.

3.1 Motivation

As noted previously, nonnegative integer roots of the chromatic polynomial describe the graph's noncolorability with a certain number of colors. For graphs with degree bounded by D, it is easy to see that $D + 1$ colors are always sufficient. This can be rephrased as the lack of positive integer roots greater than D.

Sokal [18] has found a similar bound for complex roots:

Theorem 3.1 (Sokal)**.** *There exists a universal constant c such that for all simple graphs with maximum degree $\leq D$ the roots of the chromatic polynomial lie in the disc $|q| \leq cD$. Also, if the second-largest degree is $\leq D$, all complex roots have $|q| \leq cD+1$. Furthermore, $c \leq 7.963907$.* □

Remark 3.1. Sokal's proof was simplified by Borgs [8] and recently improved by Fernández and Procacci to $c \leq 6.91$ [11]. For real roots, Dong and Koh have shown $c \leq 5.67$ [10].

3.2 Preliminaries

We'll need some continuous structure on graphs, so we'll allow edge-weighted graphs with weights between 0 and 1. This is analogous to the linear relaxation quite common in integer programming. To work with this model, we first have to extend the definition of the chromatic polynomial and some related concepts to weighted graphs.

Let $G = (V, E)$ be a simple graph with edge weights $0 \leq w_e \leq 1$. We may assume that G is complete, attaching a weight of 0 to non-edges. For a nonnegative integer k, define the chromatic polynomial as

$$\mathsf{chr}(G, k) = \sum \prod (1 - w_e) \tag{3.1}$$

where the sum enumerates all possible k-colorings of G while the product iterates over edges connecting vertices of the same color. The proof that we indeed get a polynomial carries over from the unweighted case.

The *degree* of a vertex v is modified to mean the sum of weights on all edges incident to v. Edge *deletion* $G\backslash e$ is handled by zeroing the corresponding weight. After an edge *contraction* G/e, a new edge emanating from the resulting vertex will have a weight of $w_1 \oplus w_2 := w_1 + w_2 - w_1 w_2$ where w_1 and w_2 denote the weights of the original edges from the endpoints of e to the same vertex. The usual reduction changes to

$$\mathsf{chr}(G,q) = \mathsf{chr}(G\backslash e,q) - w_e \mathsf{chr}(G/e,q)\ . \tag{3.2}$$

Changing the weight of a single edge will be marked as $G[e : w_e]$.

During the proof, we'll assume that the set of vertices does not change. In this sense, edge contraction does no good to us, so let's define *contraction with compensation* as the contraction of an edge followed by the addition of a new isolated vertex, using the notion of $G \upharpoonright e$. we may think of it as moving all edges from a given vertex to another one. Obviously $\mathsf{chr}(G \upharpoonright e) = q\mathsf{chr}(G/e)$, so the reduction can be written as

$$\mathsf{chr}(G,q) = \mathsf{chr}(G\backslash e,q) - \frac{w_e}{q}\mathsf{chr}(G \upharpoonright e,q)\ . \tag{3.3}$$

3.3 Proving the theorem

Lemma 3.2. *Let $0 \le s,t \le 1$ and $0 \le x < 1$. Recall the definition $s \oplus t = s + t - st$. Then*

$$\log(1 - sx) - \log(1 - (s \oplus t)x) \le -t \log(1 - x)\ . \tag{3.4}$$

Proof. Consider both sides of the inequality as a function of t. It is easy to see that the claim holds for $t = 0$ and $t = 1$. Since the right-hand side is linear in t, it suffices to prove that the left-hand side is convex.

