

# Cobordism and elimination of singularities

Master's Thesis  
Revised version

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# Preface

This work is mainly about the cobordism theory of smooth manifolds, an element of differential topology. We give a new proof for a theorem by J. F. Hughes [Hu], demonstrating the use of singularity theory in cobordism theory along the way.

Our main result will be that in the cobordism group of 4-dimensional oriented manifolds,  $\Omega_4 \cong \mathbb{Z}$ , exactly the even elements have representatives that can be immersed into  $\mathbb{R}^6$ . Using Herbert's multiple point formula, this gives Hughes's result as a corollary: an immersion  $M_{or}^4 \looparrowright \mathbb{R}^6$ , if it is in a generic position (i. e. self-transverse), must have an even number of triple points. This is in parallel with Banchoff's classical theorem for immersions from 2-surfaces to  $\mathbb{R}^3$ .

We review most of the results in differential topology that we use: in Chapter 1, one can find an overview of basic facts about "generally positioned" (generic) smooth maps, while in Chapter 2, we recite two multiple point formulas – that is, partial characterizations of the multiple point sets of certain maps between manifolds – and prove one of them. (The algebraic topology we will need is standard, so we will use it without reference.) Chapter 3 is more detailed, and is devoted to the proof of the main result.

I wish to express my immense gratitude to Professor András Szűcs for the interesting pieces of mathematics that he called my attention to and his tireless and patient advising.

# Contents

<b>1</b>	<b>Generic differentiable maps between manifolds</b>	<b>1</b>
1.1	Genericity . . . . .	1
1.2	Global behavior: transversality . . . . .	2
1.3	Local behavior: singularities . . . . .	3
<b>2</b>	<b>Multiple point formulas</b>	<b>8</b>
2.1	Multiple point manifolds . . . . .	8
2.2	Herbert's and Ronga's formulas . . . . .	9
<b>3</b>	<b>The image of <math>Imm^{SO}(4, 2) \rightarrow \Omega_4</math></b>	<b>14</b>
3.1	The obstruction $[\Sigma(f)]$ . . . . .	14
3.2	Elimination of double points in $Imm^{SO}(4, 3)$ . . . . .	27
3.3	Elimination of singularities using classifying spaces . . . . .	31
3.4	Elimination of singularities by a geometric constuction . . . . .	41
3.5	Hughes's theorem . . . . .	45
	<b>References</b>	<b>47</b>

# 1 Generic differentiable maps between manifolds

## 1.1 Genericity

In this chapter, we shall review a few classical results about differentiable maps between manifolds. We shall only deal with manifolds and mappings smooth enough. In this spirit, from now on, the word "smooth" always stands for  $C^\infty$  class (infinitely differentiable)<sup>1</sup>. All manifolds and maps are assumed to be differentiable and all manifolds are without boundary in this chapter, unless otherwise noted.

We know from real analysis that continuous maps can exhibit very bad behavior. Even differentiable maps can have extremely complicated local and global structures. A powerful idea to make considerations simpler is to restrict our attention to a *generic* class of maps: informally speaking, we disregard any phenomenon that can be eliminated by a "small perturbation" of the map.

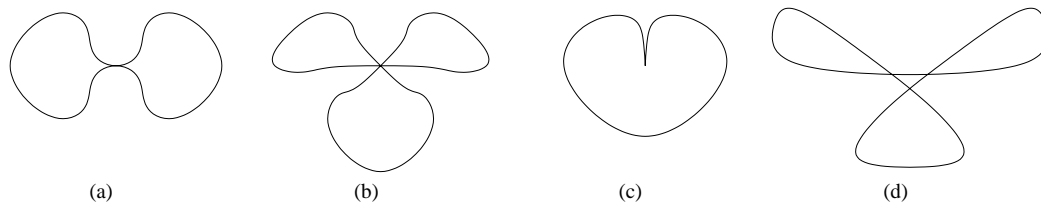


Figure 1: *Examples of smooth maps  $S^1 \rightarrow \mathbb{R}^2$ . (a), (b) and (c) are non-generic: (a) is not self-transverse, (b) has a triple point, (c) has a singularity of non-generic type; (d) is generic by all aspects that we will consider.*

This idea may have come from physics: if a function is ultimately computed from the readings of a measuring instrument, there is no point in dealing with "unstable" behavior of the function; since the readings already involve some error, we should usually feel comfortable to alter the function a little bit further.

To give a mathematically precise sense to the notion of the "small change" of a map, we first have to define a suitable topology on the set of differentiable maps. For the sake of simplicity, we only give a formal definition for the space of  $C^r$  class maps. The idea is that for two functions to qualify as "close" to each other, not only their values should be close, but also the partial derivatives of order at most  $r$ :

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<sup>1</sup>There are general results (see [W]) which assert that from a complete  $C^1$  atlas on a manifold, one can uniquely choose a  $C^\infty$  atlas up to  $C^\infty$  diffeomorphism; furthermore, in the space of  $C^1$ -differentiable maps, the  $C^\infty$ -smooth mappings form a dense subset. This means that we do not really lose generality by only dealing with the smooth case.

**Definition 1.1.** Let  $M$  and  $N$  be differentiable manifolds, and let  $M$  be compact. Then let us denote the set of  $r$  times differentiable mappings from  $M$  to  $N$  by  $C^r(M, N)$ . Furthermore, let us fix a standard topology on this set: the subbase shall be formed by all sets of the form

$$N(f, (\varphi, U), (\psi, V), K, \varepsilon) = \{g \in C^r(M, N) : g(K) \subset V, |\partial^\alpha(\psi \circ f \circ \varphi^{-1})(\varphi(x)) - \partial^\alpha(\psi \circ g \circ \varphi^{-1})(\varphi(x))| < \varepsilon, \forall x \in K\},$$

where  $(\varphi, U)$  is a map in the atlas of  $M$ ,  $(\psi, V)$  is a map in the atlas of  $N$ ,  $K \subset U$  is a compact set such that  $f(K) \subset V$ , and  $\alpha = (\alpha_1, \dots, \alpha_n)$  is a multiindex,  $|\alpha| \leq r$ .

We will not deal with the case when  $M$  is not compact (in such a case, one has to consider the *strong* topology on  $C^r(M, N)$ , where the subbase neighborhoods are defined using a countable family of locally finite coordinate neighborhoods).

This space  $C^r(M, N)$  can easily be given a metric with which it is complete, therefore  $C^r(M, N)$  is a Baire space. As a consequence, a countable intersection of open and dense subsets is still dense. Of course, a finite intersection of open and dense subsets continues to be open and dense. This fits in well with our intuitive idea of "genericity": if the generic maps form an open and dense subset in  $C^r(M, N)$ , then there will be a generic map as near as we want to any  $C^r$  map, but a generic map will stay generic under small modifications. This justifies the following

**Definition 1.2.** In this paper, we will call a property  $G$  of  $C^r$  functions (that is, a subset  $G \subset C^r(M, N)$ ) generic if it is open and dense with respect to the  $C^r$  topology.

We will use this concept much like one uses "almost every" in measure theory or "a typical continuous function" in real analysis; for example, "a typical generic function  $f : M \rightarrow N$  satisfies condition  $G$ " means  $G$  is a generic property.

It is important to note that genericity is often understood in a more lenient way (for example, allowing dense  $G_\delta$  sets). Also note that there is a natural, continuous embedding  $C^s(M, N) \hookrightarrow C^r(M, N)$  whenever  $s > r$ .

## 1.2 Global behavior: transversality

Let us recall the concept of *transversality* and state R. Thom's transversality lemma. Transversality is the differentiable equivalent of the elementary geometric concept of "general positioning"; intuitively speaking, it asserts that two objects in a figure "touch as little as possible". One feels that this can be achieved if the map is generic, and Thom's lemmas show that this is indeed the case.

**Definition 1.3.** Two submanifolds  $M_1, M_2 \subset N$  are said to be transverse if for each  $p$  in  $M_1 \cap M_2$ , the tangent space  $T_p N$  is generated by the tangent spaces of the submanifolds:  $T_p N = T_p M_1 + T_p M_2$ .

A smooth map  $f : V \rightarrow N$  is called *transverse to the submanifold*  $M \subset N$  if for each point  $p = f(q) \in M \cap f(V)$ ,  $f_*(T_qV) + T_pM = T_pN$ .

It is easy to see that if  $M_1$  and  $M_2$  are transverse, then  $M_1 \cap M_2$  is a submanifold of  $N$  (furthermore,  $\text{codim } M_1 \cap M_2 = \text{codim } M_1 + \text{codim } M_2$  – if this is larger than  $\dim N$ , no intersection is allowed at all). Similarly, if  $f$  is transverse to  $M$ , then  $f^{-1}(M)$  is a submanifold of  $V$  of codimension  $\text{codim } M$ .

We define the transversality of more than two submanifolds so that each is transverse to the intersection of any number of others. Transversality to a manifold with boundary  $W$  means transversality to both  $W$  and  $\partial W$ .

The basic version of Thom's lemma says:

**Theorem 1.4** (Thom). *For a fixed submanifold  $M \subset N$  and a manifold  $V$  ( $\dim V < \dim N$ ), those maps  $f : V \rightarrow N$  that are transverse to  $M$  form a generic subset of  $C^r(V, N)$  ( $r \geq 1$ ).*

We will also need a sharper result that is only true for immersions:

**Definition 1.5.** *An immersion  $f : M^m \looparrowright N^n$  ( $m < n$ ) is called *self-transverse* if its "leaves" intersect transversally. That is, whenever  $q = f(p_1) = \dots = f(p_k)$  for distinct points  $p_1, \dots, p_k \in M$ ,  $f_*(T_{p_i}M) + \bigcap_{j \neq i} f_*(T_{p_j}M) = T_qM$ .*

**Theorem 1.6** (Thom). *Given manifolds  $M^m$  and  $N^n$  ( $m < n$ ), self-transverse immersions form a generic subset of all immersions  $M \looparrowright N$  (which themselves form an open subset in all smooth maps  $M \rightarrow N$ ).*

### 1.3 Local behavior: singularities

Our goal here is to classify the local behavior of generic differentiable functions. We assume  $m < n$  for simplicity. It should be noted that this is still not well-understood for certain pairs of dimensions  $(m, k)$ , where  $k = n - m$  is the codimension. However, we shall present a partial characterization due to *J. Boardman* in all cases, as well as a complete description in the case where the codimension  $k$  is large, that is,  $2n > 3m - 2$ . (Another important and simple special case is  $n = 1$ , when we are looking at Morse functions.) Notations and statements that follow are mostly taken from [AGV].

We first need to define what we wish to classify:

**Definition 1.7.** *A map germ  $M \rightarrow N$  at a point  $p \in M$  is an equivalence class of maps  $\varphi : U_p \rightarrow N$ , where  $p \in U_p \subset M$  is an open neighborhood, and – as usual with germs – two maps are equivalent if they agree on a (possibly smaller) neighborhood of  $p$ .*

**Definition 1.8.** *Two map germs are called differentially left-right equivalent or  $\mathcal{A}$ -equivalent if there is a diffeomorphism between their source and target manifolds that conjugate one germ to the other. More formally, let  $f_i : U_i \rightarrow V_i$  ( $i = 1, 2$ ) be germs, where  $p_i \in U_i \subset M_i$ ,  $f_i(p) \in V_i \subset N_i$  are open neighborhoods. If there are diffeomorphisms  $h : U_1 \rightarrow U_2$  and  $k : V_1 \rightarrow V_2$  such that  $k \circ f_1 \circ h^{-1} = f_2$ , then the germs represented by  $f_1$  and  $f_2$  are  $\mathcal{A}$ -equivalent.*

A set of map germ classes (by  $\mathcal{A}$ -equivalence) is called generic for a pair of dimensions  $(m, n)$  if a generic map in  $C^\infty(M^m, N^n)$  gives a germ from the named set of classes in each point  $p \in M$ . In general, there may not be a unique smallest generic set of germ classes. However, certain classes can not be left out:

**Definition 1.9.** *A map germ  $f : M \rightarrow N$  at a point  $p$  is said to be stable if there is a neighborhood  $p \in U$  and a neighborhood  $f \in E \subset C^\infty(M, N)$  such that  $\forall f' \in E \exists p' \in U$  for which the germ of  $f'$  in  $p'$  is  $\mathcal{A}$ -equivalent to that of  $f$  in  $p$ . (This is obviously a property of the  $\mathcal{A}$ -class of  $f$ .)*

In other words, stable germ classes are those that cannot be essentially altered by arbitrarily small perturbations. In small pairs of dimensions  $m$  and  $n$ , the set of stable germs are generic; unfortunately, for some larger dimensions, this ceases to be true.

Let us fix the compact differentiable manifolds  $M^m$  and  $N^n$ , where  $\dim M = m$  and  $\dim N = n$ . Let  $TM$  and  $TN$  be the respective tangent bundles. If  $f \in C^1(M, N)$ , then the differential  $f_* : T_p M \rightarrow T_{f(p)} N$  is a linear map in each point  $p \in M$ . Naturally, the rank of this map is at most  $\min(m, n)$ . If  $\text{rk}(T_p f) = \min(m, n)$ , then the germ of  $f$  around  $p$  is uniquely determined up to  $\mathcal{A}$ -equivalence; depending on whether  $m \geq n$  or  $m \leq n$ :

If  $m \geq n$ , then  $f$  is a submersion near to  $p$ , that is, it is locally a projection of a fiber bundle to its base: by the implicit function theorem, in a small neighborhood of  $p$  and  $f(p)$ , there are coordinate functions  $x_1, \dots, x_m$  and  $y_1, \dots, y_n$  respectively such that  $y_i(f(q)) = x_i(q)$  ( $i = 1, 2, \dots, n$ ) if  $q$  is near to  $p$ .

If  $m \leq n$ , then  $f$  is an immersion near to  $p$ , which is to say it is locally a differentiable embedding: again with local coordinates  $x_i$  and  $y_j$ ,  $y_i(f(q)) = x_i(q)$  ( $i = 1, 2, \dots, m$ ) and  $y_j(f(q)) = 0$  ( $j = m + 1, \dots, n$ ).

These are the points  $p$  where  $f$  behaves in the simplest possible way; they are usually called the regular points of  $f$ . The map germ of  $f$  at a regular point is  $\mathcal{A}$ -equivalent to the standard projection or embedding, and it is easy to see that this map germ class is stable.

All the other points are called the *critical* or *singular points* of  $f$ . For a generic  $f$ , the critical points form a "thin" set, and we can partially classify them (more precisely, the map germ classes around them) further according to  $\mathcal{A}$ -equivalence. From the above discussion, one invariant property of singular germ classes is already apparent: the co-rank  $i$  of  $T_p f$ .

**Definition 1.10.** Let  $\Sigma^i(f) = \{p \in M : \text{rk}(T_p f) = \min(n, m) - i\}$ , that is, the set of points where the rank of  $T_p f$  drops by  $i$  compared to the maximum possible.

For example,  $\Sigma^0(f)$  is the set of regular points. The co-rank is clearly an invariant under  $\mathcal{A}$ -equivalence: the rank of the derivative is unchanged under left and right composition with linear isomorphisms (coming from the chain rule). By *Thom's Jet transversality theorem*, it can easily be seen that for generic maps  $f$ , the set  $\Sigma^i(f)$  is a (not necessarily compact) submanifold of  $M$ , with codimension  $\text{codim } \Sigma^i(f) = i(|m - n| + i)$ .

**Remark 1.11.** From this, we get that if  $n = 2m$ , a generic map  $f : M \rightarrow N$  has no singular points at all, thus it is an immersion. Also, such an  $f$  has a 0-dimensional double point manifold. If  $n \geq 2m + 1$ , a generic  $f$  will be an embedding.

The idea that leads to a finer classification of singular map germs is the following:

Let us restrict  $f$  to the singularity manifold  $\Sigma^i(f)$ , and examine the rank of its derivative. Let  $\Sigma^{i,j}(f) = \Sigma^j(f|_{\Sigma^i(f)})$ . This new set may no longer be a submanifold; however, if it is, we can proceed and inductively define  $\Sigma^{i_1, \dots, i_l}(f) = \Sigma^{i_l}(f|_{\Sigma^{i_1, \dots, i_{l-1}}(f)})$ , where  $I = (i_1, \dots, i_l)$  is a non-increasing sequence of nonnegative integers. The endpoints of this classification are those sets where  $i_l = 0$ ; in this case,  $f$  is a maximal rank map when restricted to  $\Sigma^I(f) = \Sigma^{i_1, \dots, i_l}(f)$ .

J. M. Boardman proved that for a generic  $f$ , these sets are indeed submanifolds. (He defined the submanifolds  $\Sigma^I(M, N)$  in the infinite dimensional jet space over  $(M, N)$ , and used a jet transversality in the proof.) Boardman also gave the codimension of each stratum.

