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Bevezetés

Az affin teljes algebrák vizsgálata a 60-as évek elejéig nyúlik vissza. Grätzer György ismerte fel először a fogalom jelentőségét az univerzális algebrában, és az ő nevéhez köthetők az első jelentős eredmények is a témában. A csoporthatások univerzális algebrai szempontból szintén fontos szerepet töltenek be. A két fogalom kapcsolatára világít rá John Snow egy tétele, miszerint ha egy véges rendpolinomteljes háló előáll mint egy véges algebra kongruenciahálója, akkor előáll úgy is, mint egy vektortér vagy egy affin teljes csoporthatás kongruenciahálója. Ez utóbbi eredmény motiválja az affin teljes csoporthatások vizsgálatát.

Dolgozatomban többek között ismertetek néhány fontos csoportosztályt, amikből affin teljes csoporthatások származtathatók. Ilyenek például a véges egyszerű csoportok, a szimmetrikus csoportok vagy a Frobenius-csoportok. Szerepelni fognak olyan csoportok is, melyek sosem adnak meg affin teljes csoporthatásokat, például azon kommutatív csoportok, melyek nem izomorfak egy elemi Abel 2-csoporttal. Ilyen esetekben különböző kompatibilis függvényeket, például hatványfüggvényeket kell konstruálnunk, melyek cáfolják az algebra affin teljességét.

Megvizsgáljuk, milyen esetekben terjed ki az affin teljesség a részcsoportokról a csoportra. Végül bevezetünk egy új definíciót, ami minden csoportnak egy karakterisztikus részcsoportját adja meg. Bizonyos értelemben ez a részcsoport méri azt, hogy a csoport mennyire áll közel ahhoz, hogy 1-affin teljes csoporthatást adjon meg.

Affine complete G-sets

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Abstract

György Grätzer was the first to investigate affine complete algebras. He realized the importance of this topic in universal algebra. Recently, John Snow observed that if the congruence lattice of a finite algebra is order polynomially complete, then this lattice is isomorphic to the congruence lattice of a vector space or of an affine complete G-set.

We take this as motivation to study affine complete G-sets in general. Recall that, for a group G acting on a set Ω, the algebra $\langle \Omega, G \rangle$ is called a G-set. We will investigate faithful actions only. An algebra is affine complete if all its congruence preserving functions are polynomials.

We present some families of affine complete G-sets, like regular actions of certain groups. We will prove that the regular G-set corresponding to a Frobenius group or a non-abelian group that is generated by involutions is affine complete. For some other classes of groups, including finite abelian groups (except elementary abelian 2-groups), we can construct several compatible functions, in particular, power endomorphisms, that are not polynomial functions.

We will investigate the heredity of affine completeness in regular G-sets. Then we will define the characteristic subgroup $C(G)$ of G. This subgroup is suitable to indicate how close the group is to produce a 1-affine complete G-set.

1 Introduction

1.1 Definitions

As usual, the expression *algebra* refers to a pair $\mathcal{A} = (A, F)$. Here, A is called the underlying set of $\mathcal{A}$ and F contains the fundamental operations. Many well-known algebraic structures such as groups, rings, fields, Boolean algebras, lattices or modules are all considered as algebras now. The algebra $\mathcal{A}$ is finite if both A and F are finite.

Let $\mathfrak{C}$ be a set of functions on $\mathcal{A}$, i.e. for any $f \in \mathfrak{C}$ there is a $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$ such that we have $f : A^k \to A$. Such $\mathfrak{C}$ is called a *clone* if it contains all projections and it is closed under composition. The smallest clone containing the fundamental operations and the constant functions is the set of polynomials. In other words, a polynomial is a function on $\mathcal{A}$ that can be obtained from the fundamental operations, constant functions and projections by finitely many compositions. For a group G it is hard to describe the polynomials of G. Typically, a polynomial can be written as a product of some elements of G, some variables and the inverse of some variables. Note that a polynomial can be written in many different forms here. However, if G is an abelian group then there is a canonical form of the polynomials, namely these are the functions $(x_1, x_2, \ldots, x_k) \mapsto a_1x_1 + a_2x_2 + \ldots a_kx_k + g$, where $a_i \in \{0, 1, \ldots exp(G) - 1\}$ for any $i \leq k$ and $g \in G$. If we have a finite field F then according to the Lagrange interpolation every function on F is a polynomial. Such algebras are called functionally complete or polynomially complete. Another important example for a polynomially complete algebra is the 2-element Boolean algebra.

A congruence on $\mathcal{A}$ is an equivalence relation ρ satisfying the following condition: for any fundamental operation f with arity k, and elements $a_1, b_1, \ldots, a_k, b_k \in \mathcal{A}$ such that $(a_i, b_i) \in \rho$ we have that $(f(a_1, a_2, \ldots, a_k), f(b_1, b_2, \ldots, b_k)) \in \rho$. As ρ is a binary relation, it is a subset of $A \times A$. It is easy to see that an equivalence relation ρ is a congruence if and only if it is a subalgebra of $A \times A$. The congruences of a group G are well studied. Every congruence corresponds to a normal subgroup of G in the following way. For a given $N \lhd G$, the congruence ρ_N is the binary relation $\{(a, b) \in G \times G \mid aN = bN\}$. Similarly, the congruences of a ring R are the equivalence relations corresponding to the ideals of R. Here, the equivalence classes of ρ_I for a given ideal I are the cosets of I. Usually, we abbreviate $(a, b) \in \rho$ with $a \equiv_\rho b$ or simply $a\rho b$. The congruences of an algebra $\mathcal{A}$ form a lattice denoted by $Con(\mathcal{A})$.

An n-ary function f on $\mathcal{A}$ is called compatible with the relation $R \subseteq A^k$ if the following holds:

$$\forall a_{i,j} \in A \quad (1 \leq i \leq n, 1 \leq j \leq k) :$$

$$((a_{1,1}, \ldots, a_{1,k}) \in R, \ldots, (a_{n,1}, \ldots, a_{n,k}) \in R) \Rightarrow$$

$$\Rightarrow ((f(a_{1,1}, \ldots, a_{n,1}), \ldots, f(a_{1,k}, \ldots, a_{n,k})) \in R)$$

An equivalence relation is a congruence if and only if every fundamental operation is compatible with it. Note that every function is compatible with the smallest congruence (equality) and the biggest congruence (any two element is congruent). We call a function f on $\mathcal{A}$ compatible if it is compatible with all the congruences of $\mathcal{A}$. According to the fact that every function is compatible with the smallest and the biggest congruence in $Con(\mathcal{A})$ we have that on a simple algebra every function is compatible. Thus for any prime p and any $\alpha \in \mathbb{N}$, the cyclic group Z_p (and all other finite simple groups), the finite field F_{p^α} and the 2-element Boolean algebra have the property that every function is compatible on them. In general, it is clear that constant functions, projections and the fundamental operations are compatible. It is also easy to see that the composition of compatible functions is also compatible. Thus the polynomials are compatible. This observation yields the following definitions.

Definition 1. *An algebra $\mathcal{A}$ is affine complete if every compatible function on $\mathcal{A}$ is a polynomial. $\mathcal{A}$ is k-affine complete if every at most k-ary compatible function is a (k-ary) polynomial.*

Clearly $\mathcal{A}$ is affine complete if and only if $\mathcal{A}$ is k-affine complete for all $k \in \mathbb{N}$. Note that a polynomially complete algebra is always affine complete, as it only has polynomial functions, thus it is trivial that every compatible function is a polynomial on them. This holds for finite fields or the 2-element Boolean algebra. For the cyclic group Z_p, we have p^{p^n} n-ary functions, and these are all compatible. However, according to the canonical form of polynomials on abelian groups, we have that there are p^{n+1} n-ary polynomials. Thus for $p \geq 3$ the cyclic group Z_p is not k-affine complete for any $k \in \mathbb{N}$.

1.2 Known Results

It was György Grätzer in the early 60's who posed the following question:
What are the affine complete algebras?