We know that $\log t$ is concave on its entire domain and thus $\log(a + bt)$ is also concave for any real a and b. Substituting $a = 1 - sx$ and $b = sx - x$ yields that $\log(1 - (s \oplus t)x)$ is concave too. This proves the proposition because $\log(1 - sx)$ is constant. □

Theorem 3.3. *Let G be a simple weighted graph as defined above with an edge $e = ij$ selected. Suppose that all vertices, possibly except one, have a degree of at most D and*

$$|q| \ge \left(1 + \frac{1}{2D}\right)^{2D} (2D + 1)\ . \tag{3.5}$$

Let $0 \le w_1, w_2 \le 1$ be some weights. We claim that for any $q \in \mathbb{C}$

$$a. \quad \left|\log \frac{\mathsf{chr}(G \upharpoonright e,q)}{\mathsf{chr}(G\backslash e,q)}\right| \le 2D \log\left(1 + \frac{1}{2D}\right), \text{ and} \tag{3.6}$$

$$b. \quad \left|\log \frac{\mathsf{chr}(G[e : w_1],q)}{\mathsf{chr}(G[e : w_1 \oplus w_2],q)}\right| \le w_2 \log\left(1 + \frac{1}{2D}\right) \tag{3.7}$$

where $\log$ *is the principal branch of the complex logarithm function.*

Proof. Apply induction based on the number of nonzero weighted edges, including the edge e if $w_1 > 0$ or $w_2 > 0$. If all weights are zero, the claims are trivial.

a. We morph the graph $G \backslash e$ to $G \nearrow e$ by a set of successive edge deletions and additions. Possibly swapping i and j, we may assure that the degree of j is at most D. For each edge jk, reset its original weight w_{jk} to zero and add it to the edge ik so that its weight becomes $w'_{ik} = w_{ik} \oplus w_{jk}$. Intermediate graphs have fewer nonzero edges than G, and the degree criterion also holds for them, so we can apply part *b.* of the induction hypothesis, obtaining that each step results in a difference of no more than $\log\left(1 + \frac{1}{2D}\right)$ in the logarithm of the chromatic polynomial. Adding up yields that the total difference is at most $2D \log\left(1 + \frac{1}{2D}\right)$.

b. Let w_e denote the current weight of the edge e and consider the partial logarithmic derivative of $\mathsf{chr}(G, q)$ with respect to w_e, expanding its absolute value using the reduction formula (3.3):

$$\left|\frac{\partial}{\partial w_e} \log \mathsf{chr}(G, q)\right| = \left|\frac{\frac{1}{q}\mathsf{chr}(G \nearrow e, q)}{\mathsf{chr}(G \backslash e, q) - \frac{w_e}{q}\mathsf{chr}(G \nearrow e, q)}\right| = \left|\frac{K}{1 + w_e K}\right| \tag{3.8}$$

where

$$K = -\frac{1}{q}\frac{\mathsf{chr}(G \nearrow e, q)}{\mathsf{chr}(G \backslash e, q)} . \tag{3.9}$$

From part *a.* we have

$$\begin{aligned}\left|\frac{\mathsf{chr}(G \nearrow e, q)}{\mathsf{chr}(G \backslash e, q)}\right| = e^{\Re \log \frac{\mathsf{chr}(G \nearrow e, q)}{\mathsf{chr}(G \backslash e, q)}} \le e^{\left|\log \frac{\mathsf{chr}(G \nearrow e, q)}{\mathsf{chr}(G \backslash e, q)}\right|} \le \\ \le e^{2D \log\left(1 + \frac{1}{2D}\right)} = \left(1 + \frac{1}{2D}\right)^{2D}\end{aligned} \tag{3.10}$$

and therefore

$$|K| \le \frac{\left(1 + \frac{1}{2D}\right)^{2D}}{|q|} \le \frac{1}{2D+1} \tag{3.11}$$

according to our supposition for $|q|$. Note that $|K| < 1$ and therefore $|w_e K| < 1$, which we'll need shortly.