**Theorem 1.12** (Boardman). For a generic smooth map  $f : M^m \rightarrow N^n$  ( $m < n$ ) and an index sequence  $I = (i_1, \dots, i_l)$ , the set  $\Sigma^I(f)$  is a submanifold of  $M$ , and its codimension is  $\text{codim } \Sigma^I = \nu_I(m, n)$ ;

$$\nu_I(m, n) = (n - m + i_1)\mu(i_1, \dots, i_1) - (i_1 - i_2)\mu(i_2, \dots, i_2) - \dots - (i_{l-1} - i_l)\mu(i_l)$$

and  $\mu(i_1, \dots, i_l)$  is the number of non-increasing sequences  $j_1 \geq \dots \geq j_l$  of integers such that  $i_r \geq j_r \geq 0 \forall r$ ,  $j_1 > 0$ .



Of particular importance for us is the case  $I = (1, 0)$ . We get that  $\text{codim } \Sigma^{1,0} = k+1$ , where  $k = n-m$  is the codimension of the map. Any higher class  $I = (i_1, \dots, i_l)$  has either  $i_1 > 1$  or  $i_1 = i_2 = 1$ . If  $i_1 > 1$ , then  $\text{codim } \Sigma^I \geq \text{codim } \Sigma^{i_1} \geq 2(k+2)$ . If  $i_1 = i_2 = 1$ , then  $\text{codim } \Sigma^I \geq \text{codim } \Sigma^{1,1} = 2(k+1)$ .

**Corollary 1.13.** *If  $2(k+1) > m$ , that is,  $2n > 3m - 2$ , then a generic map  $f$  has only  $\Sigma^{1,0}$  singularities (besides regular points).  $\Sigma = \Sigma^{1,0}(f) \hookrightarrow M$  is a closed submanifold with  $\text{codim } \Sigma = k+1$  and  $f|_{\Sigma}$  is an embedding. Furthermore,  $f|_{M \setminus \Sigma}$ , the restriction of  $f$  to its regular points, can be assumed to be self-transverse. Finally, if  $2n > 3m - 1$  also holds, then  $f^{-1}(f(\Sigma)) = \Sigma$ , or in other words,  $p \in \Sigma$  and  $f(q) = f(p)$  implies  $q = p$ . (This is because  $f(\Sigma)$  has codimension  $2k+1$  in  $N$ , and  $2k+1 > m$ , so  $f(\Sigma)$  has no intersection with other points in  $f(M \setminus U(\Sigma))$ , where  $U(\Sigma)$  is any open neighborhood of  $\Sigma$ .)*

The following classical theorem by Whitney demonstrates that in the case of  $\Sigma^{1,0}$ , Boardman's classification is complete with respect to  $\mathcal{A}$ -equivalence (a relatively simple proof for this can be found in [Ha]).

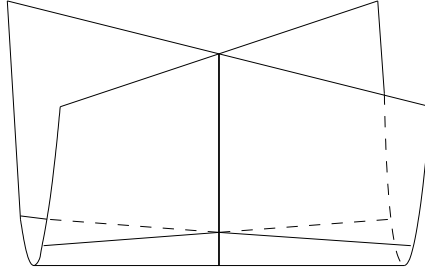


Figure 2: Image of the 2-dimensional Whitney umbrella  $wh_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ . To imagine the  $k+1$ -dimensional case, one should think of the "horizontal" lines as  $\mathbb{R}^k$ 's.

**Theorem 1.14** (Whitney normal form). *If  $p \in \Sigma^{1,0}(f)$  for a generic map  $f$ , then there are neighborhoods  $p \in U$ ,  $f(p) \in V$  and coordinates  $\varphi = (\varphi_1, \dots, \varphi_m) : U \rightarrow \mathbb{R}^m$ ,  $\psi = (\psi_1, \dots, \psi_n) : V \rightarrow \mathbb{R}^n$  such that*

$$\psi \circ f|_{U \circ \varphi^{-1}}(x, y_1, \dots, y_k, z_{k+1}, \dots, z_{m-1}) = (x^2, xy_1, \dots, xy_k, y_1, \dots, y_k, z_{k+1}, \dots, z_{m-1}),$$

or in other words,

$$\psi \circ f|_{U \circ \varphi^{-1}} = wh_k \times \text{id}_{\mathbb{R}^{m-k-1}},$$

where  $wh_k : \mathbb{R}^{k+1} \rightarrow \mathbb{R}^{2k+1}$ :

$$wh_k(x, y_1, \dots, y_k) = (x^2, xy_1, \dots, xy_k, y_1, \dots, y_k).$$

(Such a map is called a Whitney umbrella.)

**Remark 1.15.** B. Morin proved that  $\Sigma^{1,1,\dots,1,0}$  singularities have normal forms as well, so these Boardman classes contain a unique map germ class each with respect to  $\mathcal{A}$ -equivalence. For  $\Sigma^{2,0}$ , this is already false:  $\Sigma^{2,0}$  contains many  $\mathcal{A}$ -different map germs. For some higher multiindices  $I$ , a number of continuous parameters are needed to describe the  $\mathcal{A}$ -equivalence class of a map germ.

## 2 Multiple point formulas

Given a smooth map  $f : M^m \rightarrow N^n$  between closed manifolds ( $m = n - k$ ), a natural question to ask is to describe the multiple points of  $f$ . One can immediately define the  $r$ -fold intersection sets

$$N_r = \{p \in N : |f^{-1}(p)| \geq r\}$$

$$M_r = f^{-1}(N_r)$$

However, these sets will not usually be manifolds even for self-transverse immersions  $f$ , except for the simplest cases.

To remedy this situation in the case when  $f : M^m \looparrowright N^n$  is a self-transverse immersion, we will introduce closed manifolds  $\Delta_r$  and  $\tilde{\Delta}_r$  and immersions  $\psi_r : \Delta_r \looparrowright N$  and  $\mu_r : \tilde{\Delta}_r \looparrowright M$  such that  $\psi_r(\Delta_r) = N_r$  and  $\mu_r(\tilde{\Delta}_r) = M_r$ . We will even have a map  $\hat{f}_r : \tilde{\Delta}_r \rightarrow \Delta_r$  that lifts  $f$ , that is,  $\psi_r \circ \hat{f}_r = f \circ \mu_r$ . The spaces  $\Delta_r$  and  $\tilde{\Delta}_r$  (together with the maps  $\mu_r$  and  $\psi_r$ ) will be called the  *$r$ -tuple point manifolds* of  $f$ .

A similar construction can be given for the double points in the case where  $2n > 3m - 2$  and  $f$  is a generic singular map, as described in Corollary 1.13, to be detailed below.

In this chapter, we shall review two theorems about the  $Z_2$  homology class represented by multiple point manifolds: Herbert's classical result where  $f$  is a self-transverse immersion [He] and Ronga's double point formula [R]. The former is proved via a nice geometric argument in [EG]; we shall present a proof of the latter by the modification of this proof, suggested by Professor A. Szűcs. Most notations are taken from [EG], too.

### 2.1 Multiple point manifolds

First, assume that  $f : M \looparrowright N$  is a self-transverse immersion. Let

$$\hat{M}_r(f) = \{(p_1, \dots, p_r) \in M \times \dots \times M : p_i \neq p_j, f(p_i) = f(p_j)\}$$

be the set of formal  $r$ -tuple points in the source. By self-transversality,  $\hat{M}_r(f)$  is a compact manifold.  $\hat{M}_r(f)$  admits a free action of  $S_r$  by permuting the coordinates, and a sub-action of this of  $S_{r-1}$ , permuting only the last  $r - 1$  coordinates. If we factor out by these free actions, the factor spaces will also be closed manifolds.

**Definition 2.1.** *Let us define the multiple point manifolds and the relevant maps as:*

$$\Delta_r(f) = \hat{M}_r(f)/S_r,$$

$$\begin{aligned}
\psi_r(f)([p_1, \dots, p_r]) &= f(p_1), \\
\widetilde{\Delta}_r(f) &= \widehat{M}_r(f)/S_{r-1}, \\
\mu_r(f)(p_1, [p_2, \dots, p_r]) &= p_1, \\
\widehat{f}_r(p_1, [p_2, \dots, p_r]) &= [p_1, p_2, \dots, p_r].
\end{aligned}$$

From our definition of self-transversality (Definition 1.5), it follows easily that  $\mu_r$  and  $\psi_r$  are immersions. Also note that if a value  $q \in N$  is taken exactly  $r$  times, then it is covered once by  $\psi_r$ . The maps  $\psi_r$  and  $\mu_r$  only fail to be embeddings at values that are taken  $r + 1$  times or more by  $f$ .

Now let us define the double point manifolds for generic maps  $f : M^m \rightarrow N^n$ , where  $2n > 3m - 2$ . Such an  $f$  may well be singular, but by Corollary 1.13, all of its singularities are Whitney umbrellas: in a suitably chosen local coordinate system,  $f$  has the standard form  $f(x, \mathbf{y}, \mathbf{z}) = (x^2, x\mathbf{y}, \mathbf{y}, \mathbf{z})$  ( $\mathbf{y} \in \mathbb{R}^k$ ,  $\mathbf{z} \in \mathbb{R}^{m-k-1}$ ). We also know that  $\Sigma = \Sigma^{1,0}(f)$  is a submanifold of  $M$  and  $f|_\Sigma$  is an embedding.

The definition of  $\Delta_2$  and  $\widetilde{\Delta}_2$  are the same as above, the only difference being that they might not be closed manifolds. However, if we make

$$\Delta_2 \subset \overline{\Delta}_2 = \{\{p_1, p_2\} : p_i \in M, f(p_1) = f(p_2), p_1 \neq p_2 \text{ or } p_1 = p_2 \in \Sigma\}$$

$$\widetilde{\Delta}_2 \subset \widetilde{\overline{\Delta}}_2 = \{(p_1, p_2) : p_i \in M, f(p_1) = f(p_2), p_1 \neq p_2 \text{ or } p_1 = p_2 \in \Sigma\}$$

then  $\widetilde{\overline{\Delta}}_2$  is made a closed manifold and  $\overline{\Delta}_2$  a compact manifold with boundary. This can be checked by examining the standard form of  $f$  near a singular point  $p \in \Sigma$ : this shows  $\Sigma$  to be an  $m - k - 1$ -dimensional submanifold of  $M$  that is embedded in the closure of  $\mu_2(\widetilde{\Delta}_2)$ , locally an embedded  $m - k$ -manifold in  $M$ .

The extension of  $\mu_2$ ,  $\psi_2$  and  $\widehat{f}_2$  to  $\widetilde{\overline{\Delta}}_2$  and  $\overline{\Delta}_2$  is obvious;  $\mu_2$  and  $\psi_2$  continue to be immersions. We also introduce a smooth involution  $T : \widetilde{\overline{\Delta}}_2 \rightarrow \widetilde{\overline{\Delta}}_2$  for which  $\widehat{f}_2 \circ T = \widehat{f}_2$ : simply  $T(p_1, p_2) = (p_2, p_1)$ . The fixed points of  $T$  are exactly  $\mu_2^{-1}(\Sigma)$ .

## 2.2 Herbert's and Ronga's formulas

We state Herbert's formula without proof:

**Theorem 2.2** (Herbert). *Let  $f : M^m \looparrowright N^n$  be a self-transverse immersion. Denote*

$$m_r = D_M((\mu_r)_*[\widetilde{\Delta}_r]) \in H^{(r-1)k}(M; Z_2)$$

$$n_r = D_N((\psi_r)_*[\Delta_r]) \in H^{rk}(N; Z_2)$$

where  $[V^v] \in H^v(V, Z_2)$  is the mod 2 fundamental class of the closed manifold  $V$  and  $D_V$  is the mod 2 Poincaré dual. Then we have, for  $r \geq 1$ ,

$$f^*(n_r) = m_{r+1} + e \cup m_r \in H^{rk}(M, Z_2)$$

where  $e = e(\nu(f)) = w_k(\nu(f))$  is the mod 2 Euler class of the normal bundle  $\nu(f)$ .

**Remark 2.3.** If  $M$  and  $N$  are oriented and  $k$  is even then, Theorem 2.2 holds with  $Z$  coefficients as well.

And the singular version for double points:

**Theorem 2.4** (Ronga). Let  $f : M^m \rightarrow N^n$  be a generic smooth map,  $2n > 3m - 1$ . Denote

$$m_2 = D_M \left( (\mu_2)_* \left[ \widetilde{\Delta}_2 \right] \right) \in H^k(M; Z_2)$$

Then,

$$m_2 = w_k(f^*TN - TM) + f^*(f_!(1)) \in H^k(M, Z_2)$$

where  $f^*TN - TM$  is the virtual normal bundle (in the  $K$ -group of  $M$ ; or we can just say  $w(f^*TN - TM) := f^*w(TN) \cup \bar{w}(M)$ ), and  $f_! = D_N \circ f_* \circ D_M$  is the Gysin homomorphism.

Of course,  $f^*(f_!(1)) = f^*(n_1)$ , so this is a generalization of Theorem 2.2 for  $r = 1$ .

To prove Theorem 2.4, let us recall the concept of unoriented bordism and cobordism: a pair of extraordinary homology and cohomology theories that can be obtained from the spectrum  $\{(MO_k = T\gamma^k) : k \in \mathbb{N}\}$ , the Thom spaces of the universal (unoriented)  $k$ -bundles.

$$\mathfrak{N}^k(X, Y) = \lim_{q \rightarrow \infty} [S^{q-k}(X/Y), MO_q]$$

$$\mathfrak{N}_k(X, Y) = \lim_{q \rightarrow \infty} \pi_{k+q}((X/Y) \wedge MO_q)$$

Also, by the Thom-Pontryagin construction,  $\mathfrak{N}_k$  is identified with the geometrically defined bordism group:

$$\mathfrak{N}_k(X, Y) = \{f : (V^k, \partial V) \rightarrow (X, Y)\} / \text{bordism}$$

In this theory, the Poincaré duality for a closed manifold  $X = M^m$  can be readily seen: for  $k \geq 1$  and  $[\varphi] \in \mathfrak{N}^k(M) = \mathfrak{N}^k(M, \{*\}) = [S^{q-k}M, MO_q]$  ( $q \gg k$ ), where the representative  $\varphi$  is transverse to  $BO_q \hookrightarrow MO_q$ ,

$$D_M([\varphi]) = [\pi^{(q-k)} : (V^{n-k} = \varphi^{-1}(BO_q)) \rightarrow M]$$

where  $\pi : SN \setminus \{-1, 1\} \rightarrow N$  is a projection, the undoing of the suspension. Since  $BO(q) \hookrightarrow MO(q)$  is an embedding of codimension  $q$ , we can perturb  $\varphi$  so that it maps to the domain  $\mathbb{R}^{q-k} \times M \subset S^{q-k}M$  of  $\pi^{(q-k)}$ .

$BO(q) \hookrightarrow MO(q)$  is the universal embedding of codimension  $q$ , so all embeddings  $V^{n-k} \hookrightarrow \mathbb{R}^{q-n} \times M$  can be obtained this way. Moreover, all maps  $V^{n-k} \rightarrow M$  can be subjected to smooth approximation and then lifted out to  $\mathbb{R}^{q-n} \times M$  to get an embedding, for  $q$  large enough. Homotopy in  $[S^{q-k}M, MO_q]$  corresponds to bordism of the map. So in this case, Poincaré duality is simply a natural isomorphism  $D_M$ :

$$\mathfrak{N}^k(M) \xleftarrow{D_M} \mathfrak{N}_{m-k}(M).$$

We have a version for manifolds  $M$  with boundary, too, from essentially the same construction:

$$\mathfrak{N}^k(M) \xleftarrow{D_M} \mathfrak{N}_{m-k}(M, \partial M).$$

We can reduce the  $\mathfrak{N}_k$  theory to the usual  $Z_2$  coefficient homology: for

$$[f : (V, \partial V) \rightarrow (X, Y)] \in \mathfrak{N}_n(X, Y),$$

denote

$$h([f]) = f_*([V, \partial V]) \in H_n(X, Y; Z_2).$$

This reduction can be induced by the spectrum map

$$\{h_k : MO_k \rightarrow K(Z_2, k) : k \in \mathbb{N}\},$$

where  $(h_k)^*$  carries the fundamental class  $l_k \in H^k(K(Z_2, k); Z_2)$  to  $\Phi_{\gamma^k}(1_{BO_k})$ , and  $\Phi_{\gamma^k}$  denotes the Thom isomorphism for the universal bundle  $\gamma^k$  over  $BO_k$ . Being induced by a spectrum morphism, the reduction  $h$  extends naturally (functorially) to the respective cohomology theories.

The advantage of working with  $\mathfrak{N}^k(X, Y)$  is that it is contravariant in  $(X, Y)$ . If  $g : (M, \partial M) \rightarrow (N, \partial N)$  is a smooth map between manifolds and  $f : V^{n-k} \rightarrow N^n$  is transverse to  $g$ , then it is easy to check that if " $W = f^{-1}(g(V))$ ", more precisely,

$$W^{m-k} = \{(x, y) : x \in M, y \in V, f(x) = g(y)\} \subset M \times V$$

(a manifold, by transversality), then the pullback of  $[f] \in \mathfrak{N}^k(N, \partial N)$  is

$$g^*[f] = [\pi_1 : W \rightarrow M] \in \mathfrak{N}^k(M, \partial M),$$

where  $\pi_1 : M \times V \rightarrow M$  is the projection to the first coordinate.