Of course, the aim is not to give a general description for such algebras, but to characterize them according to the different types. It was Grätzer himself who achieved the first significant results in this area. He proved that every Boolean algebra is affine complete in [1]. He also characterized the affine complete bounded distributive lattices. According to [2] a bounded distributive lattice is affine complete if and only if it does not contain proper Boolean intervals (intervals containing at least two elements). Hence there are no finite affine complete distributive lattices. Note that in the point of view of affine completeness, the main difference between Boolean algebras and lattices is that the type of a Boolean algebra contains an extra unary function, the complement. Hence while a polynomial function is order-preserving on a lattice, it need not to be order-preserving on a Boolean algebra.

As all polynomial functions of a lattice are order-preserving, it is natural to investigate the following property. A lattice $\mathcal{L}$ is order affine complete, if every order-preserving compatible function is a polynomial on $\mathcal{L}$. In case of lattices it is also natural to study lattices satisfying the condition that every order-preserving function is a polynomial, rather than lattices with the property that every function is a polynomial. These lattices are called order polynomially complete. Here is a description of order polynomially complete modular lattices.

R. Wille: [9] A finite modular lattice $\mathcal{L}$ is order polynomially complete if and only if $\mathcal{L}$ is simple and relatively complemented.

The following theorem is the classification of affine complete semilattices.

K. Kaarli, L. Márki, E. T. Schmidt: [5] A semilattice $\mathcal{S}$ is affine complete if and only if the following conditions hold for $\mathcal{S}$:

- $\mathcal{S}$ does not contain any atoms;
- $\forall a \in \mathcal{S}\ \nexists b \in \mathcal{S}$ such that $0 \neq b < a$ and $(a] = (b] \cup [b, a]$ hold simultaneously;
- if a proper ideal I of $\mathcal{S}$ satisfies that $\forall a \in \mathcal{S}$ the ideal $(a] \cap I$ is a principal ideal, than I is a principal ideal.

It was K. Kaarli who characterized affine complete abelian groups in [3]. According to his result, a finite abelian group G, written in the form $Z_{n_1} \oplus Z_{n_2} \oplus \ldots \oplus Z_{n_k}$ with $n_k|n_{k-1}|\ldots|n_2|n_1$, is affine complete if and only if $n_1 = n_2$. Affine complete modules were investigated by A. Saks.

For a more detailed discussion of the topic, see [4] or [7]. In what follows we will present some new results for G-sets from the point of view of affine completeness.

2 1-affine completeness

To decide whether a G-set $\langle \Omega, G \rangle$ is affine complete, we have to investigate all k-ary compatible functions and also all k-ary polynomials for all $k \in \mathbb{N}$. However, there aren't any interesting k-ary polynomials of a G-set $\langle \Omega, G \rangle$ for $k \geq 2$. This observation leads to the following result.

Theorem 1. *Let G be a non-trivial permutation group on a set Ω, that is $|\Omega|, |G| \geq 2$. Assume that $\langle \Omega, G \rangle$ is 1-affine complete. Then exactly one of the following holds:*

(1) $\langle \Omega, G \rangle$ is affine complete.

(2) $|\Omega| = 2$ *or there exists a division ring* K *and a* K*-module* V *of rank greater than* 1 *over* K *such that* $\langle \Omega, G\rangle$ *is isomorphic to*

$$\langle V, \{x \mapsto kx + v \mid k \in K, k \neq 0, v \in V\}\rangle.$$

For finite Ω the result above is immediate from Pálfy's classification of minimal algebras. We will give a self-contained, elementary proof for arbitrary cardinalities at the end of this section. First we state an auxiliary lemma.

Lemma 2. *Let* G *be a semidirect product of a normal subgroup* $T \neq \{1\}$ *and* $H \leq G$*. We assume that* H *acts faithfully on* T *by conjugation and that every* H*-invariant subgroup of* T *is normal in* T*.*

(1) Then $\langle G/H, G\rangle$ *is not* 2*-affine complete.*

(2) $\langle G/H, G\rangle$ *is* 1*-affine complete if and only if* $|G| = 2, H = 1$ *or* T *forms a module of rank greater than* 1 *over some division ring* K*, and* $\langle G/H, G\rangle$ *is isomorphic to*

$$\langle T, \{x \mapsto kx + t \mid k \in K, k \neq 0, t \in T\}\rangle.$$

Proof: We claim that the function

$$f(aH, bH) := abH \text{ for } a, b \in T \text{ is compatible.} \tag{1}$$

For $H \leq U \leq G$ we have an $N \leq T$ such that $U = NH$ and N is H-invariant, hence normal in T. Let $a_1, a_2, b_1, b_2 \in T$ be such that $a_1NH = a_2NH, b_1NH = b_2NH$. Using the normality of N in T we obtain $f(a_1H, b_1H)NH = a_1b_1NH = a_1b_2NH = a_1Nb_2H = a_2Nb_2H = a_2b_2NH = f(a_2H, b_2H)NH$ and (1) is proved. Since f is essentially binary for $T \neq 1$, $\langle G/H, G\rangle$ is not 2-affine complete.

Next assume that $\langle G/H, G\rangle$ is 1-affine complete. We will show that

$$K := \{c_h : T \to T, x \mapsto x^h \mid h \in H\} \cup \{o : T \to T, x \mapsto 1\}$$

is closed under pointwise multiplication $\cdot$ of functions. For $g, h \in H$ the function

$$p(aH) := a^g a^h H \text{ for } a \in T$$

is compatible by a similar argument as above. By the 1-affine completeness and $p(H) = H$, either p is constant or we have $r \in H$ such that $p(aH) = raH$ for all $a \in T$. In the former case $a^g a^h \in H$ yields $a^g a^h = 1$ for all $a \in T$ and $c_g \cdot c_h = o$. In the latter case we have $(a^{-1})^r a^g a^h \in H$ yields $(a^{-1})^r a^g a^h = 1$ for all $a \in T$ and $c_g \cdot c_h = c_r$. For $h \in H$ we define $i(c_h) : T \to T, x \mapsto (x^h)^{-1}$ and $i(o) := o$. As above we find $i(c_h) = c_s$ for some $s \in T$. Hence $\langle K, \cdot, i, o\rangle$ is an abelian group. Clearly $\langle K \setminus \{o\}, \circ, ^{-1}, \mathrm{id}_T\rangle$ is a group isomorphic to H. For $u, v, w \in K$ we have the distributive laws $(u \cdot v) \circ w = (u \circ w) \cdot (v \circ w)$ and $u \circ (v \cdot w) = (u \circ v) \cdot (u \circ w)$ by the definition of the pointwise multiplication $\cdot$ of functions. Thus $\langle K, \cdot, \circ\rangle$ is a division ring. T forms a K-module. Denote the group operation in T by $+$. Clearly G is isomorphic to

$$A := \{x \mapsto kx + t \mid k \in K, k \neq 0, t \in T\}, \tag{2}$$

and $\langle G/H, G\rangle$ is isomorphic to $\langle T, A\rangle$. If T has dimension 1 over K, then $\langle T, A\rangle$ is simple and not 1-affine complete except if $|T| = 2$.

Conversely it is straightforward that $\langle G, G\rangle$ is 1-affine complete if $|G| = 2$. Now assume that T is a K-module of rank greater than 1 over some division ring K. Let A be as in (2). Let $f : T \to T$ be congruence preserving on $\langle T, A\rangle$ such that $f(0) = 0$. Then f preserves cosets of all K-submodules of T.

Let $x \in T$. Then we have $k \in K$ such that $f(x) = kx$. Likewise for $y \in T \setminus Kx$ we have $f(y) = ly$ and $f(x + y) = m(x + y)$ for some $l, m \in K$. Then $f(x + y) \equiv f(x) \mod Ky$ and $f(x + y) \equiv f(y) \mod Kx$ yield $mx \equiv kx \mod Ky$ and $my \equiv ly \mod Kx$. Hence $m = k = l$ and $f(y) = ky$ for all $y \in T \setminus Kx$. Similarly, $f(z) = kz$ for all $z \in T \setminus Ky$ and in particular for all $z \in Kx \setminus \{0\}$. Then $f(x) = kx$ for all $x \in T$. Thus f is in A. $\square$

For $H = 1$ we obtain the following consequence.