Now we may bound the multiplicative change in the chromatic polynomial:

$$\begin{aligned}\left|\log \frac{\mathsf{chr}(G[e\!:\!w_1], q)}{\mathsf{chr}(G[e\!:\!w_1 \oplus w_2], q)}\right| \le \int_{w_1}^{w_1 \oplus w_2} \left|\frac{\partial}{\partial w_e} \log \mathsf{chr}(G, q)\right| \mathrm{d}w_e = \int_{w_1}^{w_1 \oplus w_2} \left|\frac{K}{1 + w_e K}\right| \mathrm{d}w_e \le \\ \le \int_{w_1}^{w_1 \oplus w_2} \frac{|K|}{1 - w_e|K|} \mathrm{d}w_e = \log(1 - w_1|K|) - \log(1 - (w_1 \oplus w_2)|K|) \overset{(*)}{\le} \\ \overset{(*)}{\le} -w_2 \log(1 - |K|) \le -w_2 \log\left(1 - \frac{1}{2D+1}\right) = w_2 \log\left(1 + \frac{1}{2D}\right)\end{aligned} \tag{3.12}$$

where $(*)$ follows from lemma 3.2 and thus we have obtained the claim. $\square$

Consequence 3.4. *Under the same circumstances* $\mathsf{chr}(G, q) \neq 0$ *also holds.*

Proof. The claim is trivial for the empty graph. For an arbitrary graph, we may subsequently change each edge weight of the empty graph to the desired value, causing only a bounded multiplicative change to $\mathsf{chr}(G, q)$ in each step. It follows that the result cannot be zero either. □

Consequence 3.5. *If D denotes the second-largest degree in G, then all roots of the chromatic polynomial lie within the disc $|q| < (2D + 1)e$.*

Proof. Suppose the contrary. A root that violates the claim satisfies

$$|q| \geq (2D + 1)e > (2D + 1)\left(1 + \frac{1}{2D}\right)^{2D} \tag{3.13}$$

so we may apply the previous theorem, resulting in a contradiction. □

Remark 3.2. We proved an asymptotic factor of $2e \approx 5.436564$, which, despite the slightly larger additive constant, gives a stronger result than Sokal's for any positive integer D. This is also an improvement compared to the articles referenced in remark 3.1, and as to our knowledge, is the sharpest bound known at the moment.

Chapter 4

Tutte's polynomial

Tutte defined his two-variable dichromatic polynomial as a generalization of the chromatic polynomial and the deletion-contraction argument observed in section 1.3. Many graph polynomials coming from different areas of mathematics and even physics have been identified as special cases of the Tutte polynomial. We'll go through some of them in this chapter, as a gallery of applications. No proofs are included here, the interested reader is referred to [7] and [12].

So far we have worked with simple graphs, but in this more general context it will be useful to consider multigraphs with loops instead.

4.1 Flow polynomial

Orient the edges of the graph G in both directions and assign some amount of "flow" f_e to each edge e in such a way that the values given to the two orientations of the same edge are the negative of each other. We imagine these values as the amount of goods being transported over the links. If there are no specific suppliers, consumers and depots, we'll have to expect that the incoming and outgoing amount are the same for each node, as expressed by Kirchhoff's current law:

$$\sum_{ij \in E} f_{ij} = 0 \qquad \forall i \in V \; . \tag{4.1}$$

If the values assigned to the edges are real numbers or integers, such a structure is called a *circulation*, a central concept in the theory of network flows.

Here we take another aproach and choose the values from $\mathbb{Z}_k$, or more generally, an Abelian group A. In this case, an assignment adhering to Kirchhoff's law is called a k*-flow* or an A*-flow*, respectively. We call an A-flow *nowhere-zero* if it has a nonzero value on each edge.

Let $\mathsf{flow}(G, A)$ denote the number of nowhere-zero A-flows. It can be shown that $\mathsf{flow}(G, A)$ is a polynomial of $|A|$ for any graph G and therefore it is enough to consider the special case $\mathsf{flow}(G, k) = \mathsf{flow}(G, \mathbb{Z}_k)$.

For planar graphs, the flow polynomial $\mathsf{flow}(G, k)$ is the chromatic polynomial $\mathsf{chr}(G^*, k)$ of the dual graph G^*.