Now let us prove Ronga's result for the case described. We resume the use of all notations in the statement (Theorem 2.4). Let us allow the slightly ambiguous notation that for  $f : (V^{w-l}, \partial V) \rightarrow (W^w, \partial W)$ ,  $[f]$  denotes both  $[f] \in \mathfrak{N}_{w-l}(W, \partial W)$  and its Poincaré dual  $[f] = D_W[f] \in \mathfrak{N}^l(W)$ . Denote  $\Sigma = \Sigma^{1,0}(f) \subset M$ .

Let  $U(f(\Sigma))$  be an open tubular neighborhood of  $f(\Sigma)$  such that  $f$  and  $\psi_2$  are transverse to  $\partial U(f(\Sigma))$ . Let  $N_0 = N \setminus U(f(\Sigma))$ . Since  $f^{-1}(f(\Sigma)) = \Sigma$  (there are no "far-away" points in  $M$  that  $f$  takes close to  $\Sigma$ ), the preimage  $U\Sigma := f^{-1}(U(f(\Sigma)))$  is a tubular neighborhood of  $\Sigma$ . Let  $M_0 = M \setminus U\Sigma = f^{-1}(N_0)$ , and  $B = \partial U\Sigma = \partial M_0$ . This way,  $f|_B : B \rightarrow \partial N_0$  is an embedding. Since  $\psi_2$  is transverse to  $\partial N_0$ ,  $\mu_2$  is transverse to  $B$ .

**Proposition 2.5.** *Denote by  $\nu = \nu_{f|M_0}$  the normal bundle of  $f$  over the regular set  $M_0$ , and by  $D(\nu)$  its disc bundle. Introduce a submersion  $F : D(\nu) \rightarrow N$  continuing  $f_0 = f|_{M_0}$  that is locally injective: for each  $x \in M_0$ , there is an open (open in  $M_0$ ) neighborhood  $U$  containing  $x$  such that  $F|_{D(\nu|_U)}$  is a diffeomorphism. Then,*

$$F^*[f_0] = [i] + [d] \in \mathfrak{N}^k(M_0, B)$$

where  $i : M_0 \rightarrow D(\nu)$  is the zero section and  $d : (D(\xi), 0) \rightarrow (D(\nu), 0)$  is the other leaf of  $f$  (the one transverse to  $i$ ) in double points ( $\xi$  is a  $k$ -bundle over  $\tilde{\Delta}_2^0 = \mu_2^{-1}(M_0)$ ).

*Proof.*  $F$  is a submersion from  $D(\nu)$  (which has the boundary  $\partial D(\nu) = D(\nu|_B) \cup S(\nu)$ , where  $S(\nu)$  is the sphere bundle of  $\nu$ ), so it is transverse to everything in  $N$ ; thus,  $F^*[f_0] = [W \rightarrow D_\nu]$ , where  $W = \{(x, y) : x \in M, y \in D(\nu), f(x) = F(y)\}$ . Obviously,  $W$  includes  $M = \{(x, (x, \mathbf{0}))\} \subset W$ .

The remainder of  $W$  is  $D(\xi)$ , where  $\xi$  is the  $k$ -bundle  $\xi = (\mu_2^0)^*\nu$  over  $\tilde{\Delta}_2^0$  (where  $\mu_2^0 = \mu_2|_{\tilde{\Delta}_2^0}$ ), as  $f$  is self-transverse. The pull-back of  $[f : M \rightarrow F(D(\nu))]$  is clearly  $[i]$ , while  $F^*[D(\xi) \rightarrow F(D(\nu))] = [d : D(\xi) \rightarrow D(\nu)]$ .  $\square$

Now let us apply the homomorphism  $i^*$  induced by the embedding  $i : M_0 \rightarrow D_\nu$  to the equation  $F^*[f_0] = [i] + [d]$ :

$$i^*F^*[f_0] = i^*[i] + i^*[d]$$

$$f^*[f_0] = i^*[i] + [\mu_2^0]$$

as  $d$  is transverse to  $i$ . Since  $i$  is not transverse to itself, we choose a generic section  $s : M_0 \rightarrow D(\nu)$ . Then  $[i] \cong [s]$ , and

$$i^*[i] = s^*[i] = [\iota : s^{-1}(0) \hookrightarrow M_0]$$

At this point, let us reduce our equation to the  $H(\bullet; Z_2)$  theory by the functor  $h$ . If  $\varphi : (V, \partial V) \rightarrow (M_0, B)$ , then (as the Poincaré duality is functorial in the spectral construction of homologies and cohomologies):

$$D_{M_0}(h([\varphi])) = h(D_{M_0}([\varphi])) = h([\varphi]) = \varphi_*[V, \partial V]$$

Similarly, for  $\varphi : (V, \partial V) \rightarrow (N_0, \partial N_0)$ :

$$D_{N_0}(h([\varphi])) = h(D_{N_0}([\varphi])) = h([\varphi]) = \varphi_*[V, \partial V].$$

Using these equalities:

$$h(f^*[f_0] - [\iota] - [\mu_2^0]) = 0$$

$$D_{M_0}(h(f^*[f_0] - [\iota] - [\mu_2^0])) = 0$$

$$D_{M_0}f^*D_{N_0}((f_0)_*[M_0, B]) - [s^{-1}(0)] - (\mu_2^0)_*[\widetilde{\Delta}_2^0] = 0 \in H_{m-k}(M_0, B; Z_2) = H_{m-k}(M, U\Sigma; Z_2)$$

the last equality by excision.

As for the zeros of the section  $s$ ,

$$[s^{-1}(0)] = D_{M_0}(e(\nu) \bmod 2) = D_{M_0}(w_k(\nu)) \in H^k(M_0, B; Z_2) = H^k(M, U\Sigma; Z_2)$$

For  $j : (M, \emptyset) \rightarrow (M, U\Sigma)$ :

$$j^*(w_k(f^*TN - TM)) = w_k(\nu)$$

since the Stiefel-Whitney classes are stable and natural and  $\nu$  is stably the restriction of  $f^*TN - TM$  to  $(M, U\Sigma) \sim (M_0, B)$ .

Now let us examine the exact homology sequence of the pair  $(M, U\Sigma)$ :

$$\dots \rightarrow H^{m-k}(U\Sigma; Z_2) \rightarrow H^{m-k}(M; Z_2) \rightarrow H^{m-k}(M, U\Sigma; Z_2) \rightarrow H^{m-k-1}(U\Sigma; Z_2) \rightarrow \dots$$

$U\Sigma \sim \Sigma$  is homotopically  $m-k-1$  dimensional, so  $H^{m-k}(U\Sigma; Z_2) = 0$ , which means  $H^{m-k}(M; Z_2) \xrightarrow{j_*} H^{m-k}(M, U\Sigma; Z_2)$  is a monomorphism. Using the functoriality of the Poincaré dual:

$$\begin{aligned} 0 &= D_{M_0}f^*D_{N_0}((f_0)_*[M_0, B]) - D_{M_0}w_k(\nu) - (\mu_2^0)_*[\widetilde{\Delta}_2^0] = \\ &= j_* \left( D_M f^*(D_N(f_*[M])) - D_M w_k(f^*TN - TM) - (\mu_2)_*[\overline{\Delta}_2] \right) \in \\ &\in H^{m-k}(M_0, B; Z_2) = H^{m-k}(M, U\Sigma; Z_2) \end{aligned}$$

But  $j_*$  is injective, so this implies

$$D_M f^*(D_N(f_*[M])) - D_M w_k(f^*TN - TM) - (\mu_2)_*[\overline{\Delta}_2] = 0$$

Or, applying  $D_M$  one last time, we get in the cohomologies:

$$f^*(f_!(1)) - w_k(f^*TN - TM) - m_2 = 0 \in H^k(M; Z_2)$$

We did everything in  $Z_2$ -linear spaces, so the signs do not really count. QED.



### 3 The image of $Imm^{SO}(4, 2) \rightarrow \Omega_4$

In this chapter,  $M$  will denote a closed connected oriented 4-manifold.  $Emb^{(SO)}(n, k)$  and  $Imm^{(SO)}(n, k)$  will stand for the cobordism group of embeddings and immersions respectively from (oriented)  $n$ -manifolds to  $\mathbb{R}^{n+k}$ .

Our primary goal is to determine which oriented cobordism classes of 4-manifolds have a representative that can be immersed in  $\mathbb{R}^6$ , that is, the image of the map

$$\varphi : Imm^{SO}(4, 2) \rightarrow \Omega_4$$

where  $\varphi$  is the natural forgetting mapping: it takes a class  $[f : M \looparrowright \mathbb{R}^6]$  to the class  $[M]$ . Clearly,  $\varphi$  is a group homomorphism. To handle  $\Omega_4$  better, we shall use the well-known fact that the *signature*  $\sigma : \Omega_4 \rightarrow Z$  gives an isomorphism between  $\Omega_4$  and  $Z$ . The answer that we will prove is that  $\text{im}(\sigma \circ \varphi) = 2Z \leq Z$ . The strategy we pursue is roughly as follows:

We identify  $\Omega_4$  with the bordism group  $\Omega_4(\mathbb{R}^6)$ . Then we take a generic smooth map  $f : M \rightarrow \mathbb{R}^6$ ; this map may only have  $\Sigma^{1,0}$ -type singularities (Whitney umbrellas), as shown in Chapter 1. We then try to describe the obstruction to the elimination of the singular stratum  $\Sigma(f) = \Sigma^{1,0}(f)$ .

Due to technical difficulties, we will not handle all generic maps  $f : M \rightarrow \mathbb{R}^6$ , only the so-called *prim* maps (to be defined below); this will be sufficient, since there are enough prim maps in a certain sense (there are prim maps from any  $M$  to  $\mathbb{R}^6$ ). We will, by a geometrical construction, define an obstruction  $[\Sigma(f)] \in Z_2$  to the elimination of singularities for prim maps and show that this is indeed the only obstruction.

The general picture of this chapter and most proofs are based on Professor András Szűcs's ideas. The idea underlying the proof in Section 3.4 and part of Theorem 3.31's proof comes from Endre Szabó.

#### 3.1 The obstruction $[\Sigma(f)]$

We define *prim* (projected immersion) maps as:

**Definition 3.1.** A map  $f : N^n \rightarrow \mathbb{R}^{n+k}$  is said to be a *prim map* if there is an immersion  $g : N^n \looparrowright \mathbb{R}^{n+k+1}$  such that  $f = \bar{\pi}_{n+k+1} \circ g$ , where  $\bar{\pi}_{n+k+1} : \mathbb{R}^{n+k+1} \rightarrow \mathbb{R}^{n+k}$  is the projection that eliminates the  $n+k+1$ -st (last) coordinate.

Cobordism of prim maps (of the same dimensions) is simply defined as the (oriented) immersion cobordism of the respective immersions  $g$ . Hence the cobordism group of prim maps from (oriented)  $n$ -manifolds to  $\mathbb{R}^{n+k}$  can be identified with  $Imm^{(SO)}(n, k+1)$ .

**Remark 3.2.** *The prim cobordism class of  $f$ , the class  $[g] \in Imm^{(SO)}(n, k+1)$ , is determined by  $f$  and an orientation of the line bundle  $\ker Tf$  over  $\Sigma(f)$ ; the map  $f$  in itself may be insufficient. From now on, whenever we speak of a prim map  $f$ , we assume that an orientation of  $\ker Tf$  is also given.*

Let us now focus on the case when  $n = 4$ ,  $k = 2$  and the manifold  $M$  is oriented. As always, we can assume the genericity of  $g$  and  $f$ . The only singular stratum  $\Sigma(f) = \Sigma^{1,0}(f)$  of such a map will be the set of points  $x \in M$  where the seventh coordinate axis is in the tangent space  $\text{im}(T_x g)$ . By Corollary 1.13, the set  $\Sigma(f)$  is a compact 1-manifold embedded in  $M$ , in other words, a collection of finitely many embedded circles ( $S^1$ -s). Moreover,  $f$  restricted to  $\Sigma(f)$  is an embedding.

Let us also examine the double point manifolds in the source,  $\mu = \mu_2(f) : \widetilde{\Delta}_2(f) \looparrowright M$  and in the target,  $\psi = \psi_2(f) : \Delta_2(f) \looparrowright \mathbb{R}^6$ . The double point manifolds have dimension 2. (From now on, we usually omit  $f$  from the notation of these maps and manifolds.) As we saw in Chapter 2, adding the singular curve  $\Sigma \hookrightarrow M$  to  $\widetilde{\Delta}_2$  in the source and  $f(\Sigma) \hookrightarrow \mathbb{R}^6$  to  $\Delta_2$  in the target, we get a closed manifold  $\widetilde{\overline{\Delta}}_2$  with an immersion  $\mu : \widetilde{\overline{\Delta}}_2 \looparrowright M$  in the source and a compact manifold  $\overline{\Delta}_2$  (with boundary  $f(\Sigma)$ ) with an immersion  $\psi : \overline{\Delta}_2 \looparrowright \mathbb{R}^6$ .

We have a smooth map  $\hat{f} : \widetilde{\overline{\Delta}}_2 \rightarrow \overline{\Delta}_2$  with  $f \circ \mu = \psi \circ \hat{f}$ . Finally, we have a smooth involution  $T : \widetilde{\overline{\Delta}}_2 \rightarrow \widetilde{\overline{\Delta}}_2$ , such that  $\hat{f} \circ T = \hat{f}$  and the fixed points of  $T$  are exactly  $\Sigma$ . We will not differentiate between  $\Sigma \hookrightarrow \widetilde{\overline{\Delta}}_2$  and  $\mu(\Sigma) \hookrightarrow M$  in notation; since everything is an embedding here, this should not cause confusion.

According to the *Pontryagin construction*, if we choose a *framing* (a trivialization of the normal bundle – that is to say, 5 linearly independent sections) of  $f(\Sigma)$  in the target space  $\mathbb{R}^6$ , we get an invariant in  $\pi_6(S^5) \cong Z_2$ ; more precisely, the framed cobordism classes of dimension 1 and codimension 5 framed embeddings are identified with the elements of  $\pi_6(S^5)$ :  $Emb^{\text{fr}}(1, 5) = \pi_6(S^5)$ .

We shall therefore attempt to fix a trivialization of the normal bundle  $\nu(f(\Sigma) \hookrightarrow \mathbb{R}^6)$  (in the end, we will opt for a curve  $S \hookrightarrow \psi(\Delta_2)$  near to  $f(\Sigma)$  for technical reasons.) The trivialization will involve an arbitrary choice, so we will also need to prove that the resulting class in  $Emb^{\text{fr}}(1, 5)$  does not depend on this choice. Once we have established this, we will have defined the class  $[\Sigma(f)] \in \pi_6(S^5) \cong Z_2$  for generic  $f = \bar{\pi}_7 \circ g : M \rightarrow \mathbb{R}^6$  prim maps.

Let us see a few easy facts about framed cobordisms, particularly in dimensions (1, 5):

**Lemma 3.3.** *Let  $N^n \hookrightarrow \mathbb{R}^{n+k}$  be a submanifold with framing  $[u_1(x), \dots, u_k(x)]$  for  $x \in N$ . Let  $H : N \times [0, 1] \rightarrow GL^+(k)$  be a homotopy in linear transformations*

starting from  $H(x, 0) = I_k$ . Then,  $[w_1(x), \dots, w_k(x)] = H(x, 1) \cdot [u_1(x), \dots, u_k(x)]$  is a valid framing on  $N$  and the two framings are cobordant:  $[N; \mathbf{u}] = [N; \mathbf{w}] \in \text{Emb}^{\text{fr}}(n, k)$ .