Lemma 3. *Let G be an abelian group with $|G| > 1$. Then $\langle G, G\rangle$ is not 2-affine complete, and $\langle G, G\rangle$ is 1-affine complete if and only if G has exponent 2.*

Proof: As we have already proved the binary function $h : (x, y) \mapsto xy$ is compatible, thus $\langle G, G\rangle$ is not 2-affine complete.

The function $f : x \mapsto x^2$ is compatible on G. Indeed, let $H \leq G$, and $a, b \in G$ such that $aH = bH$. As H is a normal subgroup in G, we have $a^2H = aHa = bHa = b^2H$. Thus this power map is compatible. We only have to check whether it is a polynomial function. It is clearly not a left translation, and it is constant only if the exponent of G is 2. Thus an abelian group that is not an elementary abelian 2-group is not 1-affine complete. It is easy to see that an elementary abelian 2-group is 1-affine complete. Later, we will prove a more general theorem, namely that every group that is generated by involutions is 1-affine complete.

$\square$

In this proof, we only used that every subgroup of an abelian group is normal. Hence the same argument works for Hamiltonian groups, as well.

We recall some well known facts about transitive permutation actions.

Lemma 4. *Let G be a transitive permutation group on a set Ω, let $\alpha \in \Omega$, and let $H := \{g \in G \mid g\alpha = \alpha\}$.*

(1) $\langle \Omega, G\rangle$ is isomorphic to $\langle G/H, G\rangle$ where $g * xH := gxH$ for all $g, x \in G$.

(2) For $H \leq U \leq G$ define

$$\rho_U := \{(xH, yH) \in (G/H)^2 \mid xU = yU\}.$$

Then $\mathrm{Con}(\langle G/H, G\rangle) = \{\rho_U \mid H \leq U \leq G\}$.

Proof of Theorem 1: By a result of Salomaa in [8] it suffices to show that every binary compatible function is constant or essentially unary or that the conditions of (2) in Theorem 1 are satisfied.

Let $e \in \mathrm{Comp}_2(\Omega, \langle \Omega, G\rangle)$. Since $|G| > 1$ by assumption, we have $\alpha \in \Omega$ such that $|G\alpha| > 1$. By the 1-affine completeness of $\langle \Omega, G\rangle$ the functions $e_1(x) := e(x, \alpha)$ and $e_2(x) := e(\alpha, x)$ are polynomial. Hence they are either constant or in G. If, for example, $e_1 \in G$, then $f(x, y) := e(e_1^{-1}(x), y)$ satisfies $f(x, \alpha) = x$ for all $x \in \Omega$. Hence it suffices to consider $f \in \mathrm{Comp}_2(\Omega, \langle \Omega, G\rangle)$ such that the unary functions $x \to f(x, \alpha)$ and $x \to f(\alpha, x)$ are constant or the identity function, respectively. Now, it is clear that we may assume $f(\alpha, \alpha) = \alpha$, as well. For $\omega \in \Omega$ we define

$$f_\omega(x) := f(\omega, x) \text{ for } x \in \Omega.$$

We have four cases, now.

Case 1: $\underline{f(x, \alpha) = \alpha, f(\alpha, x) = \alpha \text{ for all } x \in \Omega.}$ Since $|G\alpha| > 1$, we have $g \in G$ such that $g\alpha \neq \alpha$. Then $h(x) := f(x, gx)$ satisfies $h(\alpha) = h(g^{-1}\alpha) = f(\alpha, \alpha)$. As a non-injective polynomial function, h is constant and

$$f(x, gx) = f(\alpha, \alpha) \text{ for all } x \in \Omega.$$

For $\beta \in \Omega, \beta \neq g^{-1}(\alpha)$, we have $f_\beta(\alpha) = f_\beta(g(\beta)) = f(\alpha, \alpha)$. Consequently f_β is constant and

$$f(x, y) = f(\alpha, \alpha) \text{ for all } x, y \in \Omega, x \neq g^{-1}\alpha.$$

If $|\Omega| > 2$, then the functions of the form $x \mapsto f(x, \gamma)$ for $\gamma \in \Omega$ are not injective and hence f is constant. Otherwise, if $|\Omega| = 2$, then $|G| = 2$ and $\langle \Omega, G\rangle$ satisfies the conditions of (2).

Case 2: $\underline{f(x, \alpha) = x, f(\alpha, x) = \alpha \text{ for all } x \in \Omega.}$ Assume that we have $\beta \in \Omega$ such that f_β is not constant. Then $f_\beta \in G$ and $f_\beta(\alpha) = \beta$. The function $r(x) := f(x, f_\beta^{-1}(\alpha))$ satisfies $r(\alpha) = \alpha$ and $r(\beta) = f_\beta(f_\beta^{-1}(\alpha)) = \alpha$. Hence r is constant and

$$f(x, f_\beta^{-1}(\alpha)) = \alpha \text{ for all } x \in \Omega.$$

Let $\omega \in \Omega \setminus \{\alpha\}$. Then f_ω is not constant but in G. In particular, $\omega = f_\omega(\alpha)$ is in $G\alpha$ and $G\alpha = \Omega$.

Let $g \in G$ be such that $g(\alpha) \neq f_\beta^{-1}(\alpha)$. Then $h(x) := f(x, g(x))$ satisfies $h(\alpha) = \alpha$ and $h(g^{-1} f_\beta^{-1}(\alpha)) = f(g^{-1} f_\beta^{-1}(\alpha), f_\beta^{-1}(\alpha)) = \alpha$. Hence h is constant and $h(g^{-1}(\alpha)) = f(g^{-1}(\alpha), \alpha) = g^{-1}(\alpha)$ yields $g^{-1}(\alpha) = \alpha$. Hence $G\alpha = \{\alpha, f_\beta^{-1}(\alpha)\}$ and both Ω and G have size 2. Then $\langle \Omega, G\rangle$ satisfies the conditions of (2).

Case 3: $\underline{f(x,\alpha) = \alpha, f(\alpha, x) = x. \text{ for all } x \in \Omega}$: This case is essentially the same as the previous one.

Case 4: $\underline{f(x,\alpha) = f(\alpha, x) = x \text{ for all } x \in \Omega.}$ We will show that G satisfies the conditions of (2). First suppose that there exists $\beta \in \Omega \setminus G\alpha$. Since $f_\beta(\alpha) = \beta \notin G\alpha$, f_β is not in G but constant. Hence

$$f(x,y) = x \text{ for all } x \in \Omega \setminus G\alpha, y \in \Omega,$$

and similarly

$$f(x,y) = y \text{ for all } x \in \Omega, y \in \Omega \setminus G\alpha.$$

Now $x = y$ for all $x, y \in \Omega \setminus G\alpha$ yields $\Omega = G\alpha \cup \{\beta\}$. Let $\gamma \in G\alpha, \gamma \neq \alpha$. The function $c : \Omega \to \Omega$ defined by $c(G\alpha) := \alpha$ and $c(\beta) = \gamma$ is compatible because the total congruence is the unique congruence ρ such there exists $\delta \in G\alpha$ with $(\beta, \delta) \in \rho$. Since c is obviously not polynomial, we have a contradiction. Hence $\Omega = G\alpha$, that is, G is transitive on Ω. Let H be the stabilizer in G of α. By Lemma 4 we may assume that $\Omega = \{xH \mid x \in G\}$.

If there exists an $a \in G$ such that $f_{a\alpha}$ is constant, then $f'(x,y) := a^{-1} f(ax, y)$ satisfies the assumptions of the previous case and consequently $|\Omega| = |G| = 2$. Hence we assume $f_\omega \in G$ for all $\omega \in \Omega$ in the following. Note that $T := \{f_\omega \in G \mid \omega \in \Omega\}$ is a transversal through G/H. It is clear that $f(xH, yH) = xyH$ for any $x, y \in T$.