4.2 Reliability polynomial

Suppose we have a network of workstations interconnected with faulty links that only work with probability p independently of each other. We are interested in whether the network stays connected, in the sense that the number of connected components is no more than it would have been if all the links were fully functional.

For a graph G and a real number $0 \leq p \leq 1$, let $\mathsf{rel}(G, p)$ denote the probability that the selected random subgraph has the same number of components as G. This is a polynomial in p which we call the *reliability polynomial* of G and which satisfies a relation similar to our deletion-contraction argument for any non-bridge edge e:

$$\mathsf{rel}(G, p) = p\ \mathsf{rel}(G/e, p) + (1 - p)\ \mathsf{rel}(G\backslash e, p)\ . \tag{4.2}$$

4.3 Statistical mechanics and the Potts model

Ferromagnets can be thought of as a set of interacting spins on a crystalline lattice. In the Potts model, each spin can assume one of q possible states. If two neighboring spins (those joined by an edge e) are in the same state, it adds some value $-J_e$ to the energy H of the system.

The *Boltzmann weight* of a configuration (i.e. assignment of states to spins) is $e^{-\beta H}$ where $\beta = \frac{1}{kT} \geq 0$ is the inverse temperature calculated from the temperature T and the Boltzmann constant k. The probability of the configuration is proportional to its Boltzmann weight.

These weights, however, do not form a probability distribution, so in order to calculate the probabilities, we have to normalize with the sum of the Boltzmann weights of all configurations, known as the *partition function* of the Potts model. It is a polynomial in terms of q, denoted by $Z_G(q, \{J_e\})$ or $Z_G(q, \beta)$.

The behavior of a coupling $\{J_e\}$ is determined by the willingness of spins to become aligned: it is said to be ferromagnetic if $J_e > 0$, antiferromagnetic if $J_e < 0$ and non-interacting if $J_e = 0$. Interaction is strengthened by decreasing the temperature, and as we approach the limit of $T = 0$ only configurations with no adjacent spins sharing a common state will have nonzero energy. This explains the intimate relationship between the partition function of the Potts model and the chromatic polynomial. The Ising model is obtained as the special case $q = 2$.

Phase transitions, of particular importance for statistical physicists, are closely related to the roots of the partition function, and therefore to the roots of the chromatic polynomial as well. For more details on this topic, see the introduction of [18].

4.4 Jones polynomial of alternating knots

Knot theory, an area of topology, tries to provide a classification of (tame) mathematical knots, which are the equivalence classes of piecewise linear simple closed curves in euclidean 3-space under deformation (called ambient isotropy). They are usually represented by their projections into the two dimensional plane with finitely many crossings, each consisting of only two intersecting lines and the over/under relationship properly marked. Such a drawing is called a *knot diagram*.

Unfortunately the diagram of the same knot can be drawn in many different ways and it is highly nontrivial to check that they are really equivalent. Actually the fundamental problem of knot theory is to determine whether two diagrams represent the same knot, or especially, whether a given diagram is equivalent to the unknot.

Although there exist algorithms to solve this problem, we know nothing about their complexity, and would also like some kind of witness that convinces us about the result. For a positive answer, a set of so-called Reidemeister moves proving the equivalence can be provided, while for a negative one an *invariant* is used: some property that is preserved during deformation and is indeed different for the two knots.

Simple invariants such as tricolorability inevitably map many knots into the same class, degrading their usefulness. Mappings to polynomials or groups are less prone to such coincidences. One of the most famous knot polynomials is the *Jones polynomial* $V_L(t)$, which is the unique Laurent polynomial in the variable $\sqrt{t}$ characterized by the so-called skein relation

$$t^2 V_L(\nwarrow\!\!\!\!\nearrow) - t^{-2} V_L(\nwarrow\!\!\!\!\nearrow) = (t^{-1} - t) V_L(\curvearrowright\curvearrowleft) \tag{4.3}$$

and being 1 on the unknot. Note the similarity between this identity and our deletion-contraction argument. Also note that some of these modifications will produce a *link* instead of a knot, meaning the union of multiple, possibly interconnected closed curves. But this is no problem, our definitions extend to links as well.