*Proof.* Let us take the submanifold  $N^n \times [0, 1] \hookrightarrow \mathbb{R}^{n+k} \times [0, 1]$  and equip it with the framing  $H(x, t) \cdot [u_1(x), \dots, u_k(x)]$  in each point  $(x, t) \in N^n \times [0, 1]$ . (Of course, we need to complement each normal vector with a component in the vertical direction; this component can be set to 0.) Since  $H(x, t) \in GL^+(k)$  is invertible, the new framing spans the normal space at each point, so we gave a framed cobordism between  $[N; \mathbf{u}]$  and  $[N; \mathbf{w}]$ .  $\square$

**Corollary 3.4.** *Given a Riemannian metric on the normal bundle  $\nu(N^n \hookrightarrow \mathbb{R}^{n+k})$ , we can always assume that any framing is orthonormal.*

*Proof.* The procedure of the *Gram-Schmidt orthogonalization* can be implemented as a deformation that is continuous in the starting vectors, that is, a homotopy in  $GL^+(k)$ .  $\square$

**Proposition 3.5.** *In  $\text{Emb}^{\text{fr}}(1, 5)$ , exchanging two normal fields in the framing does not change the cobordism class:  $[N; u_1, u_2, \dots, u_5] = [N; u_2, u_1, \dots, u_5] \in \text{Emb}^{\text{fr}}(1, 5) = \pi_6(S^5)$ . Therefore, we can assume all framings to be oriented (after fixing an orientation on the curve  $N$ ) in this case.*

*Proof.* Clearly, exchanging two normal fields gives an automorphism  $\text{Emb}^{\text{fr}}(1, 5) \rightarrow \text{Emb}^{\text{fr}}(1, 5)$ . However,  $Z_2$  only has the identity as automorphism.  $\square$

The following proposition is both easy and well-known:

**Proposition 3.6.**  $\pi_1(SO(k)) \cong Z_2$  for  $k \geq 3$ .

**Lemma 3.7.** *Let  $N \cong S^1$ ,  $N \hookrightarrow \mathbb{R}^6$ , and  $\nu = \nu(N \hookrightarrow \mathbb{R}^6)$  be its normal bundle. Let us choose two positively oriented framings  $[N, \mathbf{u}]$  and  $[N, \mathbf{w}]$  ( $u_i, w_i \in \nu$ ). Let  $A(x) = \mathbf{w}\mathbf{u}^{-1} \in GL^+(5) \sim SO(5)$  be the "ratio" of the two framings: the extent to which  $\mathbf{u}(x)$  needs to be twisted so that we get  $\mathbf{w}(x)$ . Then, the two framings are cobordant if and only if  $[A] = 0 \in [N, GL^+(5)]$ .*

*Proof.* If  $[A] = 0 \in [N, GL^+(5)]$ , then we are given a homotopy  $H : N \times [0, 1] \rightarrow GL^+(5)$  starting in the identity and ending in  $A$ , so by Lemma 3.3 we have a framed cobordism.

For the other direction, we make use of the fact that  $[A] \in [N, GL^+(5)] = \pi_1(GL^+(5)) \cong Z_2$ . By contradiction, assume that  $[N, \mathbf{u}]$  and  $[N, \mathbf{w}]$  are cobordant even though their ratio  $[A]$  is not null-homotopic. But in this case, let us take any

other positively oriented framing  $[N, \mathbf{v}]$ : if  $B(\mathbf{w}) = \mathbf{v}$ , then  $[B] \neq [BA] = [B] + [A]$ , using the fact that pointwise multiplication in a topological group (like  $GL^+(5)$ ) can be replaced by addition in the homotopy groups.

Since  $[N, GL^+(5)] = \pi_1(GL^+(5))$  has only 2 elements, this means that  $[N, \mathbf{v}]$  is in the same class as either  $[N, \mathbf{u}]$  or  $[N, \mathbf{w}]$ . However, by Proposition 3.5, all framings can be assumed to be positively oriented, so this would mean that all framings are cobordant, in contradiction with the fact  $\#\{Emb^{fr}(1, 5)\} = 2 > 1$ .  $\square$

(Of course, all of this works for dimensions  $(1, k)$  with  $k \geq 3$ .)

**Remark 3.8.** *In the above proof, we conveniently used the well-known fact that  $Emb^{fr}(1, k) = \pi_{k+1}(S^k) \cong Z_2$  for  $k \geq 3$ . However, we can prove this as a by-product at this point.*

*Proof.* From the above considerations, it follows that there are at most 2 non-cobordant framed embeddings in dimension 1, codimension  $k \geq 3$ . To prove that there are 2, we only need to show a non-nullcobordant framing.

Let us equip the standard  $S^1 \subset \mathbb{R}^2 \times \{0\} \times \cdots \times \{0\}$  with the standard null-cobordant framing  $\mathbf{u}$  and a framing  $\mathbf{v}$  so that the ratio  $[A] = [\mathbf{v}\mathbf{u}^{-1}] \in [S^1, SO(k) \cdot B] \cong [S^1, SO(k)] = \pi_1(SO(k))$  (where  $SO(k) \cdot B$  is one of the two cosets of  $SO(k)$  in  $O(k)$ ) is not null-homotopic.

Assume that  $\mathbf{u}$  can be extended to a connected surface  $W$  with boundary  $S^1 = \partial W = W \cap (\mathbb{R}^{k+1} \times \{0\})$ , to form a framed null-cobordism.  $W$  must be orientable, since its normal bundle is trivialized, so  $W = A_p \setminus D^2$  for some canonical oriented surface  $A_p$ . It is easy to construct the extension of  $\mathbf{v}$  to such a  $W$ , so the ratio  $A = \mathbf{v}\mathbf{u}^{-1} : W \rightarrow SO(k) \cdot B$  is extended, too. If

$$\pi_1(A_p) = \left\langle a_1, b_1, \dots, a_p, b_p \left| \prod_{i=1}^p [a_i, b_i] = 1 \right. \right\rangle$$

for the generating loops  $a_i, b_i \in \pi_1(A_p)$ , then from the CW structure of  $W$ , we can see that  $S^1$  represents the product  $[a_1, b_1] \dots [a_p, b_p]$ . But this means that  $A_* : \pi_1(W) \rightarrow \pi_1(SO(k))$  sends  $[S^1]$  to  $A_*([a_1, b_1] \dots [a_p, b_p]) = 0$  in the Abelian group  $\pi_1(SO(k))$ , which gives contradiction.  $\square$

Let us return to the prim maps  $f : M \rightarrow \mathbb{R}^6$ . In practice, it will be more convenient to work with an embedded 1-manifold  $S \subset \psi(\Delta_2)$  in the double points that is near to  $f(\Sigma)$ , than with  $f(\Sigma)$  itself. More precisely, let  $\vartheta = \nu(\Sigma \hookrightarrow \overline{\Delta_2})$  be the normal bundle of  $\Sigma$  in  $\overline{\Delta_2}$ .  $\vartheta$  is a 1-dimensional

bundle over  $\Sigma$ . We can equip  $\vartheta$  with a Riemannian metric and take its closed disc bundle,  $D(\vartheta) = \{v \in \vartheta : \|v\| \leq 1\}$ .

Denote by  $U(\Sigma)$  a  $T$ -invariant closed tubular neighborhood of  $\Sigma$  in  $\widetilde{\Delta}_2$ , small enough that it does not contain a triple point ( $\nu|_{U(\Sigma)}$  is an embedding). By genericity,  $\Sigma$  is far away from the triple points, so this can be achieved. Let  $\tau : D(\vartheta) \rightarrow U(\Sigma)$  be a  $Z_2$ -equivariant diffeomorphism:  $T(\tau(v)) = \tau(-v)$  for  $v \in \vartheta$  above each point of  $\Sigma$ .

The boundary  $\partial U(\Sigma)$  is a disjoint union  $\tilde{S}_2 \cup \tilde{S}_1$  of the upper ( $\tilde{S}_2$ ) and the lower ( $\tilde{S}_1$ ) components of the boundary (with respect to  $\pi_7 \circ g$ , where  $\pi_7$  is the projection to the 7-th coordinate). Let  $S = \psi(\hat{f}(\tilde{S}_1)) = \psi(\hat{f}(\tilde{S}_2))$ .

We now try to give a framing of the normal bundle  $\nu_S = \nu(\psi|_S : S \hookrightarrow \mathbb{R}^6)$ . Denote  $\nu(\Delta_2) = \nu(\psi|_{\Delta_2} : \Delta_2 \hookrightarrow \mathbb{R}^6)$ ; this will later serve well to establish a framed cobordism.

Let us define a pair of oriented 2-plane bundles over  $S$ . For any  $q \in S$ , let  $\tilde{q}_i$  ( $i = 1, 2$ ) be its lower and upper preimages;  $\tilde{q}_i \in \tilde{S}_i$ . Define the bundles  $\xi_i(q) = f_*((T_{\tilde{q}_i} \mu(T_{\tilde{q}_i} \widetilde{\Delta}_2))^\perp)$ , the 2-dimensional remainders (the parts orthogonal to  $\psi(\Delta_2)$ ) of the tangent spaces of the two leaves of the mapping  $f$  at the double point  $q$ . Since the codimension  $6 - 4 = 2$  is even, the bundles  $\xi_i$  inherit an orientation from that of  $M$  and  $\mathbb{R}^6$ .

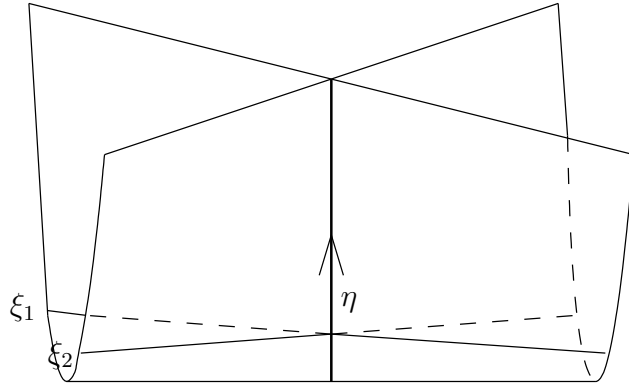


Figure 3:  $(\xi_1)_q$  and  $(\xi_2)_q$  collapse into each other as  $q \rightarrow \Sigma$

**Lemma 3.9.**  $\nu_S$  is globally decomposed:  $\nu_S = \eta \oplus \xi_1 \oplus \xi_2$ , where  $\eta$  is a trivial line bundle.

*Proof.* In the normal bundle  $\vartheta = \{(p, x) : p \in \Sigma, x \in [-1, 1]\}$ , consider the tangent vectors  $\tilde{u}_2 = (\partial/\partial x)|_{\tilde{S}_2}$  in the "upper" boundary and  $\tilde{u}_1 = -(\partial/\partial x)|_{\tilde{S}_1}$  in the "lower" boundary. By the  $Z_2$ -equivariance,  $(\psi \circ \hat{f} \circ \vartheta)_* \tilde{u}_1 = (\psi \circ \hat{f} \circ \vartheta)_* \tilde{u}_2$ ; let us denote this vector field by  $u$  and the trivial bundle it spans by  $\eta$ . In effect,  $u$  is the inner normal vector of  $S$  in  $\psi(\Delta_2 \setminus \hat{f}(U(\Sigma)))$ .

What remains is to decompose the normal space of the double point manifold  $\psi(\Delta_2)$  over  $S$  into  $\xi_1$  and  $\xi_2$ . Obviously,  $\xi_1$  and  $\xi_2$  are normal to  $\psi(\Delta_2)$ , as  $f$  is an immersion in  $\tilde{q}_i$ . By self-transversality of  $f$  outside  $\Sigma$ , the bundles  $\xi_1$  and  $\xi_2$  are transverse, so the decomposition of  $\nu_S$  is complete.  $\square$

For further use, let us introduce a Riemannian metric on the 2-bundle  $\nu(\mu|_{U(\Sigma)} : U(\Sigma) \rightarrow M)$ ; denote by  $J$  the rotation by  $+\pi/2$ . Restricting this metric to the boundaries  $\tilde{S}_i$  of  $U(\Sigma)$  and pushing them forward, we get Riemannian metrics over  $\xi_1$  and  $\xi_2$  too; here, let the rotations by  $+\pi/2$  be denoted by  $J_1$  and  $J_2$ , respectively. We can extend the Riemannian metric of  $\xi_1$  and  $\xi_2$  over  $S$  to the whole normal bundle  $\nu_S$  such that  $\nu_S = \eta \oplus \xi_1 \oplus \xi_2$  is an *orthogonal* decomposition.

**Remark 3.10.** *We also have a homotopically well-defined bundle isomorphism between  $\xi_1$  and  $\xi_2$ , which is given by the normal bundle  $\nu(\mu : \overline{\Delta}_2 \rightarrow M)$  over  $U(\Sigma) \cong \Sigma \times [-1, 1]$ . So we have a canonical decomposition of  $\nu_S$  into a trivial line bundle and a pair of canonically isomorphic oriented plane bundles. By the general Thom construction, such a structure already gives an invariant in  $\pi_6(T\xi)$ , where  $T\xi$  is the Thom space for the bundle  $\xi = \varepsilon^1 \oplus 2\gamma_{SO}^2$ , and  $\gamma_{SO}^2$  is the universal oriented 2-dimensional bundle.*

Let us choose a generic section  $\hat{s} : \overline{\Delta}_2 \rightarrow \nu(\mu : \overline{\Delta}_2 \rightarrow M)$ . It is a section of a 2-dimensional bundle over a 2-dimensional manifold, so typically it will not vanish over the curve  $\Sigma$ . If we took the tubular neighborhood  $U(\Sigma)$  small enough,  $\hat{s}$  does not vanish even on  $U(\Sigma)$ . Thus,  $\hat{s}$  defines a framing: if  $s_i = f_*\mu_*(\hat{s}|_{\tilde{S}_i})$ , then  $[u, s_1, J_1(s_1), s_2, J_2(s_2)]$  is a framing of  $\nu_S$ .

**Lemma 3.11.** *The framed cobordism class  $[S; u, s_1, J_1(s_1), s_2, J_2(s_2)]$  does not depend on the section  $\hat{s}$ , as long as  $\hat{s}$  does not vanish over  $U(\Sigma)$ . Therefore, it is an invariant of the prim map  $f$ . (By the Pontryagin construction, this invariant lies in  $\pi_6(S^5) \cong Z_2$ ).*

**Definition 3.12.** *Let us denote this framed cobordism class by  $[\Sigma(f)] \in Emb^{\text{fr}}(1, 5) = \pi_6(S^5)$ .*

*Proof.* Let  $C \cong S^1$  be an arbitrary connected component of the embedded 1-manifold  $S$ . Clearly,  $[\Sigma(f), \hat{s}]$  (temporarily mentioning the possible dependence on  $\hat{s}$  in the notation) is the sum of the framed cobordism classes of the framings on the components, so we only need to deal with a change of the section over  $\vartheta(C \times [-1, 1]) \subset U(\Sigma)$ . Therefore, we can assume without loss of generality that  $S = S^1$ .

Now we examine what happens if  $\hat{s}$  is replaced by another nonzero section

$$\hat{t} : U(\Sigma) \rightarrow \nu(\mu : U(\Sigma) \rightarrow M).$$

Let us norm both  $\hat{s}$  and  $\hat{t}$  to unit length with respect to the Riemannian metric that we took on  $\nu(\mu : U(\Sigma) \rightarrow M)$ ; this way,  $[\hat{s}, J(\hat{s})]$  and  $[\hat{t}, J(\hat{t})]$  are both positive orthonormal bases in each fiber.

Then we can take, in the fiber above each point  $x \in U(\Sigma)$ , the ratio  $[\hat{s}, J(\hat{s})]_x^{-1}[\hat{t}, J(\hat{t})]_x$ , that is, the unique special orthogonal transformation  $A(x) \in SO(2)$  that takes  $\hat{s}(x)$  into  $\hat{t}(x)$ . These matrices  $A(x)$  form a continuous map  $A : U(\Sigma) \rightarrow SO(2)$ . Let us define  $A_1, A_2 : S \rightarrow SO(2)$  as follows:  $A_i(q) = A(\tilde{q}_i)$ ; this way,  $A : U(\Sigma) \cong \Sigma \times [-1, 1] \rightarrow SO(2)$  gives a homotopy between  $A_1$  and  $A_2$ .

With respect to the Riemannian metric that we took on  $\nu_S$ , the framings  $[u, s_1, J_1(s_1), s_2, J_2(s_2)]$  and  $[u, t_1, J_1(t_1), t_2, J_2(t_2)]$  are orthonormal bases of the fiber of  $\nu_S$  over each point  $q \in S$ , so we can take their ratio  $B(q) \in SO(5)$  in a similar manner to what we have seen in Lemma 3.7. In fact, we already know  $B$ :

$$B(q) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & A_1(q) & 0 \\ 0 & 0 & A_2(q) \end{bmatrix}$$

Thus,  $B : S \rightarrow SO(5)$  expresses the twisting one has to do to get the framing  $[\Sigma(f), \hat{t}]$  from  $[\Sigma(f), \hat{s}]$ .