Let $p \in \mathrm{Pol}_1(\mathbf{G})$. We claim that

$$q(aH) := p(a)H \text{ for } a \in T \text{ preserves all congruences of } \langle \Omega, G\rangle. \tag{3}$$

Let $\rho \in \mathrm{Con}(\langle \Omega, G\rangle)$. By Lemma 4 we have $H \leq U \leq G$ such that $\rho = \rho_U$. Let $a, b \in T$ such that $aU = bU$. For $x \in G$, $f(aH, xH) = axH$ and $f(bH, xH) = bxH$ are congruent modulo ρ_U since f is congruence preserving. Hence

$$(b^{-1}a)^x \in U. \tag{4}$$

We have $l \in \mathbb{N}$, $g_1, \ldots, g_l \in G$, and $e_1, \ldots, e_{l-1} \in \{-1, 1\}$ such that $p(x) = g_1 x^{e_1} g_2 \cdots g_{l-1} x^{e_{l-1}} g_l$ for all $x \in G$. Then

$$\begin{aligned} p(b)^{-1} p(a) &= g_l^{-1} b^{-e_{l-1}} g_{l-1}^{-1} \cdots b^{-e_1} g_1^{-1} g_1 a^{e_1} g_2 \cdots g_{l-1} a^{e_{l-1}} g_l \\ &= (b^{-e_{l-1}} a^{e_{l-1}})^{g_l} (g_{l-1}^{-1} \cdots b^{-e_1} a^{e_1} g_2 \cdots g_{l-1})^{a^{e_{l-1}} g_l}. \end{aligned}$$

By induction on l we obtain that $p(b)^{-1}p(a)$ is a product of conjugates of $b^{-1}a$ and of ba^{-1}, that is, conjugates of $(b^{-1}a)^{-1}$. Hence by (4) we have $p(b)^{-1}p(a) \in U$. Thus q preserves ρ_U.

We show that

$$H \cap H^x = 1 \text{ for all } x \in G \setminus H. \tag{5}$$

Let $b \in T \setminus \{1\}$, and let $c \in bHb^{-1} \cap H$. The function $g(aH) := a^{-1}caH$ for $a \in T$ is compatible by (3). Since $g(bH) = H$ and $g(H) = H$, we have that g is constant and $c^a \in H$ for all $a \in T$. Hence $c \in \bigcap_{a \in T} aHa^{-1} = \bigcap_{x \in G} xHx^{-1} = 1$. Therefore $bHb^{-1} \cap H = 1$ which proves (5).

Next we show

$$[a^h, b] = 1 \text{ for all } a, b \in T, h \in H. \tag{6}$$

We fix $b \in T, h \in H$. The function $m(aH) := [a^h, b]H$ for $a \in T$ is compatible by (3). Since $m(b^{h^{-1}}H) = m(H) = H$, we have that m is constant and $[a^h, b] \in H$ for all $a \in T$. By a similar argument $b(a^h)^{-1}b^{-1}a^h$ is in H for all $a \in T$. Then $[a^h, b] = (b(a^h)^{-1}b^{-1}a^h)^b$ is in $H \cap H^b$ and $[a^h, b] = 1$ by (5).

Now $\langle T \rangle$ is abelian by (6) and $H \cap \langle T \rangle = 1$ by (5). Hence $\langle T \rangle = T$. For $a \in T$, $h \in H$ we have $u \in T, v \in H$ such that $a^h = uv$. By (6) uv commutes with u^{-1}. Hence $v = uvu^{-1}$ which implies $v = 1$ by (5). Thus T is an abelian normal subgroup of G.

Finally we see that every binary compatible function on $\langle \Omega, G \rangle$ is constant, essentially unary or that G is a semidirect product of a nontrivial abelian normal subgroup T and $H \leq G$ such that $\langle \Omega, G \rangle$ is isomorphic to $\langle G/H, G \rangle$. In the latter case $\langle \Omega, G \rangle$ satisfies the conditions of (2) and is not 2-affine complete by Lemma 2. $\square$

According to Theorem 1 we only have to consider unary functions, from now on. In other words, we study 1-affine completeness of G-sets.

3 Regular group actions

3.1 Compatible functions

For a group G let $R(G)$ denote the regular G-set $\langle G, G \rangle$. According to Lemma 4 a function $f : G \to G$ is compatible on the regular G-set $R(G)$ if and only if for any $H \leq G$ and $a, b \in G$ such that $aH = bH$, it holds that $f(a)H = f(b)H$. The G-set $R(G)$ is 1-affine complete if every compatible function on $R(G)$ is constant or is a multiplication with an element of G. Let f be a function on G such that $f(1) = g$ for some $g \in G$. For an arbitrary $h \in G$ let m_h be the function

$$m_h : G \to G$$

$$x \mapsto hx$$

These are compatible functions on $R(G)$ and $(m_{g^{-1}} \circ f)(1) = 1$. It is also clear that $m_g \circ (m_{g^{-1}} \circ f) = f$. As the composition of compatible functions is again a compatible function this implies that f is compatible if and only if $m_{g^{-1}} \circ f$ is compatible. Thus $R(G)$ is 1-affine complete if and only if the compatible functions on $R(G)$ fixing 1 are 1_G and id_G, where 1_G denotes the constant 1 map on G and id_G denotes the identity of G. This simplifies the matter of the 1-affine completeness of $R(G)$. However, the most important property of a compatible function f satisfying $f(1) = 1$ is that it leaves invariant the subgroups of G. Indeed, let $H \leq G$ and $h \in H$, then $1H = hH$, thus $H = 1H = f(1)H = f(h)H$, hence $H = f(h)H$ and $f(h) \in H$. As these functions leave invariant the subgroups of G we can restrict them to any subgroup $H \leq G$. It is clear that if f is compatible on $R(G)$ such that $f(1) = 1$ then $f|_H$ is compatible on $R(H)$ such that $f|_H(1) = 1$.

The following definition is a useful tool in our investigations. It is a way to construct new compatible functions if we are given a compatible function f.

Definition 2. *Let f be a compatible function on G. Then f_g is the following function: $f_g : x \to f(g)^{-1}f(gx)$*

Lemma 5. *Given a compatible function f on $R(G)$ and an arbitrary $g \in G$, the function f_g is also compatible on $R(G)$ such that $f_g(1) = 1$.*

Proof: Note that $f_g = m_{f(g)^{-1}} \circ f \circ m_g$. As the composition of compatible functions is also compatible, we proved that f_g is compatible. It is also easy to see that $f_g(1) = f(g)^{-1}f(g) = 1$. □

Semidirect products and direct products of groups often occur among the subgroups of a given group. The following lemma shows how a compatible function is determined by its restriction to the components.

Lemma 6. *Let $G = A \rtimes B$ and f be a compatible function on $R(G)$ such that $f(1) = 1$. Then $f(ab) = f(a)f(b)$ holds for any $a \in A$ and $b \in B$.*

Proof: Every $g \in G$ is uniquely determined by its A-coset and B-coset. Thus we only have to show that $f(ab)A = f(a)f(b)A$ and $f(ab)B = f(a)f(b)B$.

We have that $ab = ba^b$ where $a^b \in A$, thus $abA = bA$. As f is compatible, we have $f(ab)A = f(b)A$. Since $f(1) = 1$ we know that f preserves the subgroups of G, hence $f(a) \in A$. As A is a normal subgroup of G, we have that $f(a)^{f(b)} \in A$, thus $A = f(a)^{f(b)}A$. This implies that $f(ab)A = f(b)A = f(b)f(a)^{f(b)}A = f(a)f(b)A$.

As f is compatible and $abB = aB$, we have that $f(ab)B = f(a)B$. As f fixes 1, we know that $f(b) \in B$. Thus $f(ab)B = f(a)B = f(a)f(b)B$ and we are ready. □

Corollary 7. *Let $n \geq 1$ and $\alpha \geq 2$. Then the compatible functions of $R(\mathbb{Z}_n^\alpha)$ fixing 1 are exactly the following:*

$$p_k : \mathbb{Z}_n^\alpha \to \mathbb{Z}_n^\alpha, \quad k \in \mathbb{N}$$

$$x \to x^k$$

Proof: The function p_k is a homomorphism of $\mathbb{Z}_n^\alpha$ that leaves invariant the subgroups. Thus given a $H \leq \mathbb{Z}_n^\alpha$ and $a, b \in \mathbb{Z}_n^\alpha$ such that $aH = bH$, we have that $b^{-1}a \in H$. Thus $p_k(b^{-1}a) \in H$, hence $p_k(b)^{-1}p_k(a) \in H$. This means $p_k(a)H = p_k(b)H$, thus p_k is indeed a compatible function fixing 1.