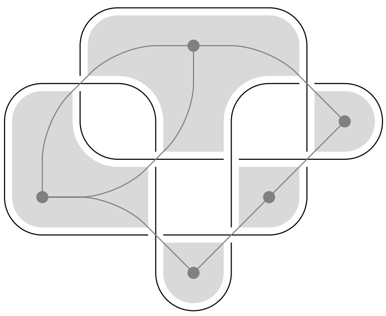

Figure 5: an alternating knot (Alexander-Briggs 7_6) with its corresponding graph

In order to make the connection with our previous graph polynomials more explicit, we'll need to find some relationship between graphs and knots. Observe that diagrams are closed curves in the plane and therefore their regions can be 2-colored. Now consider the graph whose vertices are, say, the white regions, and two vertices are connected if the corresponding regions share a common crossing. We get a planar graph which can be used to reconstruct the diagram except the over/under relationships.

This shortcoming can be addressed by restricting ourselves to some special classes of diagrams. For example, if one who travels along the curve alternatingly traverses overcrossings and undercrossings, we have an *alternating diagram.* Planar graphs are in one-to-one correspondance with alternating link diagrams, so their Jones polynomial defines another graph polynomial $V_{L(G)}(t)$.

4.5 A common generalization

At this point we have already defined five different graph polynomials. All of them have some kind of reduction in a spirit similar to the deletion-contraction argument. Let's define a more general polynomial that is actually based on deletion-contraction. For the empty graph let $T_{\overline{K_n}}(x,y)=1$ while for an edge e use the recursion

$$T_G(x,y)=\begin{cases} xT_{G\setminus e} \text{ (or } xT_{G/e}) & \text{if } e \text{ is a bridge,}\\ yT_{G\setminus e} \text{ (or } yT_{G/e}) & \text{if } e \text{ is a loop,}\\ T_{G\setminus e}+T_{G/e} & \text{otherwise.}\end{cases} \tag{4.4}$$

For bridges and loops $T_{G\setminus e}$ and $T_{G/e}$ are equal and thus the distinction is unimportant. In fact, the first two recursions could have been replaced by the starting condition that $T_G(x,y)=x^b y^l$ if G is a forest made up of b edges with l loops added. The resulting T_G is a bivariate polynomial called the *Tutte polynomial* of G.

We could have been even more general. Define the *universal polynomial* of a graph as $U_{\overline{K_n}}(x,y,\alpha,\sigma,\tau)=\alpha^n$ and

$$U_G(x,y,\alpha,\sigma,\tau)=\begin{cases} xU_{G\setminus e} \text{ (or } \alpha xU_{G/e}) & \text{if } e \text{ is a bridge,}\\ yU_{G\setminus e} \text{ (or } yU_{G/e}) & \text{if } e \text{ is a loop,}\\ \sigma U_{G\setminus e}+\tau U_{G/e} & \text{otherwise.}\end{cases} \tag{4.5}$$

But this isn't substantially more general as it can be shown that

$$U_G(x,y,\alpha,\sigma,\tau)=\alpha^{c(G)}\sigma^{|E|-|V|+c(G)}\tau^{|V|-c(G)}T_G\Big(\frac{\alpha x}{\tau},\frac{y}{\sigma}\Big) \tag{4.6}$$

where $c(G)$ denotes the number of components in G.

The Tutte polynomial can also be expressed in a more explicit way:

$$T_G(x,y)=\sum_{F\subseteq E}(x-1)^{c(F)-c(E)}(y-1)^{c(F)+|F|-|V|} \tag{4.7}$$

where $c(F)$ is the number of components in the spanning subgraph (V,F). By writing $c(F)=|V|-r(F)$ with $r(F)$ being the rank function of the graphic matroid, we obtain that the Tutte polynomial depends only on the matroid of G, i.e. graphs sharing the same matroid have the same Tutte polynomial too. Note that this claim cannot be reversed: the Tutte polynomial does not uniquely determine the graphic matroid.