**Sublemma 3.13.** *B is homotopically trivial:  $[B] = 0 \in \pi_1(SO(5))$ .*

*Proof.* Recall again the fact that in a topological group (like  $SO(5)$ ), multiplication in the fundamental group and pointwise multiplication of the loops is equivalent; thus,

$$B(q) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & A_1(q) & 0 \\ 0 & 0 & I_2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & A_2(q) \end{bmatrix} = B_1(q) \cdot B_2(q); \quad [B] = [B_1] + [B_2] \in \pi_1(SO(5))$$

Now observe that  $B_2(q) = CB'_2(p)C^{-1}$  for every  $p$ , where

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_2 \\ 0 & I_2 & 0 \end{bmatrix} \in SO(5), \quad B'_2(q) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & A_2(q) & 0 \\ 0 & 0 & I_2 \end{bmatrix}.$$

Since  $SO(5)$  is connected, there is a path  $H : [0, 1] \rightarrow SO(5)$ ,  $H(0) = I_5$ ,  $H(1) = C$ . By taking  $D_t(p) = H(t)B_2(p)H(t)^{-1}$ ,  $D_t$  is a homotopy between  $B_2$  and  $B'_2$ . We already know that  $A_1$  and  $A_2$  are homotopic; thus so are  $B'_2$  and  $B_1$ . Therefore,  $B_1$

and  $B'_2$  represent the same element in  $\pi_1(SO(5)) \cong Z_2$ . Whichever this element be, its double  $[B] = [B_1] + [B'_2]$  is zero in  $\pi_1(SO(5))$ . □

We now see that  $B$  is null-homotopic. Using Lemma 3.3, the two framings give the same framed cobordism class in  $Emb^{fr}(1, 5) = \pi_6(S^5)$ . □

**Remark 3.14.** *Technically, we did not prove that  $[\Sigma(f)]$  is independent on the tubular neighborhood  $U(\Sigma)$  and the map  $\tau : \vartheta \rightarrow U(\Sigma)$ . However, the proof for this is essentially identical to the above one about independence on  $\hat{s}$  (the only extra notion we need is that for two tubular neighborhoods, there is a third that is contained in both).*

We will now prove that  $[\Sigma(f)]$  is indeed an invariant of the prim cobordism class  $[f]$ :

**Theorem 3.15.** *For a prim map  $f = \bar{\pi}_7 \circ g$ , ( $g : M \looparrowright \mathbb{R}^7$ ), the class  $[\Sigma(f)]$  only depends on the prim cobordism class  $[f] \in Imm^{SO}(4, 3)$ . So we get a homomorphism  $[\Sigma] : Imm^{SO}(4, 3) \rightarrow Z_2$ .*

*Proof.* First we prove that if  $[f] = 0$ , then  $[\Sigma(f)] = 0$ .

Suppose  $G : W_{or}^5 \looparrowright (\mathbb{R}^8)^+$  is an immersion such that  $\partial W = M$ , the tangent space  $G_*(T_p W)$  is not horizontal (not a subspace of  $\mathbb{R}^7 \times \{0\}$ ) in  $p \in M$  (this can be ensured by adding a "collar" to  $G$ ), and  $G$  and its projection  $F = \bar{\pi}_7 \circ G$  are generic. (It is the 7-th coordinate that we collapse, the 8-th coordinate is brought in by the cobordism.)

Fortunately, the condition  $2n > 3m - 2$  still holds, so  $F$  satisfies the conditions of Corollary 1.13. (We only stated this for manifolds without boundary, but the version with boundary can easily be obtained by using the closed manifold  $W \cup_{\partial W} (-W)$ .)

The singular points of  $F$ , the set  $\Sigma_F = \Sigma^{1,0}(F)$  is a 2-manifold with boundary  $\partial \Sigma_F = \Sigma_F \cap M = \Sigma(f) = \Sigma$ . Let us discard any connected components of  $\Sigma_F$  that are not connected to the boundary.

$\Sigma_F$  is embedded in the 3-manifold  $\widetilde{\Delta}_2(F)$ . Let  $U(\Sigma_F)$  be a  $T$ -invariant tubular neighborhood extending  $U(\Sigma)$  with  $\tau : \vartheta_F = D\left(\nu\left(\Sigma_F \hookrightarrow \widetilde{\Delta}_2(F)\right)\right) \rightarrow U(\Sigma_F)$ , a  $Z_2$ -invariant diffeomorphism. Let  $S_F = f(\mu(\partial U(\Sigma_F)))$ .

We can break up the normal bundle of  $S_F$  continuing the decomposition of  $\nu(S \hookrightarrow \mathbb{R}^6)$ , and in much the same way:

$$\nu(S_F \hookrightarrow \mathbb{R}^7) = \eta \oplus \xi_1 \oplus \xi_2,$$



where  $\eta = \langle u \rangle$ ,  $u \in \nu(S_F \hookrightarrow \psi(\Delta_2(F) \setminus \hat{f}(U(\Sigma_F))))$  ( $u$  is the normal vector pointing away from  $\Sigma_F$ ), and for  $q \in S_F$ ,  $\tilde{q}_i \in \partial U(\Sigma(F))$ ,  $f(\mu(\tilde{q}_i)) = q$ :

$$\xi_i(q) = f_*((T_{\tilde{q}_i}\mu(T_{\tilde{q}_i}\widetilde{\Delta}_2(F)))^\perp).$$

This decomposition is globally consistent over  $S_F$ , as we have an ordering between the pre-images  $\tilde{q}_{1,2}$ . Also, the bundles  $\xi_{1,2}$  inherit an orientation from the double point manifold  $\widetilde{\Delta}_2$ , which is oriented because  $k = 2$  is even.

Now we want to find a nonzero section  $\hat{s}_F : U(\Sigma_F) \rightarrow \nu(U(\Sigma_F) \hookrightarrow W)$ . We need to be more careful here: the bundle is 2-dimensional, as is  $\Sigma_F$ . However,  $\Sigma_F$  is a surface with boundary, and we made sure that each of its connected components have nontrivial boundary.

Since all closed 2-manifolds have a CW structure that has a single 2-cell, all connected compact 2-manifolds with nontrivial boundary are homotopically equivalent to a 1-complex. This implies that  $[\Sigma_F, BSO_2] = 0$ , as  $sk_1(BSO_2) = \{*\}$ , so all oriented plane bundles over  $\Sigma_F$  are trivial. Thus we can give a nonzero section  $\hat{s}_F$  over  $\Sigma_F$ , which we can extend to the tubular neighborhood  $U(\Sigma_F)$ .

Taking  $s_i(q) = f_*\mu_*(\hat{s}_F(\tilde{q}_i))$ , and introducing Riemannian metrics on  $\xi_1$  and  $\xi_2$  that continue the ones already given over  $S$ , we have a framing  $[u, s_1, J_1(s_1), s_2, J_2(s_2)]$  over  $S_F$  that continues the one given on  $S$ . The surface  $S_F$  with this framing is exactly a framed cobordism that shows  $[\Sigma(f)] = [S; u, s_1, J_1(s_1), s_2, J_2(s_2)] = 0$ .

We have proved that if  $[f] = 0$ , then  $[\Sigma(f)] = 0$ , thus,  $[\Sigma(f)]$  only depends on  $[f]$ . Since everything is additive, we also get a homomorphism  $[\Sigma] : Imm^{SO}(4, 3) \rightarrow \pi_6(S^5) = Z_2$ .  $\square$

We now prove a key result that connects the global properties of  $M$  with the class  $[\Sigma(f)]$ . Due to geometric difficulties, this proof will only work for an even smaller class of maps  $f$ .

**Definition 3.16.** *A mapping  $f : N^n \rightarrow \mathbb{R}^{n+k}$  is said to be a prem (projected embedding) if there is an embedding  $g : N^n \hookrightarrow \mathbb{R}^{n+k+1}$  such that  $f = \bar{\pi}_{n+k+1} \circ g$ , where  $\bar{\pi}_{n+k+1}$  is the projection that eliminates the last coordinate.*

As with prims, codimension  $k$  prims from (oriented)  $n$ -manifolds can be identified with representatives of the elements of  $Emb^{(SO)}(n, k+1)$ , and we define cobordism of prims via this identification.

**Remark 3.17.** *As with prims,  $f$  in itself does not determine the prem cobordism class  $[g] \in Emb^{(SO)}(n, k+1)$ . However,  $f$  and an orientation of the line bundle*

$\ker Tf$  over  $\Sigma(f)$  is sufficient to reconstruct the cobordism class  $[g]$  of the embedding  $g$ . From now on, whenever we speak of a prem map  $f$ , we understand that an orientation of the line bundle  $\ker Tf$  is also given.

We shall later prove that there are, in a sense, "enough" prem maps so that the general result is implied by the prem case.

**Theorem 3.18.** *For every generic prem map  $f = \bar{\pi}_7 \circ g$  ( $g : M_{or}^4 \hookrightarrow \mathbb{R}^7$ ), we have  $[\Sigma(f)] = \sigma(M) \bmod 2 \in Z_2$ , where  $\sigma(M)$  is the signature of the oriented 4-manifold  $M$ .*

*Proof.* In the prem case, the double point manifold  $\bar{\Delta}_2$  is still only immersed in  $\mathbb{R}^6$  (as  $f$  might have triple points), but now we have a global ordering of the pre-images; that is,  $\hat{f} : \tilde{\Delta}_2 \rightarrow \Delta_2$  is a trivial double covering. Using the self-transversality of  $f$  at the double points, the four-dimensional normal bundle  $\nu(\psi : \Delta_2 \looparrowright \mathbb{R}^6)$  decomposes to two 2-dimensional bundles: in the double point  $q \in \Delta_2$ , we have the orthogonal spaces  $f_*((T_{\tilde{q}_i} \mu(T_{\tilde{q}_i} \tilde{\Delta}_2))^{\perp})$  ( $i = 1, 2$ ), the normal spaces of the double surface in each leaf.

The ordering of the two pre-images  $\tilde{q}_{1,2} \in \tilde{\Delta}_2$  is globally consistent – this is why we needed  $f$  to be prem! –, so we have a global decomposition  $\nu(\Delta_2 \looparrowright \mathbb{R}^6) = \xi_1 \oplus \xi_2$ . Very importantly, this decomposition is the extension of the one we chose over  $S$ . These plane bundles are still oriented, as seen from the same argument.

Our general plan is to use  $W' = \Delta_2 \setminus \hat{f}(\text{int } U(\Sigma))$  itself as a cobordism, with suitable framing, to establish a relation between the framed curve  $[S; u, s_1, J_1(s_1), s_2, J_2(s_2)]$  and a standard framed curve whose cobordism class can be computed. To this end, we prove a useful technical statement:

**Lemma 3.19.** *Let  $N^n \hookrightarrow \mathbb{R}^{n+k}$  be a compact manifold with boundary, with the normal framing  $[v_1, \dots, v_k]$ . Let  $u$  be an inner normal vector for  $N$  in  $\partial N$ . Then, its boundary  $\partial N$  with the framing  $[u, v_1, \dots, v_k]$  is null-cobordant:  $[\partial N; u, v_1, \dots, v_k] = 0 \in \text{Emb}^{\text{fr}}(n-1, k+1)$*

*Proof.* By definition, we need a manifold  $N' \hookrightarrow (\mathbb{R}^{n+k+1})^+$  with boundary  $\partial N' = N' \cap (\mathbb{R}^{n+k} \times \{0\}) = \partial N$ , and a framing  $[u', v'_1, \dots, v'_k]$  on it, such that the framing's restriction on  $\partial N$  is the desired  $[u, v_1, \dots, v_k]$ . What we do is "lift" the interior of  $N$  out of the plane  $\mathbb{R}^{n+k} \times \{0\}$ . For this, take a smooth function  $h : N \rightarrow \mathbb{R}^+$  such that  $h|_{\partial N} \equiv 0$ , but the inner derivative  $\partial_u(h)(x)$  is positive,  $\forall x \in \partial N$ .  $N'$  shall be the same as  $N$ , but its embedding to  $(\mathbb{R}^{n+k+1})^+$  is lifted by  $h$ :

$$N' = \{(x, h(x)) : x \in N\} \hookrightarrow (\mathbb{R}^{n+k+1})^+.$$

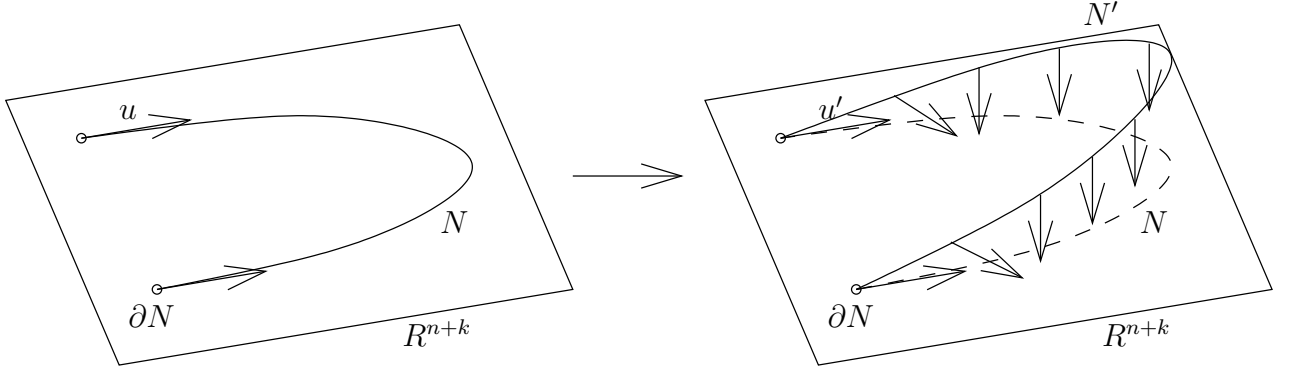


Figure 4: *Lifting the cobordism in Lemma 3.19.*

Denote by  $U = U(\partial N) = \partial N \times [0, \varepsilon)$  an open tubular neighborhood of  $\partial N$ . Then,

$$u'(x) = \begin{cases} \frac{\tau}{\varepsilon}u(y) - (1 - \frac{\tau}{\varepsilon})\mathbf{e}_{n+k+1} & \text{if } x \in U, x \sim (y, \tau) \\ -\mathbf{e}_{n+k+1} & \text{otherwise} \end{cases}$$

where  $-\mathbf{e}_{n+k+1} = (0, \dots, 0, -1)$  is the vector pointing straight down. All the other vectors need not change:  $v'_i((x, h(x))) = v_i((x, 0)), \forall x \in N$ .

The vectors  $v'_i$  stay normal, as even their projections  $v_i$  are not tangent to the projection of  $N'$ , namely  $N$ . If  $U$  is chosen to be small enough,  $u'$  is also normal:  $h$  "starts rising" near in  $U$  to the direction of  $u$  (for  $y \sim (x, \tau)$ ,  $\partial_u(h)(y) > 0$  if  $\tau$  is small enough, since  $h \in C^1$  and  $\partial N$  is compact). Finally, the vectors  $u', v'_1, \dots, v'_k$  are linearly independent: on  $\partial N$ , this is because  $u$  is in the tangent space of  $N$ , while in other points, the linear hull  $\langle v'_1, \dots, v'_k \rangle$  is in  $\mathbb{R}^{n+k} \times \{0\}$ , while  $u'$  has a downward component.

So we succeeded in giving a framed cobordism  $[N'; u', v'_1, \dots, v'_k]$  that demonstrates  $[\partial N; u, v_1, \dots, v_k] = 0 \in Emb^{fr}(n-1, k+1)$ .

□

To use this handy result, we need to trivialize the bundles  $\xi_1$  and  $\xi_2$  over the immersed compact manifold  $W' = \Delta_2 \setminus \hat{f}(\text{int } U(\Sigma))$  with boundary to obtain a framing. As above, let us take a generic section  $\hat{s} : \widetilde{\Delta}_2 \rightarrow \nu(\mu : \widetilde{\Delta}_2 \rightarrow M)$ . This is a section of a 2-bundle over a 2-manifold, so it has isolated zeroes. Again by genericity of  $\hat{s}$ , we can assume that the zeroes do not fall in  $U(\Sigma)$  and that they are not equivalent by  $T$ . Finally we can assume, for simplicity, that  $\hat{s}$  does not vanish at the triple points of  $f$  (where  $\mu$  is not an embedding).

**Lemma 3.20.** *Let  $\hat{s} : \widetilde{\Delta}_2 \rightarrow D(\nu(\mu))$  be as above. Then,  $\#\{\hat{s}^{-1}(0)\} = [\Sigma(f)] \bmod 2$ .*

*Proof.* We will try to use  $\hat{s}$  to trivialize the oriented 2-bundles  $\xi_{1,2}$  over  $W'$ . By Lemma 3.19, this will establish a framed cobordism starting in  $[S; u, s_1, J_1(s_1), s_2, J_2(s_2)]$ .

If, in a point  $q \in W'$ , where  $q = f(\tilde{q}_1) = f(\tilde{q}_2)$ , the section  $\hat{s}(\tilde{q}_{1,2})$  is non-zero in both pre-images, we have produced a desired trivialization of the planes  $(\xi_{1,2})_q$ : let us simply take the framing  $[s_1, J_1(s_1), s_2, J_2(s_2)]$ , where once again,  $s_i(q) = f_*\mu_*(\hat{s}(\tilde{q}_i))$  is just the projection of the section  $\hat{s}$  and  $J_i$  is the rotation by the positive angle  $+\pi/2$  in the respective bundles. (By definition, this is consistent with the framing we took on  $S$ .)

Were it not for the zeroes of  $\hat{s}$ , we would have established a framing  $[s_1, J_1(s_1), s_2, J_2(s_2)]$  over the immersed framed 2-manifold with boundary,  $W'$ . Since  $2 \dim W' < 6$ , we can perturb the immersion  $\psi|_{W'} : W' \looparrowright \mathbb{R}^6$  to have an embedded submanifold that, according to Lemma 3.19, shows  $[\Sigma(f)] = 0$ .