Now suppose that f is a compatible function on $R(\mathbb{Z}_n^\alpha)$ fixing 1. Let $\mathbb{Z}_n^\alpha = \prod_{i=1}^\alpha \langle a_i \rangle$ and let $k \in \mathbb{N}$ be such that $f(a_1) = a_1^k$. We claim that $f = p_k$. Let $b \in \langle a_j \rangle$ for some $j \geq 2$ and let $f(b) = b^l$ with $l \in \mathbb{N}$. According to Lemma 6 we have that $f(a_1 b) = f(a_1)f(b)$, thus $f(a_1 b) = a_1^k b^l$. This is also an element of $\langle a_1 b \rangle$, hence it is $(a_1 b)^t = a_1^t b^t$ for some $t \in \mathbb{N}$. Thus $a_1^t b^t = a_1^k b^l$, hence $t \equiv k \pmod{n}$. Thus $f(b) = b^k$. The same method works for a_2 and an arbitrary element of $\langle a_1 \rangle$. Thus for any $1 \leq j \leq \alpha$ and $x \in \langle a_j \rangle$, we have that $f(x) = x^k$. According to Lemma 6 it is clear that $f = p_k$, now. □

3.2 Hereditary conditions

Proposition 8. *Let G be a group with subgroups U_i $(i \in I), N$ and $H := \bigcap_{i \in I} U_i$ such that $H \leq N$ and $U_i N = G$ for all $i \in I$.*

Then $\langle G/H, G \rangle$ is 1-affine complete if $\langle N/H, N \rangle$ is 1-affine complete.

Proof: Let f be compatible on $\langle G/H, G \rangle$ with $f(H) = H$. Let $x \in G$, and let $i \in I$. Then we have $u \in U_i$ such that $x^{-1}N = uN$, that is, $xu \in N$. Since $xU_i = xuU_i$, we have

$$f(xH)U_i = f(xuH)U_i. \tag{7}$$

First we assume that f is constant H on N/H. Then $f(xuH) = H$ yields $f(xH) \subseteq U_i$ by (7). Hence $f(xH) \subseteq \bigcap_{i \in I} U_i = H$. Thus f is constant H on G/H.

Next we assume that f is the identity on N/H. Then $f(xuH) = xuH$ yields $f(xH) \subseteq xU_i$ by (7). Hence $f(xH) \subseteq x \bigcap_{i \in I} U_i = xH$. Thus f is the identity on G/H. □

Let $G := \mathrm{AGL}(n, K)$ be the group of affine transformations on K^n, and let $H := Z(\mathrm{GL}(n, K))$. If $n > 1$, then $\langle G/H, G \rangle$ is affine complete by Proposition 8. [For $v \in K^n$ define $t_v : K^n \to K^n, x \mapsto x + v$. Let $N := \{t_v \mid v \in K^n\}H$. Then $\langle N/H, N \rangle$ is 1-affine complete by Lemma 2. Let $U := \mathrm{GL}(n, K)$. Then

$G = UN$. For $v \in K^n$ we have $U^{t_v}H \cap U = \{a \in U \mid av = kv \text{ for some } k \in K\}$. Hence $\bigcap_{v \in K^n} U^{t_v}H = H$. Proposition 8 applies and yields that $\langle G/H, G\rangle$ is 1-affine complete. Since H has no abelian complement in G, Theorem 1 implies that $\langle G/H, G\rangle$ is ac.]

Corollary 9. *Let G be a group with a normal subgroup N and $U \leq G$ such that $G = UN$ and $\bigcap_{g \in G} U^g = 1$. Assume that $R(N)$ is 1-affine complete. Then $R(G)$ is 1-affine complete.*

Before our observations on how the affine completeness of proper subgroups imply the affine completeness of the group, we show a construction. The G-sets obtained by this construction are never 1-affine complete.

Proposition 10. *Let A and B be nontrivial groups such that $gcd(|A|, |B|) = 1$. Then $R(A \times B)$ is not 1-affine complete.*

Proof: The elements of the group $A \times B$ can be uniquely written in the form ab such that $a \in A$ and $b \in B$. Let f be the function that maps ab to a. Here $f(1) = 1$ holds. As $gcd(|A|, |B|) = 1$, the subgroups of $A \times B$ are of the form $C \times D$ with $C \leq A$ and $D \leq B$. Thus a function h is compatible on $R(A \times B)$ if and only if $h|_A$ is compatible on $R(A)$ and $h|_B$ is compatible on $R(B)$. The function f clearly satisfies this condition but it is neither constant 1 nor the identity. Hence $A \times B$ is not 1-affine complete. □

Lemma 11. *Let G be a group with subgroups A, B. Suppose that there exists a compatible function f on $R(G)$ such that $f(A) = 1$ and $f|_B = \mathrm{id}_B$.*

Then $[A, B] = 1$ and $\gcd(|A|, |B|) = 1$.

Proof: Let $a \in A, b \in B$. Notice that $f(1) = 1$. Since $b\langle b^{-1}a\rangle = a\langle b^{-1}a\rangle$, we have $f(b) \in f(a)\langle b^{-1}a\rangle$, that is, $b \in \langle b^{-1}a\rangle$. Then b commutes with $b^{-1}a$ and consequently with a. Hence A and B centralize each other.

Clearly $A \cap B = 1$. Suppose that a prime p divides $\gcd(|A|, |B|)$. Let $a \in A, b \in B$ both have order p. Then $\langle a, b\rangle$ is an elementary abelian group of order p^2. Since $f(ab) = f(a) \cdot f(b) = b$ is not contained in $\langle ab\rangle$, the function f does not preserve the congruence modulo $\langle ab\rangle$. This contradicts the assumption that f is compatible. Thus $\gcd(|A|, |B|) = 1$. □

Lemma 12. *Let G be a group and $A \leq G$ such that $R(A)$ is 1-affine complete. Let f be a compatible function on G such that $f(1) = 1$ and let $g \in G$ be an arbitrary element. Then $f|_{gA}$ is either constant or it is of the form $x \to h * x$ with some $h \in \langle g\rangle$.*

Proof: According to Lemma 5 we know that f_g is a compatible function on $R(G)$. The restriction of f_g to A is a compatible function on $R(A)$ with $f_g(1) = 1$. As $R(A)$ is 1-affine complete we got that $f_g|_A$ is either constant 1 or the identity.

Case 1: $\underline{f_g|_A \text{ is constant } 1}$. Then for any $x \in A$ we have that $f(g)^{-1}f(gx) = 1$, hence $f(gx) = f(g)$ for all $x \in A$. Thus $f|_{gA}$ is constant.

Case 2: $\underline{f_g|_A \text{ is the identity}}$. Then for any $x \in A$ we have that $f(g)^{-1}f(gx) = x$, hence $f(gx) = f(g)x = (f(g)g^{-1})(gx)$ for all $x \in A$. Thus $f|_{gA}$ is the multiplication with $f(g)g^{-1}$. □

The following theorem is the most important hereditary condition for 1-affine complete G-sets. It covers the case of direct and semidirect products of groups producing 1-affine complete regular G-sets.

Theorem 13. *Let G be a group with subgroups A, B such that $R(A), R(B)$ are 1-affine complete. Then $R(\langle A, B\rangle)$ is 1-affine complete except if $[A, B] = 1$ and $\gcd(|A|, |B|) = 1$.*

Proof: Let f be a compatible function on $\langle A, B\rangle$ such that $f(1) = 1$. We have four cases now.

Case 1: $\underline{f|_A = 1_A \text{ and } f|_B = id_B}$. Then by Lemma 11 we have that $[A, B] = 1$ and $\gcd(|A|, |B|) = 1$.