There is some symmetry between the two variables of the Tutte polynomial. For a planar graph G and its dual G^*, they are exchanged: $T_G(x,y) = T_{G^*}(y,x)$. The same relation holds for a matroid and its dual.

If G is the union of G_1 and G_2 where $|V(G_1) \cap V(G_2)| \le 1$ then $T_G = T_{G_1} \cdot T_{G_2}$. This is analogous to our constructions in section 1.5.

As we see, the Tutte polynomial exhibits a number of nice properties. But with all of these, it could have been just another graph polynomial. What makes it special is that it is a common generalization of the seemingly unrelated concepts mentioned in the previous sections.

The chromatic polynomial of G is equivalent to the restriction of T_G to $y = 0$:

$$\mathsf{chr}(G,k) = U_G\left(1 - \frac{1}{k}, 0, k, 1, -1\right) = (-1)^{|V|-c(G)} k^{c(G)} T_G(1-k, 0) \tag{4.8}$$

The flow polynomial of G is equivalent to T_G for $x = 0$:

$$\mathsf{flow}(G,k) = U_G(0, k-1, 1, -1, 1) = (-1)^{|E|-|V|+c(G)} T_G(0, 1-k) \tag{4.9}$$

The reliability polynomial corresponds to T_G for $x = 1$:

$$\mathsf{rel}(G,p) = U_G(p, 1, 1, 1-p, p) = p^{|V|-c(G)} (1-p)^{|E|-|V|+c(G)} T_G\left(1, \frac{1}{1-p}\right) \tag{4.10}$$

The partition function of the q-state Potts model can be expressed in terms of T_G for $(x-1)(y-1) = q$:

$$\begin{aligned} Z_G(q,\beta) = e^{-\beta|E|} U_G\left(1 + \frac{e^\beta - 1}{q}, e^\beta, q, 1, e^\beta - 1\right) = \\ = e^{-\beta|E|} q^{c(G)} (e^\beta - 1)^{|V|-c(G)} T_G\left(\frac{q + e^\beta - 1}{e^\beta - 1}, e^\beta\right) \end{aligned} \tag{4.11}$$

Remark 4.1. The Potts model has an extension called the Fortuin-Kasteleyn random cluster model, whose partition function is equivalent to the Tutte polynomial.

The Jones polynomial of the alternating knot (or link) specified by the planar graph G is related to T_G for $xy = 1$:

$$V_{L(G)}(t) = (-1)^w t^{\frac{b-a+3w}{4}} T_G\left(-t, -\frac{1}{t}\right) \tag{4.12}$$

where a and b denote the number of white and black regions respectively and w is the writhe of the knot diagram. Note that the role of black and white is asymmetric and can not be exchanged, but the correct assignment is not discussed here. For the details, see [20] or [7].

The Tutte polynomial also has some additional combinatorial meaning when evaluated at certain (x, y) pairs:

- $T_G(1, 1)$ = number of spanning trees of G
- $T_G(1, 2)$ = number of connected spanning subgraphs
- $T_G(2, 1)$ = number of acyclic subgraphs (that is, forests)
- $T_G(2, 2)$ = number of spanning subgraphs
- $T_G(2, 0)$ = number of acyclic orientations of G (in accordance with claim 2.23)
- $T_G(1, 0)$ = number of acyclic orientations where the only source is a fixed vertex
- $T_G(0, 2)$ = number of orientations of a bridgeless G such that each edge is contained in an oriented cycle
- $2T_G(3, 3)$ = number of T-tetromino tilings of a $4n \times 4m$ rectangle if G is an $n \times m$ grid
- $T_G(0, 0)$ = characteristic function of the empty graph
- $T_G(0, -1)$ = characteristic function of Eulerian graphs
- $T_G(-2, 0)$ corresponds to the number of Eulerian orientations [14]
- $T_G(-1, -1)$ corresponds to the dimension of the bicycle space of binary codes [12]