In general, however,  $\hat{s}$  has zeroes at  $\tilde{q}^1, \dots, \tilde{q}^k$ . Thus, we have a framing of  $\nu(\psi|_{W'})$ , except in the points  $q^l = f(\tilde{q}^l)$ ,  $1 \leq l \leq k$ . Let us cut out small, disjoint open disc neighborhoods  $D_l$  of  $q^l$  from  $W'$ . Denote  $S_l = \partial D_l$ . We have a legitimate framing  $[s_1, J_1(s_1), s_2, J_2(s_2)]$  on the remaining surface  $\psi(W)$ , where  $W = W' \setminus \bigcup_{l=1}^k D_l$ . By perturbing  $\psi|_W$  slightly, this will give a framed embedding. So we get:

$$[\Sigma(f)] = - \sum_{l=1}^k [S_l; u, s_1, J_1(s_1), s_2, J_2(s_2)] \in Emb^{\text{fr}}(1, 5) \cong Z_2,$$

where  $u$  is the inner normal vector of  $\psi(W)$  in each boundary component.

Since  $k = \#\{\hat{s}^{-1}(0)\}$ , it suffices to show that for all values  $l$ , the class  $[D_l; u, s_1, J_1(s_1), s_2, J_2(s_2)]$  is the nontrivial element in  $Emb^{\text{fr}}(1, 5) = Z_2$ .  $D_l$  is an embedded disc, so it can be transported by isotopy to the standard embedding of the unit disc,  $\{(x_1, \dots, x_6) : x_1^2 + x_2^2 \leq 1, x_3 = \dots = x_6 = 0\} \hookrightarrow \mathbb{R}^6$ , so that the fibers of  $\xi_1$  and  $\xi_2$  go into the coordinate planes spanned by  $\langle \mathbf{e}_3, \mathbf{e}_4 \rangle$  and  $\langle \mathbf{e}_5, \mathbf{e}_6 \rangle$ , respectively, where  $\mathbf{e}_i$  is the  $i$ 'th vector of the standard basis in  $\mathbb{R}^6$ .

By symmetry, we can assume that  $\tilde{q}^l = \tilde{q}_1^l$  is the first pre-image of  $q^l$  and  $T(\tilde{q}_1^l) = \tilde{q}_2^l$  is the second.  $T$  does not carry zeroes of  $\hat{s}$  to one another, so  $\hat{s}(\tilde{q}_2^l) \neq 0$ . If  $D_l$  was taken small enough, the turning number of  $s_2(q^l) = f_*(\tilde{q}_2^l)$  in  $\xi_2$  is zero as  $q^l$  goes round  $S_l$ , and the same is true for  $J_2(s_2)$ . This means that the sections  $s_2$  and  $J_2(s_2)$  can be twisted to  $\mathbf{e}_5$  and  $\mathbf{e}_6$  by isotopy.

So we only need to care about the nontrivial normal vectors  $[u, s_1, J_1(s_1)]$  in  $\mathbb{R}^4$ .  $\hat{s}$  has a root at  $\tilde{q}_1^l$ , but by genericity, its derivative can be assumed to be regular in  $\tilde{q}_1^l$ . This means that  $\hat{s}$  can be supposed to be the identity map  $\text{id}_{\mathbb{R}^2} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  in properly chosen local coordinates. In this case,  $s_1(q^l) = Aq^l$ , where  $A : \mathbb{R}^2 \rightarrow \mathbb{R}^6$  is the linear map  $(x_1, x_2) \rightarrow (0, 0, x_1, x_2, 0, 0)$ .

At this point, we transformed the framing into a standard one: at  $q' = (\cos \alpha, \sin \alpha, 0, 0, 0, 0) \in S_l$ , we have

$$\begin{aligned} v_1(q') &= (\cos \alpha, \sin \alpha, 0, 0, 0, 0) = u \\ v_2(q') &= (0, 0, \cos \alpha, \sin \alpha, 0, 0) = s_1 \\ v_3(q') &= (0, 0, -\sin \alpha, \cos \alpha, 0, 0) = J_1(s_1) \\ v_4(q') &= (0, 0, 0, 0, 1, 0) = s_2 \\ v_5(q') &= (0, 0, 0, 0, 0, 1) = J_2(s_2) \end{aligned}$$

over the standard circle  $S_l = S^1$ . We will now proceed to compute its class in  $Emb^{\text{fr}}(1, 5) \cong Z_2$ . (An alternative route is available here: if this class was 0, that would mean  $[\Sigma(f)] = 0$  for every  $f = \bar{\pi}_7 \circ g$ ,  $g \in Emb^{SO}(4, 3)$ ; a single example for the contrary would suffice.)

Let us now compare our framing with a standard one:

$$\begin{aligned} w_1 &= (\cos \alpha, \sin \alpha, 0, 0, 0, 0) \\ w_2 &= (0, 0, 1, 0, 0, 0) \\ w_3 &= (0, 0, 0, 1, 0, 0) \\ w_4 &= (0, 0, 0, 0, 1, 0) \\ w_5 &= (0, 0, 0, 0, 0, 1) \end{aligned}$$

This framing  $[S^1; w_1, \dots, w_5]$  is null-cobordant, easily seen by Lemma 3.19:  $w_1$  is the inner normal vector for  $\partial D^2 = S^1$ , while  $w_2, \dots, w_5$  can be extended to  $D^2$ . Now let us take the ratio of the two framings in  $SO(5)$ , just like we did in Lemma 3.11: with

$$A(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha & 0 & 0 \\ 0 & -\sin \alpha & \cos \alpha & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$A\mathbf{w} = \mathbf{v}$ .

**Proposition 3.21.** *A as a loop is homotopically nontrivial; with  $A : S^1 = [0, 2\pi]/(0 \sim 2\pi) \rightarrow SO(5)$ , we get  $[A] \neq 0 \in \pi_1(SO(5)) \cong Z_2$ .*

*Proof.* From the exact homotopical sequences for the fibrations  $SO(n) \xrightarrow{S^{n-1}} SO(n-1)$  for  $n = 3, 4$  and  $5$ , we can see that for the map

$$i : SO(2) \rightarrow SO(5), i(M) = \begin{bmatrix} M & 0 \\ 0 & I_3 \end{bmatrix},$$

$i_* : \pi_1(SO(2)) \rightarrow \pi_1(SO(5))$  is an epimorphism. Since  $\pi_1(SO(2)) \cong Z$  and  $\pi_1(SO(5)) \cong Z_2$ , this implies that the generator in  $\pi_1(SO(2))$  goes into a nontrivial element. But

the loop  $A$  is exactly the  $i$ -image of the generator in  $\pi_1(SO(2))$ . (This fact is also known as the "scarf trick".)  $\square$

Now we can conclude the proof of Lemma 3.20:

From Lemma 3.7,  $[A] \neq 0$  means that  $[S_i; \mathbf{w}] \neq [S_i; \mathbf{v}] = 0$ . So

$$1 = [S_i; \mathbf{w}] = [S_i; u, s_1, J_1(s_1), s_2, J_2(s_2)],$$

and from this,  $[\Sigma(f)] = k \cdot 1 = \#\{\hat{s}^{-1}(0)\} \bmod 2$ .  $\square$

We have transformed the geometric invariant  $[\Sigma(f)]$  to  $\#\{\hat{s}^{-1}(0)\} \bmod 2$ , an algebraic quantity. Note that  $\hat{s}$ , a section of the normal bundle  $\nu(\mu : \widetilde{\Delta}_2 \rightarrow M)$ , gives a perturbation of the immersion  $\mu : \widetilde{\Delta}_2 \looparrowright M$ . Therefore  $\#\{\hat{s}^{-1}(0)\} \bmod 2$  is the mod 2 number of "self-intersections" of  $\mu$ .

Using the fact that the intersections of immersed submanifolds correspond under the Poincaré duality  $D$  to the cup products of the corresponding cohomology classes, one obtains the following:

$$[\Sigma(f)] = \#\{\hat{s}^{-1}(0)\} \bmod 2 = D(\mu_*[\widetilde{\Delta}_2] \cup \mu_*[\widetilde{\Delta}_2]) \cap [M] = D(\mu_*[\widetilde{\Delta}_2])^2 \cap [M] \in H^0(M, \mathbb{Z}_2) = \mathbb{Z}_2.$$

By *Ronga's double point formula* (Theorem 2.4),

$$D(\mu_*[\widetilde{\Delta}_2]) = w_2(f^*(T\mathbb{R}^6) - TM) = w_2(\varepsilon^6 - TM) = \bar{w}_2(M).$$

Since  $M$  is oriented,  $w_2(M) = \bar{w}_2(M)$ .

$$[\Sigma(f)] = w_2^2(M) \cap [M] = (p_1(M) \bmod 2) \cap [M] = p_1[M] \bmod 2$$

Using the *Rokhlin-Hirzebruch signature formula*, which just says  $p_1(M) = 3\sigma(M)$  in dimension 4, we get

$$[\Sigma(f)] = w_2^2(M) \cap [M] = (p_1(M) \bmod 2) \cap [M] = p_1[M] \bmod 2 = \sigma(M) \bmod 2$$

and our theorem is proved.  $\square$

## 3.2 Elimination of double points in $Imm^{SO}(4, 3)$

Our Theorem 3.18 that says  $[\Sigma(f)] = \sigma(M) \bmod 2$  works only for prem maps, so we obviously need information that there are "enough" prem maps in some sense.

**Theorem 3.22.** *The natural map  $Emb^{SO}(4, 3) \rightarrow Imm^{SO}(4, 3)$  is an epimorphism.*

*Proof.* Let us take an immersion  $g : M \looparrowright \mathbb{R}^7$  representing  $[g] \in Imm^{SO}(4, 3)$ . We can assume that  $g$  is self-transverse, so it has an embedded 1-dimensional double point manifold  $\Delta_2 \hookrightarrow \mathbb{R}^7$ . The double point manifold in the source is  $\tilde{\Delta}_2 \hookrightarrow M$ . We claim that  $g : \tilde{\Delta}_2 \rightarrow \Delta_2$  is a trivial double covering.

Let  $C \cong S^1$  be a component of  $\Delta_2$  that is covered by  $\tilde{C} \hookrightarrow M$ ; fix an orientation of  $C$ . For each  $x \in C$ , the normal space  $(\nu_C)_x$  splits into two 3-spaces,  $(\nu_C)_x = (\xi_1)_x \oplus (\xi_2)_x$ , where  $g(\tilde{x}_i) = x$  and  $(\xi_i)_x = g_*((T_{\tilde{x}_i}\tilde{C})^\perp)$ . These vector spaces inherit an orientation from the orientations of  $M$  and  $C$ . (We have no global separation to  $\tilde{x}_1$  and  $\tilde{x}_2$  yet, only a pair above each point  $x \in C$ .)

In the 7-dimensional space  $T_x\mathbb{R}^7$ , we get an orientation by composing the orientations of  $(\xi_1)_x$ ,  $(\xi_2)_x$  and the fixed orientation of the circle  $C$ . Going around  $C$ , this orientation cannot change. However, if the covering  $g : \tilde{C} \rightarrow C$  was not trivial,  $(\xi_1)_x$  and  $(\xi_2)_x$  would change place and, both being odd dimensional, this would change the orientation in  $\mathbb{R}^7$ . So we proved that the covering  $g : \tilde{\Delta}_2 \rightarrow \Delta_2$  is trivial, and so we have *bundles*  $\xi_1$  and  $\xi_2$  over  $C$ .

Now we try to cast the double points away along one double curve,  $S^1 \cong C = g(\tilde{C}_1 \cup \tilde{C}_2)$ , where  $\tilde{C}_{1,2}$  are the two connected components of the pre-image of  $C$ . Let  $U_i \cong S^1 \times D^3$  be a tubular neighborhood for  $C_i$  and  $U \cong S^1 \times D^6$  for  $C$ . (The neighborhoods are indeed products, since an oriented bundle over  $S^1$  is always trivial.) Let  $\mu_i : U_i \rightarrow M$  and  $\mu : U \rightarrow \mathbb{R}^7$  be the respective embeddings. With suitable choice of the coordinates (and neighborhoods), we can assume that the maps  $\mu \circ g|_{U_i} \circ \mu_i^{-1} = g_i$  are  $g_1(\alpha, \mathbf{v}) = (\alpha, \mathbf{v}, \mathbf{0})$  and  $g_2(\alpha, \mathbf{u}) = (\alpha, \mathbf{0}, \mathbf{u})$ , where  $\mathbf{v}, \mathbf{u}, \mathbf{0} \in D^3$  and  $\alpha \in S^1$ .

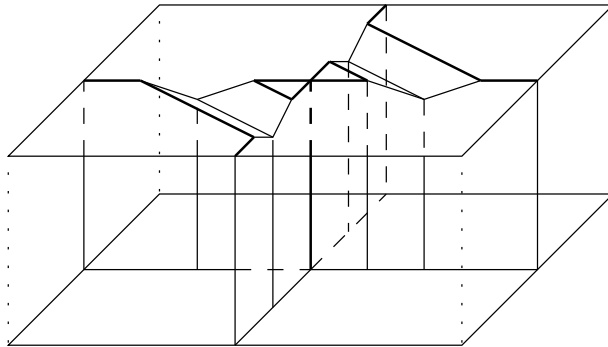


Figure 5: *Illustration of a cobordism of immersions separating a component of the double point manifold (dimension 0 in figure) from the immersed manifold.*

We now meticulously describe a cobordism  $W$  starting in  $M$  and an immersion  $h$  continuing  $g$  on it:  $h : W \rightarrow \mathbb{R}^7 \times [0, 1]$  such that  $h|_M = (g, 0)$ . However, Figure 5

should be a lot more intuitive. Unfortunately, we cannot settle with the relatively simple construction seen in the Figure, as we need  $W$  to be a smooth manifold and  $h : W \looparrowright \mathbb{R}^7 \times [0, 1]$  must be smooth on it. The last coordinate  $t = \pi_8 \circ h$  will be called *height*.

As Figure 5 shows, we want to dig a groove in the shape of  $S^3 \times [0, 1]$  between the two leaves of  $g$ . To achieve this, we start from  $g \times \text{id}_{\mathbb{R}}$  that maps from  $M \times \mathbb{R}$ , but round off the pair of vertical lines  $(\mu_1(\alpha, \mathbf{u}), t)$  and  $(\mu_2(\alpha, \mathbf{u}), t)$  (where  $\alpha \in S^1$  and  $\mathbf{u} \in D^3$  are fixed) to form a  $\cap$ -like dome, then intersect this with  $\mathbb{R}^7 \times [0, 1]$ . For vectors  $\mathbf{u}$  where the base of the dome is above 1, we get the product  $g \times \text{id}_{\mathbb{R}}$  back. For intermediate altitudes, we get a pair of lines, but they are "bended towards each other". Finally, if the top of the  $\cap$ -shape is below 1, then  $(\mu_1(\alpha, \mathbf{u}), 0)$  and  $(\mu_2(\alpha, \mathbf{u}), 0)$  will be joined by a curved line.

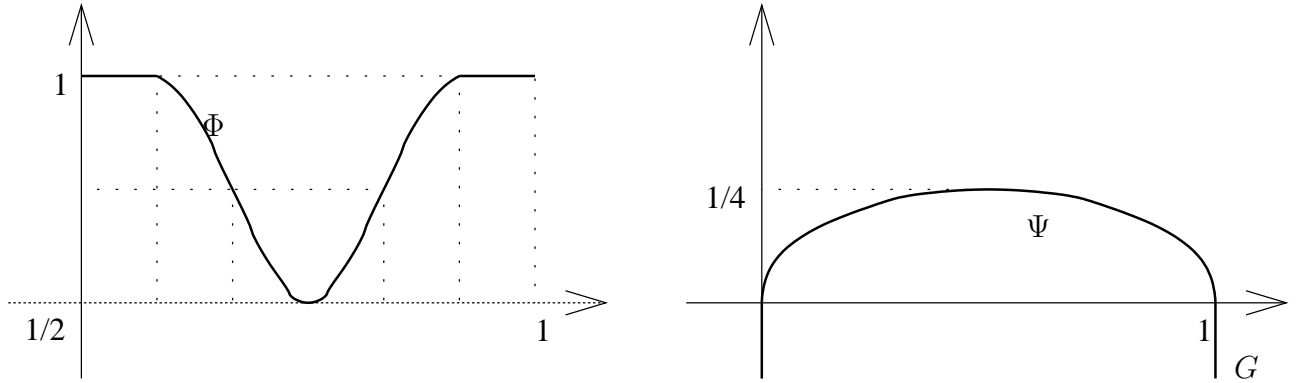


Figure 6: *The auxiliary functions  $\Phi$  and  $\Psi$  and the "dome"  $G$*

Now let us put these plans into action. Let  $\Psi : [0, 1] \rightarrow [0, 1/4]$  be a concave, continuous function that is smooth on the open interval  $(0, 1)$ , symmetric with respect to  $1/2$ , and its  $n$ -th derivative  $\Psi^n(0)$  is  $\infty$ ,  $\forall n \in \mathbb{N}$ . Further, we require that  $\Psi(1/2) = 1/4$  and  $\Psi|_{[0, 1/2]}$  be strictly increasing. Let  $\Psi^{-1} = (\Psi|_{[0, 1/2]})^{-1}$ ; this is also a smooth function. The graph of  $\Psi$ , joined by a pair of vertical semilines on both sides, will be our "dome". Let  $G = \text{graph}\Psi \cup (\{0, 1\} \times (-\infty, 0]) \hookrightarrow \mathbb{R}^2$  be the complete dome; this is a smooth curve due to the properties of  $\Psi$ .