Case 2: $\underline{f|_A = id_A \text{ and } f|_B = 1_B}$. Again by Lemma 11 we have that $[A, B] = 1$ and $\gcd(|A|, |B|) = 1$. In these two cases $\langle A, B\rangle = A \times B$ with $\gcd(|A|, |B|) = 1$, thus $R(\langle A, B\rangle)$ is not 1-affine complete according to Proposition 10.

Case 3: $\underline{f|_A = id_A \text{ and } f|_B = id_B}$. An element of $\langle A, B\rangle$ can be written as the product of elements of A and B. There is a shortest form of every element in $\langle A, B\rangle$. We prove by induction on this length that $f(x) = x$ for every $x \in \langle A, B\rangle$. For elements with length at most 1 it is clear that $f(x) = x$, because such an x is in A or in B. Now assume that the length of x is $l \geq 2$ and that the statement holds for any $x \in \langle A, B\rangle$ with length smaller than l. Without loss of generality we may assume that $x = x_1ab$ is the shortest form of x, $x_1 \in \langle A, B\rangle$, $a \in A$ and $b \in B$. Note that $x_1abB = x_1aB$ and $x_1abA^b = x_1bA^b$. Here $R(A^b)$ is also 1-affine complete, because A^b is isomorphic to A. According to Lemma 12 the function f on these cosets is either constant or it is a multiplication with some element of $\langle A, B\rangle$. Assume it is constant on both cosets. Then $f(x_1ab) = f(x_1a)$ and $f(x_1ab) = f(x_1b)$. As x_1a and x_1b has length smaller then l, we know that f fixes these elements. Thus $f(x_1ab) = x_1a = x_1b$ and consequently $a = b$. This is a contradiction, because this implies that x_1a^2 is a shorter form of x than x_1ab. Thus f is a multiplication with some element of $\langle A, B\rangle$ on at least one of the cosets. The element we multiply with can only be 1, because both cosets has an element that is fixed by f. Hence f fixes x, too.

Case 4: $\underline{f|_A = 1_A \text{ and } f|_B = 1_B}$. We will prove by induction on the length of an element of $x \in \langle A, B\rangle$ that $f(x) = 1$. For elements with length at most 1 it is clear that $f(x) = 1$ as these are the elements of $A \cup B$. Assume that $l \geq 2$ and the statement holds for elements with length smaller than l. Again, without loss of generality we may assume that $x = x_1ab$ is the shortest form of x, $x_1 \in \langle A, B\rangle$, $a \in A$ and $b \in B$. Note that $x_1abB = x_1aB$ and $x_1abA^b = x_1bA^b$, and that according to Lemma 12 the function f on these cosets is either constant or it is a multiplication with some element of $\langle A, B\rangle$. Assume that it is a multiplication with some element on both cosets. Then on the B-coset it has to be the multiplication with $(x_1a)^{-1}$ and on the A^b-coset it has to be the multiplication with $(x_1b)^{-1}$. As these cosets has a common element x, we got that $(x_1a)^{-1} = (x_1b)^{-1}$, hence $x_1a = x_1b$ and $a = b$. This is a contradiction, because if $a = b$ then x_1a^2 is a shorter form of x than x_1ab. Thus f is constant on at least one of these cosets. As both cosets has an element that is mapped to 1, the function f can only be constant 1, so we are ready. □

The following corollaries are immediate consequences of Theorem 13.

Corollary 14. *Let G be a group with subgroups A, B such that $G = AB$. Assume that $R(A), R(B)$ are 1-affine complete. Then $R(G)$ is 1-affine complete except if* $\gcd(|A|, |B|) = 1$ *and $G = A \times B$.*

Corollary 15. *Let G be a group that is generated by involutions. Then $R(G)$ is 1-affine complete.*

By Corollary 15 and Theorem 1 every nonabelian group G that is generated by involutions with the regular action on itself is affine complete. This covers the dihedral groups, nonabelian simple groups, S_n, $SL(n,q)$ for $n \geq 3$, and all nonabelian Coxeter groups. Since S_n is 1-affine complete it is clear that every group occurs as a subgroup of a 1-affine complete group. Later, we will show another way to construct such an embedding yielding a much smaller 1-affine complete group in general.

For a group G let $N := \langle x \in G \mid x^2 = 1\rangle$. If N has a complement U in G and $\bigcap_{g\in G} U^g = 1$, then $R(G)$ is affine complete by Corollary 9. This applies to $SL(n,q) \leq G \leq GL(n,q)$ for $n > 2$ and to $SL(2,q)\,\langle \mathrm{diag}(-1,1)\rangle \leq G \leq GL(2,q)$. [We note that $SL(n,q)\,\langle \mathrm{diag}(-1,1,\dots,1)\rangle$ is generated by all conjugates of a transposition matrix and by $\mathrm{diag}(-1,1,\dots,1)$.]

Corollary 16. *Let G be a finite group all of whose minimal normal subgroups are nonabelian. Then $R(G)$ is affine complete.*

Proof: Let N be the product of all minimal normal subgroups of G. Let $a \in G \backslash N$, and let $U := \langle a \rangle$. We claim that $K := UN$ satisfies the assumptions of Corollary 9. For $k \in K$ we have $U^k N = K$. Clearly $H := \bigcap_{k \in K} U^k$ is a cyclic normal subgroup of K. By assumption we have simple non-abelian subgroups $S_1, \ldots, S_l$ of G such that $N = S_1 \cdots S_l$. Hence N has no abelian normal subgroup and in particular $H \cap N = 1$. Thus $[H, N] = 1$. But the centralizer of N in G is trivial by assumption. Hence $H = 1$. Since N is generated by involutions by the Odd-Order Theorem, $R(N)$ is affine complete by Corollary 15. Hence $R(K)$ is 1-affine complete by Corollary 9.

Now let f be compatible on $R(G)$ with $f(1) = 1$. By Corollary 15 we have either $f(N) = 1$ or $f|_N = \mathrm{id}_N$. In the first case we find $f(a) = 1$ and in the second $f(a) = a$ by the 1-affine completeness of $R(K)$. Thus $f(N) = 1$ implies that f is constant on all of G and $f|_N = \mathrm{id}_N$ implies that f is the identity map on all of G. Now $R(G)$ is 1-affine complete and consequently affine complete by Theorem 1. □

Lemma 17. *Assume that G is a group with subgroups $H_1, H_2, \ldots, H_k$, $k \geq 2$, such that the set $G \backslash \{1\}$ is the disjoint union $\bigcup_{i=1}^{k} (H_i \backslash \{1\})$. Assume that f is a compatible function on G such that there exists an $x \neq 1$ in G satisfying $f(x) = 1$ or $f(x) = x$. Then f is the constant 1 map or the identity.*

Proof: Without loss of generality suppose that $x \in H_1$. Let $y \in H_j \setminus \{1\}$ for some $2 \leq j \leq k$. Let $z = x^{-1}y$ and $z \in H_l$. Note that $l \neq 1$ and $l \neq j$. Then $xH_l = xzH_l = yH_l$, thus $f(x)H_l = f(y)H_l$. Note that the intersection of any two cosets $g_1 H_{i_1}$ and $g_2 H_{i_2}$ has at most one element, hence $f(y)$ is determined by its H_l-coset and H_j-coset.

Case 1: Now if $f(x) = 1$, then $f(y)$ is in the same H_l-coset and H_j-coset as 1. Thus $f(y) = 1$ for any $y \notin H_1$. Now we can choose x from H_2 and do the same for any $y \in H_1$, which implies that f is the constant 1 map.