As a summary, here is a plot showing all special values of the Tutte polinomial:

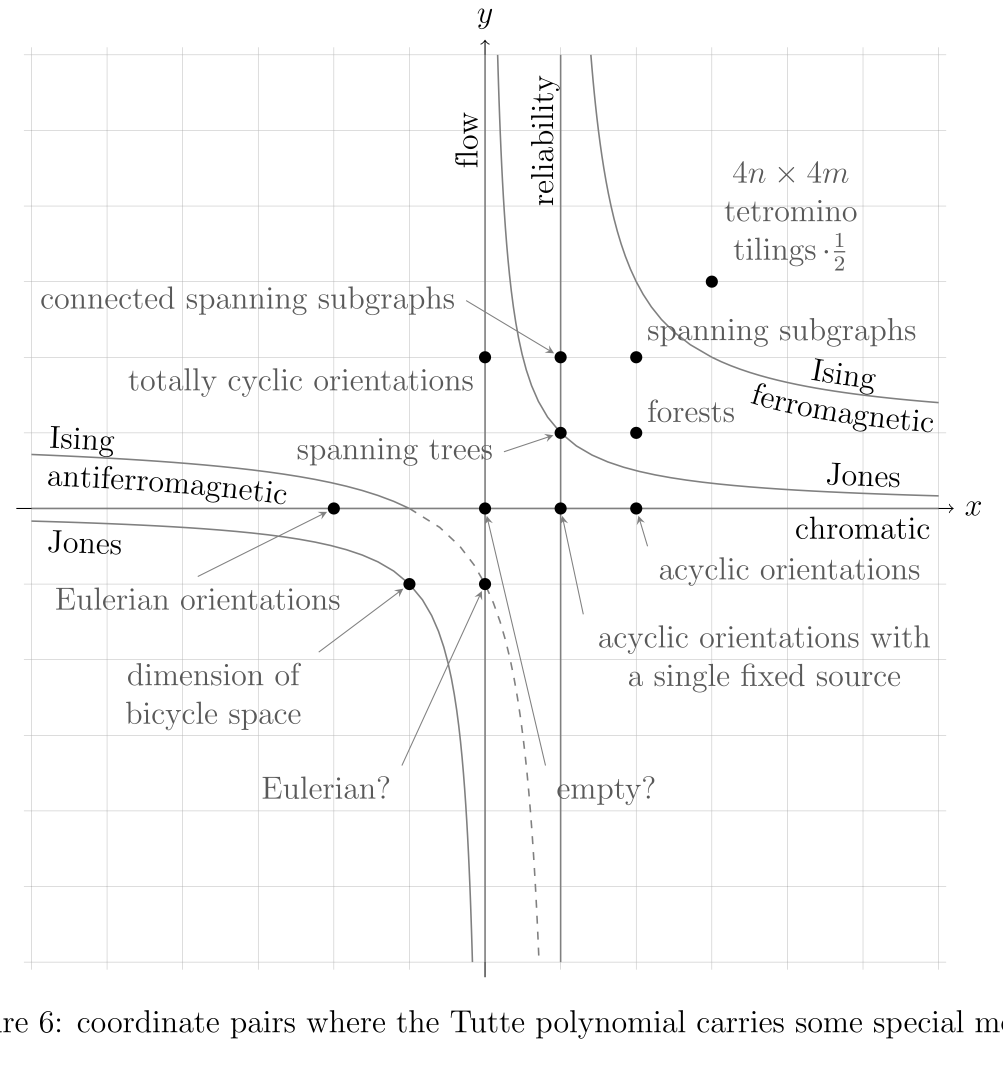

Figure 6: coordinate pairs where the Tutte polynomial carries some special meaning

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