Let  $\Phi : [0, 1] \rightarrow [1/2, 1]$  be a smooth function symmetric with respect to  $1/2$ , for which  $\Phi|_{[0, 1/6]} \equiv 1$ ,  $\Phi|_{[1/6, 1/2]}$  is strictly decreasing,  $\Phi(1/3) = 3/4$  and  $\Phi(1/2) = 1/2$ . The base of the dome over  $(\alpha, \mathbf{u})$  will be at the height  $1 - \Phi(\|\mathbf{u}\|)$ . Define the set  $W_0$  and the map  $h_0 : W_0 \rightarrow \mathbb{R}^7 \times \mathbb{R}$  as:

$$W_0 = \{(\alpha, \mathbf{u}, s, t) : \alpha \in S^1, \mathbf{u} \in D^3, (s, t - \Phi(\|\mathbf{u}\|)) \in G\} \subset S^1 \times D^3 \times \mathbb{R}^2;$$

$$h_0(\alpha, \mathbf{u}, s, t) = (\mu(\alpha, (1 - s)\mathbf{u}, s\mathbf{u}), t).$$



From the smoothness of  $\Phi$  and  $G$ ,  $W_0$  is a smooth oriented 5-manifold. Clearly,  $h_0$  is an immersion, as  $\mu$  is a diffeomorphism and  $(\alpha, \mathbf{u}, s, t) \rightarrow (\alpha, (1-s)\mathbf{u}, s\mathbf{u}, t)$  is easily of the maximal rank 5.

Let  $W = (W_0 \cap \{t \in [0, 1]\}) \cup ((M \setminus U_1 \setminus U_2) \times [0, 1])$ . There is a natural gluing here, as on  $\partial U_i$ ,  $\Phi = 1$ , so we only cut a pair of lines out of the dome.  $h = h_0|_{\{t \in [0, 1]\}}$  is also naturally extended: for  $x \notin U_i$ , let  $h(x, t) = (g(x), t)$ . This continues  $h_0|_{\{t \in [0, 1]\}}$  and  $h$  is everywhere an immersion.

Look at the two boundaries of  $W$  and  $h$ . At the level 0:

$$\partial_0 W = M, \quad h|_{\partial_0 W} = (g, 0)$$

since  $\Phi \geq 1/2$  everywhere. The upper boundary, however, decomposes into two connected components, because for  $1/3 < \|\mathbf{u}\| < 2/3$ , the whole dome is under the height 1:

$$\partial_1 W = M' \cup N,$$

where

$$M' = (M \setminus U_1 \setminus U_2) \cup \{(\alpha, \mathbf{u}, s) : \alpha \in S^1, \|\mathbf{u}\| \geq 2/3, (s, 1 - \Phi(\|\mathbf{u}\|)) \in G\},$$

or expressing  $s$  explicitly:

$$s = \Psi^{-1}(1 - \Phi(\|\mathbf{u}\|)) \text{ or } s = 1 - \Psi^{-1}(1 - \Phi(\|\mathbf{u}\|)).$$

The only essential property of  $M'$ , however, is that  $(g_1, 1) = h|_{M'}$  has lost the double curve  $C$ : indeed, for  $\mathbf{u} > 0$ ,  $h$  is injective in  $W_0 \cap \{t = 1\}$ .

If we prove that  $g_2 = \bar{\pi}_8 \circ h|_N$  is null-cobordant in immersions, we will have eliminated a double curve, and then by induction, gained an embedding. So let us examine  $N$ :

$$N = \{(\alpha, \mathbf{u}, s) : \alpha \in S^1, \|\mathbf{u}\| \leq 1/3, (s, 1 - \Phi(\|\mathbf{u}\|)) \in G\}.$$

$$N = N_1 \cup N_2 :$$

$$N_1 = \{(\alpha, \mathbf{u}, s) : \alpha \in S^1, \|\mathbf{u}\| \leq 1/3, s = \Psi^{-1}(1 - \Phi(\|\mathbf{u}\|))\}$$

$$N_2 = \{(\alpha, \mathbf{u}, s) : \alpha \in S^1, \|\mathbf{u}\| \leq 1/3, s = 1 - \Psi^{-1}(1 - \Phi(\|\mathbf{u}\|))\}$$

$$\partial N_1 = \partial N_2 = N_1 \cap N_2 = \{(\alpha, \mathbf{u}, 1/2) : \|\mathbf{u}\| = 1/3\}$$

So  $N = S^1 \times D^3 \cup S^1 \times D^3$ , glued along  $\text{id}_{\partial D^3} \times \text{id}_{S^1}$ , which means  $N = S^1 \times S^3$ .

All that remains is to prove that  $[g_2] = 0 \in \text{Imm}^{SO}(4, 3)$  (see Figure 7). For this, it suffices that  $g_2$  can be extended to an immersion  $h_2 : V = D^2 \times S^3 \looparrowright (\mathbb{R}^8)^+$  for which  $N = \partial V$ ,  $h_2|_N = (g_2, 0)$  and  $h_2(x) \in \mathbb{R}^7 \times \{0\}$  only if  $x \in N$ .

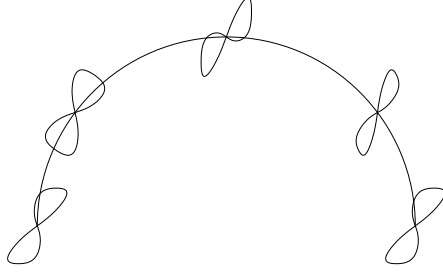


Figure 7: *Elimination of the immersion  $g_2$ .*

Let  $i : D^2 \rightarrow (\mathbb{R}^8)^+$  be a null-cobordism of  $\partial D^2 = S^1 = C$  that starts vertically: in each point  $x \in X$ , for some normal vector  $u \in \nu(S^1 \hookrightarrow D^2)|_x$ , let  $\partial_u(i) = \mathbf{e}_8$ . The tangent space  $TC$  is globally split into a pair of trivialized 3-bundles  $\xi_1$  and  $\xi_2$ . Therefore,  $\xi_1$  and  $\xi_2$  can be extended to be a splitting of the normal bundle of  $i$  into a pair of trivial bundles, giving  $\nu(i) = \xi_1 \oplus \xi_2$ .

Take an embedding  $\tau : D(\nu(i)) = D(\xi_1 \oplus \xi_2) = D^2 \times D^3 \times D^3 \rightarrow (\mathbb{R}^8)^+$  to a tubular neighborhood, so that  $\tau$  extends  $\mu : S^1 \times D^6 = D(\nu(C)) \rightarrow \mathbb{R}^7$ . The following map will do:

$$h_2(x, r) = \tau(x, (1 - s)\mathbf{u}, s\mathbf{u}),$$

where  $r = (\mathbf{u}, s)$ ,  $r \in \{(\mathbf{u}, s) : (s, 1 - \Phi(\|\mathbf{u}\|)) \in G\} = S^3$ .

This  $h_2$  is clearly an immersion and is consistent with the definition of  $g_2$ , so we proved  $[g] = [g_1] \in \text{Imm}^{SO}(4, 3)$ . Iterating this elimination procedure, we finally get  $[g] = [g_K]$ , where  $g_K \in \text{Emb}^{SO}(4, 3)$ . □

**Corollary 3.23.** *Theorem 3.18 can be extended to  $\text{Imm}^{SO}(4, 3)$ :*

$$[\Sigma(f)] = \sigma(M) \bmod 2 \text{ for generic prim maps } f = \bar{\pi}_7 \circ g \text{ (where } g : M \looparrowright \mathbb{R}^7 \text{)}.$$

*Proof.* Let us consider the homeomorphism  $\varphi : \text{Imm}^{SO}(4, 3) \rightarrow Z_2$ , where  $\varphi(f) = [\Sigma(f)] - \sigma(M) \bmod 2$ , and  $\iota : \text{Emb}^{SO}(4, 3) \rightarrow \text{Imm}^{SO}(4, 3)$  is the natural (forgetting) map.

By Theorem 3.18,  $\varphi \circ \iota = 0$ . By Theorem 3.22,  $\iota$  is an epimorphism. So  $\varphi = 0$ . □

### 3.3 Elimination of singularities using classifying spaces

Our next goal is to prove that  $[\Sigma(f)]$  is the only obstruction to the elimination of singularities by prim cobordism. This means that for any prim map  $f$ , the condition  $[\Sigma(f)] = 0$  implies that  $f$  is prim-cobordant to an immersion. To achieve this, we shall use the tool of *universal singular maps*. The following concepts and results are taken from §5 of [Sz].

The word *(mono)singularity* will stand for an  $\mathcal{A}$ -equivalence class of map germs, as in Definition 1.8. We will be considering *stable* singularities (stable in the homotopy theoretical sense – this is a concept different from the one described by Definition 1.9!), which simply means that we identify a germ  $\varphi : (\mathbb{R}^m, 0) \rightarrow (\mathbb{R}^n, 0)$  with its suspension  $\varphi \times \text{id}_{\mathbb{R}^1} : (\mathbb{R}^{m+1}, 0) \rightarrow (\mathbb{R}^{n+1}, 0)$ . A *multisingularity* means a finite multiset of (stable) singularities.

**Definition 3.24.** *For a set  $\tau$  of multisingularities, a map  $f : M^m \rightarrow P^{m+k}$  is called a  $\tau$ -map if  $\forall y \in P$ , the germ of  $f$  at  $f^{-1}(y)$  (which is a discrete set if  $f$  is generic) is from  $\tau$ .*

This way, we can describe global restrictions as well: for instance, if  $\tau$  has only singleton elements, then  $f$  must be a (topological) embedding.

**Definition 3.25.** *The cobordism group  $\text{Cob}_\tau(P^{m+k})$  consists of the classes of  $\tau$ -maps  $f : M^m \rightarrow P^{m+k}$ , where the equivalence is given by  $\tau$ -cobordisms  $g : W^{m+1} \rightarrow P^{m+k} \times [0, 1]$ .*

There is a natural partial ordering on the set of multisingularities  $\eta$ : we define  $\eta \geq \eta_0$  if for each  $y \in P$  for which the germ of  $f$  in  $f^{-1}(y)$  is  $\eta$ , there must be a point  $y_0 \in U_y$  in any neighborhood  $y \in U_y \subset P$  that gives  $\eta_0$ .

In [RSz], the authors constructed classifying spaces  $X_\tau$  that give

$$\text{Cob}_\tau(P^{m+k}) = [\dot{P}, X_\tau]$$

where  $\dot{P}$  is the one-point compactification of the manifold  $P$ .

Let  $\tau$  be a set of multisingularities and let  $\eta$  the top multisingularity in  $\tau$  ( $\eta \geq \eta_0, \forall \eta_0 \in \tau$ ). Let  $\tau' = \tau \setminus \{\eta\}$ . Then the space  $X_\tau$  can be obtained by gluing to  $X_{\tau'}$  the disc bundle  $D(\xi)$  of a vector bundle  $\xi$ , along the boundary  $S(\xi)$ . Thus, the pair  $(X_\tau, X_{\tau'})$  gives a cofibration

$$X_{\tau'} \subset X_\tau \rightarrow T\xi.$$

When the multisingularity  $\eta$  is a singleton (containing a single stable monosingularity),  $\eta = \{\varphi\}$ , then  $\xi$  here can be described as follows:

The monosingularity  $\varphi$  is represented by a map  $(\mathbb{R}^c, 0) \rightarrow (\mathbb{R}^{c+k}, 0)$ , where we take  $c$  to be minimal (in other words, 0 is an isolated  $\varphi$ -type singularity). Let  $G$  be a maximal compact subgroup of the automorphism group of  $\varphi$ .  $G$  can be conjugated by a left-right diffeomorphic action to a subgroup  $G \leq O(\mathbb{R}^c) \times O(\mathbb{R}^{c+k})$ . In the image, we get a faithful representation  $\rho : G \rightarrow O(c+k)$ . The bundle  $\xi$  is the universal bundle for this representation  $\rho$ , that is,  $\xi = EG \times_\rho \mathbb{R}^{c+k}$ , where  $EG \rightarrow BG$  is the universal principal  $G$ -bundle.

The natural map  $\text{Cob}_{\tau'}(P) \rightarrow \text{Cob}_{\tau}(P)$  can be identified with the map  $[\dot{P}, X_{\tau'}] \rightarrow [\dot{P}, X_{\tau}]$ , induced by the inclusion  $X_{\tau'} \subset X_{\tau}$ . The elimination of  $\eta$ -points in a  $\tau$ -map  $f : M^m \rightarrow P^{m+k}$  by  $\tau$ -cobordism is simply the question whether  $[f]$  is in the image.

**Definition 3.26.** A set  $\tau$  of multisingularities is complete if it only gives local restrictions; that is, if there is a set  $\Phi$  of monosingularities such that  $\tau$  consists of the multisingularities that can be composed of elements of  $\Phi$ .

**Theorem 3.27** (Szűcs). If  $\tau'$  is complete, then the sequence

$$[P, X_{\tau'}] \rightarrow [P, X_{\tau}] \xrightarrow{\alpha} \{P, T\xi\}$$

is exact, where  $\{Y, Z\}$  is the set of stable homotopy classes of maps  $Y \rightarrow Z$ :

$$\{Y, Z\} = \lim_{q \rightarrow \infty} [S^q Y, S^q Z].$$

The map  $[\dot{P}, X_{\tau'}] \rightarrow \{P, T\xi\}$  can be understood through the *universal  $\eta$ -map*. In the case when  $\eta = \{\varphi\}$  consists of a single monosingularity, this is a fiber-preserving map over  $BG$  that goes to  $\xi$  from the bundle  $\tilde{\xi} = EG \times_{\chi} \mathbb{R}^c$ , where  $\chi : G \rightarrow O(c)$  is the action of  $G$  in the source. The universal  $\eta$ -map is:

$$\begin{array}{ccc} \tilde{\xi} & \xrightarrow{(\text{id}_{EG} \times \eta)/G} & \xi \\ \downarrow & & \downarrow \\ BG & \xrightarrow{=} & BG \end{array}$$

Denote by  $\tilde{\eta}(f)$  the  $\eta$ -points in the source  $M$  and by  $\eta(f)$  the  $\eta$ -points in the target  $P$ . As  $\eta$  is the top singularity,  $\tilde{\eta}(f)$  and  $\eta(f)$  are closed submanifolds. Take a tubular neighborhood  $\tilde{U}$  over  $\tilde{\eta}$  and  $U$  over  $\eta$ , seen as neighborhoods of the 0 sections of the normal bundles  $\nu(\tilde{\eta} \hookrightarrow M)$  and  $\nu(\eta \hookrightarrow P)$ , respectively. Then the universal property of the map  $\tilde{\xi} \rightarrow \xi$  is expressed by the following commutative diagram:

$$\begin{array}{ccc} \tilde{\xi} & \xrightarrow{\quad} & \xi \\ \downarrow & \swarrow \quad \nearrow \Theta & \downarrow \\ \tilde{U} & \xrightarrow{f|_{\tilde{U}}} & U \\ \downarrow & & \downarrow \\ \tilde{\eta}(f) & \xrightarrow{f|_{\tilde{\eta}}} & \eta(f) \\ \downarrow & \swarrow \quad \searrow \vartheta & \downarrow \\ BG & \xrightarrow{=} & BG \end{array}$$

where the homotopy class  $[\vartheta] \in [\eta(f), BG]$  is uniquely determined.

From  $\Theta : U \rightarrow \xi$ , we get a map  $\theta : \dot{P} \rightarrow T\xi$  as usual in Thom-type constructions; the class  $[\theta]^s \in \{\dot{P}, T\xi\}$  is the image of  $[f] \in \text{Cob}_\tau(P) = [\dot{P}, X_\tau]$  in  $\{\dot{P}, T\xi\}$  under  $\alpha$ .

We now need a slightly modified version of this theory. Let  $\text{PrimCob}_\tau^{SO}(P^{m+k})$  be the cobordism group of prim  $\tau$ -maps with an oriented normal bundle, that is, the set of maps  $g : M^m \looparrowright P^{m+k} \times \mathbb{R}$ , with a fixed orientation ( $SO$ -structure) on  $\nu_g$ , for which the projection  $f = \pi_1 \circ g$  is a  $\tau$ -map – factorized by prim  $\tau$ -cobordisms, with orientation on the normal bundle, from manifolds with boundary. If the target manifold is  $P = \mathbb{R}^6$ , the orientation requirement only means an orientation of  $M$ .