Case 2: If $f(x) = x$, then $f(y)$ is in the same H_l-coset and H_j-coset as y. Thus $f(y) = y$ for any $y \notin H_1$. Again we can choose x from H_2 and do the same for any $y \in H_1$, hence f is the identity. □

Definition 3. *We call G a partitioned group if there exist subgroups $H_1, H_2, \ldots, H_k$ of G such that $G \backslash \{1\}$ is the disjoint union of the sets $H_1 \backslash \{1\}, H_2 \backslash \{1\}, \ldots, H_k \backslash \{1\}$.*

Corollary 18. *Assume that G is a partitioned group and f is a compatible function on G such that $f(1) = 1$. Then f is either the constant 1 map or $f|_{G \backslash \{1\}}$ is a permutation of the set $G \setminus \{1\}$ with all cycles of equal size.*

Proof: Assume that f is not the constant 1 map. Denote by f^k the function $f \circ f \circ \ldots \circ f$ with k copies of f in the composition. Note that as f is compatible,

f^k is also compatible for any $k \in \mathbb{N}$. The function f maps $G \setminus \{1\}$ to $G \setminus \{1\}$, thus f^k also maps $G \setminus \{1\}$ to $G \setminus \{1\}$. Take an arbitrary element $g \neq 1$ in G and consider the series $g = f^0(g)$, $f^1(g)$, $f^2(g)$ ... Take the minimal l such that $f^l(g) = f^k(g)$ for some $k > l$. Then $f^l(f^{k-l}(g)) = f^l(g)$, hence f^l has a fixpoint. This means that f^l is the identity, so f is bijective. The size of a cycle of f is the divisor of l. Neither of these numbers can be less then l, because otherwise there would be a $d < l$ such that $f^d(g) = g$. Thus all the cycles of f has size l. $\square$

This lemma says that if a group G is partitioned with some subgroups then it is 1-affine complete if and only if for any f compatible function on G such that $f(1) = 1$ there exists an element $x \neq 1$ satisfying $f(x) = 1$ or $f(x) = x$. It is well known that Frobenius groups, p-groups with exponent p for some prime p and certain nonabelian simple groups such as $PSL_2(q)$ and Suzuki groups are partitioned groups. Now we investigate some special Frobenius groups.

Lemma 19. *Let G be a non-abelian group of size pq for distinct primes p, q. Then $R(G)$ is 1-affine complete.*

Proof: We will use $+$ to denote the group operation on G. Let f be compatible with $f(0) = 0$. Let $G = \langle a \rangle \rtimes \langle b \rangle$ with $pa = qb = 1$. Then there is an $r \pmod p$ such that for any $x \in \langle a \rangle$ we have $-b + x + b = rx$. The order of r is $q \pmod p$. For any $x \in \langle a \rangle$ the equation $-kb + x + kb = r^k x$ holds. As G is a partitioned group it is enough to prove that there exists a $h \neq 1$ such that $f(h) = h$. By Corollary 18 we have that f is a bijection. Every element of the semidirect product can be written uniquely in the form $u + v$ such that $u \in \mathbb{Z}_p$ and $v \in \mathbb{Z}_q$. We know that $f(u+v) = f(u) + f(v)$ for such u and v. The function f restricted to $\langle b \rangle$ is also a permutation fixing 1. Thus there is a bijection $\alpha : \{1, 2, ..., q-1\} \to \{1, 2, ..., q-1\}$ such that for any $i \in \{1, 2, ..., q-1\}$ we have $f(ib) = \alpha(i)b$. Let $x \in \langle a \rangle$ and let $k \in \{1, \ldots, q-1\}$. Then $f(x+kb) = f(x) + f(kb)$ by Lemma 6. On the other hand, $f(x + kb) \in \langle x + kb \rangle$. Thus for some $n \in \mathbb{N}$ we have that

$$n(x + kb) = f(x) + \alpha(k)b \tag{8}$$

After expanding the expression on the left hand side we get

$$x + (kb + x - kb) + \ldots + (k(n-1)b + x - k(n-1)b) + knb = f(x) + \alpha(k)b \tag{9}$$

Every element has a unique form $u + v$ such that $u \in \mathbb{Z}_p$ and $v \in \mathbb{Z}_q$, hence

$$f(x) = x + (kb + x - kb) + \ldots + (k(n-1)b + x - k(n-1)b) \quad \text{and} \tag{10}$$

$$\alpha(k)b = knb \tag{11}$$

According to 11 we have $kn \equiv \alpha(k) \pmod{q}$. Substituting this to 10 yields

$$f(x) = x + (kb + x - kb) + \ldots + (k(n-1)b + x - k(n-1)b) =$$

$$= x + r^{-k}x + \ldots + r^{-k(n-1)}x = \frac{r^{-kn} - 1}{r^{-k} - 1}x = \frac{r^{-\alpha(k)} - 1}{r^{-k} - 1}x$$

Hence $f(x) = \frac{r^{-\alpha(k)}-1}{r^{-k}-1}x$. This holds for any $1 \leq k \leq q-1$ but the left hand side of the equation does not depend on k. Thus there is an element $t \pmod{p}$ such that for any $x \in \langle a \rangle$ we have $f(x) = tx$ and for any $k \in \{1, 2, ..., q-1\}$ we have $\frac{r^{-\alpha(k)}-1}{r^{-k}-1} \equiv t \pmod{p}$, that is

$$r^{-\alpha(k)} - 1 \equiv t(r^{-k} - 1) \pmod{p} \tag{12}$$

Note that α is a permutation of $\{1, 2, ..., q-1\}$, thus adding these equations for $1 \leq k \leq q-1$ we get

$$\sum_{k=1}^{q-1}(r^{-\alpha(k)} - 1) \equiv t\sum_{k=1}^{q-1}(r^{-k} - 1) \pmod{p}, \text{ hence} \tag{13}$$

$$(t-1)\sum_{k=1}^{q-1}(r^{-k} - 1) \equiv 0 \pmod{p} \tag{14}$$

As $\sum_{k=0}^{q-1} r^{-k} \equiv 0 \pmod{p}$, it is easy to calculate that $\sum_{k=1}^{q-1}(r^{-k} - 1) \equiv (-q) \pmod{p}$. Here q is not divisible by p. Thus $p|(t-1)$ and consequently $t \equiv 1 \pmod{p}$. This implies that for any $x \in \langle a \rangle$ we have $f(x) = tx = x$, thus f has a fixpoint. $\square$

The importance of these lemmata will be clear from the following theorem.

Theorem 20. *Let G be a Frobenius group. Then $R(G)$ is 1-affine complete.*

Proof: Let $a \in N$ and $b \in H$. Assume that $a, b \neq 1$ and $a^p = b^q = 1$ for distinct primes p and q. The equation $f(ab) = f(a)f(b)$ holds again, thus $f(ab) = a^l f(b) = f(b)(a^l)^{f(b)}$ with some $1 \leq l \leq p-1$. As $ab = ba^b$ we also know that $ab\langle a^b \rangle = b\langle a^b \rangle$. Thus $f(ab)\langle a^b \rangle = f(b)\langle a^b \rangle$. Hence $f(ab) = f(b)(a^b)^k = f(b)(a^k)^b$ for some $1 \leq k \leq p-1$. We got that $f(b)(a^l)^{f(b)} = f(b)(a^k)^b$ and consequently $(a^l)^{f(b)} = (a^k)^b$. Thus $(a^l)^{f(b)b^{-1}} = a^k$. Here $a^l \neq 1$, otherwise $f(a) = 1$ and $f(G) = 1$. If $f(b) = b$ then we found a fixpoint and we are ready. If $f(b) \neq b$ then $f(b)b^{-1} \neq 1$ and then the conjugation with the element $f(b)b^{-1} \in \langle b \rangle$ permutes the elements of $\langle a \rangle$. Thus $\langle a \rangle \rtimes \langle b \rangle = F$ is a non-abelian subgroup of size pq. According to Lemma 19 we have that $f|_F = \mathrm{id}_F$. Hence the elements of F are fixpoints of the function f and $f(G) = id_G$. $\square$

According to Theorem 20 $R(A_4)$ is 1-affine complete.

Definition 4. *A permutation group G acting on the finite set Ω is called a Zassenhaus group if G is the transitive extension of a Frobenius group, that is G is transitive, and the stabilizer of an element $\alpha \in \Omega$ is a Frobenius group acting on $\Omega \setminus \{\alpha\}$.*

The following corollary is a natural consequence of Theorem 20, as there are many Frobenius subgroups in a Zassenhaus group.

Corollary 21. *Let G be a Zassenhaus group. Then $R(G)$ is 1-affine complete.*

Proof: Let f be a compatible function on G such that $f(1) = 1$. Consider G as a permutation group on the set Ω. This is a Zassenhaus group, hence for any $\alpha \in \Omega$ the stabilizer G_α is a Frobenius group. Two such stabilizers intersect nontrivially and generate G. These stabilizers are 1-affine complete. According to Theorem 13 G is 1-affine complete. □

Theorem 22. *Let $p > 2$ be a prime and G be a nonabelian group with exponent p. Then $R(G)$ is 1-affine complete.*

Proof: Note that G is a p-group that is equally partitioned with its subgroups. Thus we only have to check whether the conditions of Lemma 17 hold. Let $1 < Z_1(G) < Z_2(G) < \ldots < Z_n(G) = G$ be the upper central series of the group G. As G is nonabelian, we have $n \geq 2$. $Z_2(G)$ is a p-group with nilpotence class 2. In such groups the following equation holds:

$$(xy)^k = x^k y^k [x, y]^{\frac{k(k-1)}{2}} \quad \text{for any} \quad k \in \mathbb{N} \tag{15}$$

Let f be a compatible function on G such that $f(1) = 1$. Then f preserves $Z_2(G)$. We will show that $f|_{Z_2(G)}$ is either constant 1 or the identity. The centre of this group is nontrivial, hence it contains a subgroup $\langle g \rangle$ that is isomorphic to $\mathbb{Z}_p$. Let $k \in \mathbb{N}$ be such that $f(g) = g^k$. Then given an element $h \notin \langle g \rangle$ in $Z_2(G)$ we have $\langle g, h \rangle$ is isomorphic to $(\mathbb{Z}_p)^2$ on which every compatible function is of the form $x \mapsto x^l$ with some $l \in \mathbb{N}$. Thus every element of $\langle g \rangle$ is mapped to its k^{th} power just as the elements of $G \setminus \langle g \rangle$. So we have $f(x) = x^k$ for any $x \in Z_2(G)$. Let x and y be in $Z_2(G)$ such that $[x, y] \neq 1$. As $xy\langle y \rangle = x\langle y \rangle$, we have that $f(xy) = f(x)y^l = x^k y^l$ with some $l \in \mathbb{N}$. Also $f(xy) = (xy)^k = x^k y^k [x, y]^{\frac{k(k-1)}{2}}$, where $[x, y]^{\frac{k(k-1)}{2}}$ is in the centre of $Z_2(G)$. Thus $x^k y^l = x^k y^k [x, y]^{\frac{k(k-1)}{2}}$, hence $y^{l-k} = [x, y]^{\frac{k(k-1)}{2}}$. Thus this element is in the intersection of $\langle y \rangle$ and the centre, which is the trivial subgroup $\{1\}$. Thus $[x, y]^{\frac{k(k-1)}{2}} = 1$ such that $[x, y] \neq 1$. This implies that p divides $\frac{k(k-1)}{2}$, thus either k or $k - 1$ is divisible by p. Hence f is either the constant 1 map or the identity on $Z_2(G)$. Now the conditions of Lemma 17 hold, hence f is the constant 1 map or the identity on G. Thus G is 1-affine complete. □

Proposition 23. *Assume that A and B are groups such that $R(A)$ is 1-affine complete and $exp(B)|exp(A)$. Then $R(A \times B)$ is 1-affine complete.*

Proof: Given a compatible function f on G such that $f(1) = 1$. As $R(A)$ is 1-affine complete, we have that $f|_A$ is 1_A or id_A. In other words $f|_A$ is the function $p_k|_A$ with $k = 0$ or $k = 1$, where

$$p_k : A \times B \to A \times B \tag{16}$$

$$x \to x^k \tag{17}$$

We will prove that if the compatible function f fixes 1 and $f|_A = p_k|_A$ for some $k \in \mathbb{N}$, then $f = p_k$.

First we will show that $f|_B = p_k|_B$. Let $b \in B$ an arbitrary element. Here $\langle b \rangle$ is cyclic, hence it is the direct product of cyclic groups of prime power order. Thus according to Lemma 6 it is enough to prove that $f|_H = p_k|_H$ for any $H \leq B$ cyclic with $|H|$ prime power. For this let H be such a subgroup and $c \neq 1$ an arbitrary element of H of order p^α with p prime. As $exp(B)|exp(A)$, there is an $a \in A$ with the same order as c. It is clear that $\langle a, c \rangle$ is isomorphic to $(\mathbb{Z}_{p^\alpha})^2$. According to Corollary 7 we have that any compatible function on $\langle a, c \rangle$ is of the form p_l for some $l \in \mathbb{N}$. As $f(a) = a^k$ we have that $f(c) = c^k$. Thus $f|B$ is indeed $p_k|_B$.

According to Lemma 6 it is clear that $f = p_k$, now. As $k = 0$ or $k = 1$, we have that f is either the constant 1 map or the identity. □

According to Proposition 23 for any group G we have that $R(D_{2|G|} \times G)$ is 1-affine complete. This is another construction for a group of which G is a subgroup (moreover a normal subgroup) and the G-set corresponding to it is 1-affine complete.

As we proved earlier, abelian groups are not 1-affine complete (except the elementary abelian 2-groups), because the map $x \mapsto x^2$ is a compatible function on them. The next step should be to observe when a map of the form $x \mapsto x^k$ is compatible. The following theorem gives a condition which implies that there exists such a compatible function on the group.

Proposition 24. *Assume that G is a nonabelian group such that* $\exp(G)$ *does not divide $|G : Z(G)|$. Then $R(G)$ is not 1-affine complete.*

Proof: Let $|G : Z(G)| = k$. According to the theory of transfers it is well-known that the map $f(x) = x^k$ is a homomorphism. Given a subgroup $U \leq G$ and $x, y \in G$ such that $xU = yU$ we have $y^{-1}x \in U$. Now f clearly preserves subgroups, thus $(y^{-1}x)^k \in U$. As f is a homomorphism it also holds that $(y^{-1}x)^k = y^{-k}x^k$, hence $x^kU = y^kU$. Thus f is compatible, and since $\exp(G)$ does not divide $|G : Z(G)|$, f is not the constant 1 map. G is not abelian, hence f is not the identity. Thus f is a nontrivial compatible function on G and G is not 1-affine complete. □

4 The characteristic subgroup $C(G)$

In this section we present the characteristic subgroup $C(G)$. A more detailed discussion on the topic can be found in [6].

Definition 5. *Denote by $C(G)$ the subgroup of G that is generated by all the 1-affine complete subgroups of G.*

Theorem 13 shows that $C(G)$ is the direct product of subgroups of pairwise coprime order, namely the maximal 1-affine complete subgroups of G. It also shows that the maximal 1-affine complete subgroups of G are characteristic subgroups. Indeed, for an α automorphism of G and $A \leq G$ maximal 1-affine complete subgroup, $R(\langle A, \alpha(A)\rangle)$ is 1-affine complete, since for a nontrivial A the condition $gcd(|A|, |\alpha(A)|) = 1$ never holds. According to this remark (and also by definition) it is clear that $C(G)$ is a characteristic subgroup of G. Not considering those groups which are the direct product of nontrivial groups of coprime order (we know these are not 1-affine complete), it is clear that a group G is 1-affine complete if and only if $G = C(G)$.

Corollary 25. *Let G be a group such that G is not the direct product of two nontrivial groups of coprime order. Then $R(G)$ is 1-affine complete if and only if every Sylow subgroup of G is contained in a 1-affine complete subgroup of G.*

Proof: If $R(G)$ is 1-affine complete then it is clear that every Sylow subgroup of G is contained in a 1-affine complete subgroup of G. For the other direction let $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$. Then for any $1 \leq i \leq r$ we have that $|C(G)|$ is divisible by $p_i^{\alpha_i}$, since there is a p_i-Sylow contained in $C(G)$. Thus $|C(G)|$ is divisible by $|G|$, hence $G = C(G)$ and $R(G)$ is 1-affine complete. □

It is a natural idea to look for groups with the property that $C(G) = 1$. According to our results about groups producing 1-affine complete G-sets, we know the following.

Corollary 26. *Assume that $C(G) = 1$. Then*

- *$|G|$ is odd*
- *every subgroup of G of order pq with different primes p and q is cyclic.*
- *every subgroup $H \leq G$ with exponent p for a prime p H is abelian.*

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