In this case too, we have classifying spaces  $X_\tau$  (different from the  $X_\tau$  in the unoriented theory above, of course), for which

$$[\dot{P}, X_\tau] = \text{PrimCob}_\tau^{SO}(P).$$

If  $\tau \setminus \{\eta\} = \tau'$ , where  $\eta$  is the top multisingularity, there is again a cofibration

$$X_{\tau'} \subset X_\tau \rightarrow T\xi,$$

where  $\xi$  now comes only from the automorphism group of prim representatives of  $\eta$ . Theorem 3.27 holds in this case as well. The proof of these results can be found in [Sz2].

Let us apply the theory to our case!  $\tau$  should be essentially the set of multisingularities that are generated by the  $\Sigma^{1,0}$  singularity  $wh_2 : \mathbb{R}^3 \rightarrow \mathbb{R}^5$  and the trivial (non-)singularity. In our dimensions  $m = 4$ ,  $k = 2$ , the only multisingularities that actually occur for generic maps (and cobordisms) are the singleton  $\eta = \{\varphi\}$ , where  $\varphi$  is the stable class of  $wh_2$ , and 1, 2 and 3 instances of the trivial (non-singular) stable class. Let  $\tau$  consist of  $\eta$  and the regular multisingularities,  $l$  copies of the non-singular stable class ( $l = 1, 2, \dots$ ).

Let  $\tau' = \tau \setminus \{\eta\}$ ;  $\tau'$  is a complete set of multisingularities.  $\tau'$ -maps are exactly the immersions, while all generic maps in dimensions (4, 6) and (5, 7) are  $\tau$ -maps by Corollary 1.13. This means that all prim maps are  $\tau$ -maps in these dimensions, so  $\text{PrimCob}_\tau^{SO}(S^6) = \text{Imm}^{SO}(4, 3)$ . So the following sequence is exact:

$$\begin{aligned} [P, X_{\tau'}] &\rightarrow [P, X_\tau] \rightarrow \{P, T\xi\} \\ \text{Imm}^{SO}(4, 2) &\rightarrow \text{Imm}^{SO}(4, 3) \rightarrow \{S^6 \rightarrow T\xi\} = [S^6, T\xi], \end{aligned}$$

where the last equality comes from the generalized Freudenthal theorem:  $T\xi$  is 4-connected, being the Thom space of a 5-bundle.

What is  $\xi$  in our case? Let  $G \leq O(3) \times O(5)$  be a maximal compact subgroup of the automorphism group of prim map germs  $\eta$ . As we have already established,  $wh_2$  induces a splitting of the target space  $\mathbb{R}^5$  to a pair of oriented 2-planes with a canonical isomorphism, and a trivial line bundle. A non-prim germ does not differentiate between the two 2-planes. However, if we only allow automorphisms that respect the prim structure, we have an ordering between the two. So the image  $\rho(G)$  is:

$$\rho(G) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{bmatrix} : A \in SO(2) \right\}.$$

$\rho$  is a faithful representation, so the universal space we get is  $BG = BSO_2 = BU_1 = \mathbb{C}P^\infty$ . Thus, the universal bundle is:

$$\xi = EG \times_\rho \mathbb{R}^5 = \varepsilon^1 \oplus 2\gamma_{\mathbb{C}}^1,$$

where  $\gamma_{\mathbb{C}}^1 = \gamma^1$  is the tautological  $\mathbb{C}$ -bundle over  $BU_1 = \mathbb{C}P^\infty$ . (From now on, we will only consider the tautological bundles in the complex case, so we leave  $\mathbb{C}$  from the notation.)

Now let us examine the following diagram:

$$\begin{array}{ccc} Emb^{SO}(4, 3) & \xrightarrow{[\Sigma]} & \pi_6(S^5) = Z_2 \\ \downarrow i & & \downarrow j_* \\ Imm^{SO}(4, 2) \longrightarrow Imm^{SO}(4, 3) & \xrightarrow{[\theta]} & \pi_6(T\xi) \end{array}$$

where

$$j : \begin{array}{ccc} \mathbb{R}^5 & \longrightarrow & \xi \\ \downarrow & & \downarrow \\ \{\text{point}\} & \longrightarrow & \mathbb{C}P^\infty \end{array}$$

To progress, we need to prove that the diagram is commutative:  $j_* \circ [\Sigma] = [\theta] \circ i$ . This is slightly complicated by the fact that we defined  $[\Sigma]$  through a framing on the curve  $S$ , rather than the singular curve  $f(\Sigma(f))$  itself. So we first need

**Lemma 3.28.** *Let  $f : M \rightarrow \mathbb{R}^6$  be a generic prim map and let  $\Sigma$  denote its singularity curve in the source. Then, there are diffeomorphisms  $\chi : N(\Sigma) \rightarrow D^3 \times \Sigma$  and  $\varphi : N(f(\Sigma)) \rightarrow D^5 \times f(\Sigma)$  (where  $N(\Sigma)$  and  $N(f(\Sigma))$  are tubular neighborhoods for the singular stratum in the source and the image respectively), for which  $\chi(\tilde{p}) = (0, \tilde{p})$  ( $\tilde{p} \in \Sigma$ ) and  $\varphi(p) = (0, p)$  ( $p \in f(\Sigma)$ ), and  $f$  has the standard form*

$$\varphi \circ f \circ \chi^{-1} = wh_2 \times f|_{\Sigma}.$$

Moreover, let us denote the pull-back of the coordinate vectors by  $(\tilde{u}_i)_{\tilde{p}} = \chi|_{\tilde{p}}^*(\mathbf{e}_i)$  ( $1 \leq i \leq 3$ ) and  $(u_i)_p = \varphi|_p^*(\mathbf{e}_i)$  ( $1 \leq i \leq 5$ ), respectively. Then, the framing  $[f(\Sigma); u_1, \dots, u_5]$  can be used to define  $[\Sigma(f)]$ ; that is,  $[f(\Sigma); \mathbf{u}] = [S; u, s_1, J_1(s_1), s_2, J_2(s_2)] = [\Sigma(f)] \in Emb^{\text{fr}}(1, 5)$ .

*Proof.* Recall the diagram that expresses the universal property of the map  $\tilde{\xi} \rightarrow \xi$ . In our case,  $\tilde{\eta}(f) = \Sigma$  and  $\eta(f) = f(\Sigma)$  are curves. Let us examine the map  $\vartheta : (\eta(f) = \Sigma) \rightarrow (BG = \mathbb{C}P^\infty)$ . Since  $\Sigma$  is a 1-manifold and  $sk_1(\mathbb{C}P^\infty) = \{*\}$ , we get that  $[\vartheta] = 0$ . Homotopic maps between base spaces define isomorphic pull-back bundles, so  $U \hookrightarrow \vartheta^*\xi$  is a trivial 5-bundle over  $f(\Sigma)$  and  $\tilde{U} \hookrightarrow \vartheta^*\tilde{\xi}$  is a trivial 3-bundle over  $\Sigma$ ; the map between them is  $wh_2$  in each fiber.

Since  $\Sigma$  is compact, there is a number  $\varepsilon > 0$  such that each fiber of  $U$  and  $\tilde{U}$  contains the ball  $D_\varepsilon$ . By linear rescaling, we can take  $\varepsilon$  to be 1.

We have the freedom to choose the tubular neighborhood  $U(\Sigma) \subset \widetilde{\Delta}_2$  in the definition of  $[\Sigma(f)] = [S; u, s_1, J_1(s_1), s_2, J_2(s_2)]$ . Let us choose  $U(\Sigma) = \{(\tau, 0, 0, \tilde{p}) : \tau \leq \delta, \tilde{p} \in \Sigma\} \subset N(\Sigma)$  for an arbitrary  $0 < \delta < 1$ . This gives  $S = \{(\delta^2, 0, 0, 0, p) : p \in \Sigma\}$ . (See again Figure 3.)

The normal vector of  $S$  in  $\psi(\Delta_2)$ , the vector  $u$ , can be chosen to be  $u = \varphi|_p^*(1, 0, 0, 0, 0) = u_1$  in  $q = (\delta^2, 0, 0, 0, p) = S \cap \varphi^{-1}(D^5 \times \{p\})$ .

The tangent line of the singularity curve  $f(\Sigma)$  is tangent to the double point manifold, too:  $T(f(\Sigma)) \leq \psi_*(T\widetilde{\Delta}_2)$ . Thus  $(\xi_1)_q$  and  $(\xi_2)_q$ , the normal subspaces of  $T_q\Delta_2$  in the leaves of  $f$ , can be chosen to be in the fiber; in the local coordinates given by  $\varphi|_p$ :

$$(\varphi|_p)_*(\xi_1)_q = \{(0, x_1, x_2, \delta x_1, \delta x_2, p) : x_1, x_2 \in \mathbb{R}\},$$

$$(\varphi|_p)_*(\xi_2)_q = \{(0, x_1, x_2, -\delta x_1, -\delta x_2, p) : x_1, x_2 \in \mathbb{R}\}.$$

Let us introduce the notation

$$\mathbf{w} = (w_1, \dots, w_5) : w_1 = u_1, w_2 = u_2 + \delta u_4, w_3 = u_3 + \delta u_5, w_4 = u_2 - \delta u_4, w_5 = u_3 - \delta u_5.$$

Let  $(0, c_1(p), c_2(p)) = (\chi|_{\tilde{p}})_*(\hat{s}(\tilde{q}_1))$  and  $(0, d_1(p), d_2(p)) = (\chi|_{\tilde{p}})_*(\hat{s}(\tilde{q}_2))$ . Then the framing  $[u, s_1, J_1(s_1), s_2, J_2(s_2)]$  in  $q$  can be written as  $B(p) \cdot \mathbf{w}$ , where  $B \in GL^+(5)$  is the following:

$$B(p) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C(p) & 0 \\ 0 & 0 & D(p) \end{bmatrix}, \quad C(p) = \begin{bmatrix} c_1(p) & c_2(p) \\ -c_2(p) & c_1(p) \end{bmatrix}, \quad D(p) = \begin{bmatrix} d_1(p) & d_2(p) \\ -d_2(p) & d_1(p) \end{bmatrix}.$$

$\mathbf{w} = A\mathbf{u} = A(u_1, \dots, u_5)$ , where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & \delta & 0 & -\delta & 0 \\ 0 & 0 & \delta & 0 & -\delta \end{bmatrix}$$

is a constant positive linear transformation,  $A \in GL^+(5)$ .

So the framing  $[u, s_1, J_1(s_1), s_2, J_2(s_2)]$  on  $S$  (that defines  $[\Sigma(f)]$ ) can be written as  $B(p) \cdot A \cdot \mathbf{u}(p)$ .  $\hat{s}$  gives a homotopy between  $C(p)$  and  $D(p)$ ; hence the loop  $[B(p)]$  is homotopically trivial by the same argument that we saw in Sublemma 3.13, so the loop  $[B(p) \cdot A]$  is homotopically trivial, too.

By Lemma 3.7, we can then change the framing  $[u, s_1, J_1(s_1), s_2, J_2(s_2)]$  to  $[u_1, u_2, u_3, u_4, u_5]$ , and also replace  $S$  by  $\Sigma$  (which is just a translation by  $-\delta u_1$  in these coordinates, which clearly does not change the framed cobordism class). □

Now we can prove

**Lemma 3.29.** *The diagram that we were looking at is commutative:  $j_* \circ [\Sigma] = [\theta] \circ i$ .*

*Proof.* This statement comes immediately from the previous lemma. Indeed,  $[\Sigma]$  represents a trivialization of  $\nu(f(\Sigma) \hookrightarrow \mathbb{R}^6)$  so that  $f$  is consistent with the trivialization in the sense described in Lemma 3.28.  $j$  is the natural morphism between the Thom spaces of the stronger structure  $\{I_5\} \subset O(5)$  (a framing) and the weaker structure  $\rho : G \rightarrow O(5)$ .

So  $j_*$  partially forgets the trivialization (framing) to get a  $\rho$ -structure of  $\nu(f(\Sigma) \hookrightarrow \mathbb{R}^6)$  that is consistent with  $f$ . But the homotopically unique  $[\theta]$  does just the same, so  $[\theta] = [\theta] \circ i$  must be equal to  $j_* \circ [\Sigma]$ . □

Now, if we show that  $j_* : \pi_6(S^5) \rightarrow \pi_6(T\xi)$  is an isomorphism, then  $[\theta] = [\Sigma]$  will follow, and the exact sequence of the lower row of our commutative diagram will prove that  $[\Sigma]$  is the only obstruction to the elimination of singularities by prim cobordism.

**Theorem 3.30.**  *$j_* : \pi_6(S^5) \rightarrow \pi_6(T\xi)$  is an isomorphism.*

*Proof.* First, note that  $j : T\varepsilon^5 = ST(\varepsilon^4) \rightarrow T\xi = ST(2\gamma^1)$  is a suspension:  $j = S\iota$ , where  $\iota : S^4 = T\varepsilon^4 \rightarrow T(2\gamma^1)$ . So by the generalized Freudenthal theorem, it suffices to show that  $\iota_* : \pi_5(S^4) \rightarrow \pi_5(T(2\gamma^1))$  is an isomorphism.



**Lemma 3.31.**

$$T(m\gamma^1) = \mathbb{C}P^\infty / \mathbb{C}P^{m-1},$$

and the fiber over a point  $*$   $\in \mathbb{C}P^\infty$  corresponds to  $\mathbb{C}P^m / \mathbb{C}P^{m-1}$ . Here,  $\mathbb{C}P^l = P(\mathbb{C}^{l+1}) = P(\{z_1, \dots, z_{l+1}, 0, 0, \dots\})$  ( $l \geq 0$ ), where  $P$  is the projectivization.

*Proof.* Let  $D(\mathbb{C}^l) = \{\mathbf{z} \in \mathbb{C}^l : \|\mathbf{z}\| \leq 1\}$ ;  $S(\mathbb{C}^l) = \partial D(\mathbb{C}^l)$ . Let  $N \geq 0$ . Look at the map  $f : S(\mathbb{C}^{N+1}) \times D(\mathbb{C}^m) \rightarrow S(\mathbb{C}^{N+m+1})$  that is given by

$$f(\mathbf{z}, \mathbf{w}) = (\sqrt{1 - \|\mathbf{w}\|^2} \cdot \mathbf{z}, \mathbf{w}).$$

Its restriction

$$f|_{S(\mathbb{C}^{N+1}) \times \text{int}(D(\mathbb{C}^m))} : S(\mathbb{C}^{N+1}) \times \text{int}(D(\mathbb{C}^m)) \rightarrow S(\mathbb{C}^{N+m+1}) \setminus S(\mathbb{C}^m)$$

is one-to-one. Thus we get a homeomorphism

$$\tilde{f} : \frac{S(\mathbb{C}^{N+1}) \times D(\mathbb{C}^m)}{S(\mathbb{C}^{N+1}) \times S(\mathbb{C}^m)} \rightarrow \frac{S(\mathbb{C}^{N+m+1})}{S(\mathbb{C}^m)}.$$

$\tilde{f}$  is equivariant by the  $S^1 = U_1$ -action  $\omega : (\mathbf{z} \rightarrow \omega\mathbf{z})$ . Factoring out by this action, we get:

$$T(m\gamma_N^1) = \frac{D(m\gamma_N^1)}{S(m\gamma_N^1)} = \mathbb{C}P^{N+m} / \mathbb{C}P^{m-1},$$

where  $\gamma_N^1$  is the tautological 1-bundle over  $\mathbb{C}P^N$ .

The fiber over the point  $*$   $= \mathbb{C}P^0 = P(S(\mathbb{C}^1))$  goes to  $\mathbb{C}P^m / \mathbb{C}P^{m-1}$ . Taking the direct limit of these homeomorphisms, we get  $T(m\gamma^1) = \mathbb{C}P^\infty / \mathbb{C}P^{m-1}$ .  $\square$

So the map  $\iota : S^4 \subset T2\gamma^1$  coincides with the inclusion  $S^4 = \mathbb{C}P^2 / \mathbb{C}P^1 \subset \mathbb{C}P^\infty / \mathbb{C}P^1$ . By homotopical excision, the inclusion  $\mathbb{C}P^3 / \mathbb{C}P^1 = sk_7(\mathbb{C}P^\infty / \mathbb{C}P^1) \subset \mathbb{C}P^\infty / \mathbb{C}P^1$  induces an isomorphism in  $\pi_5$ , so we only need to prove that  $\iota : S^4 \subset Y = \mathbb{C}P^3 / \mathbb{C}P^1$  induces an isomorphism in  $\pi_5$ .

We may think of the space  $Y = \mathbb{C}P^3 / \mathbb{C}P^1$  as a disc  $D^6$  glued to a sphere  $S^4$  by a gluing map  $\varphi : \partial D^6 = S^5 \rightarrow S^4$ . If  $\varphi \cong 0$ , then  $\iota_* : \pi_5(S^4) \rightarrow \pi_5(Y)$  is an isomorphism, whereas if  $[\varphi] \neq 0 \in \pi_5(S^4)$ , then the group  $\pi_5(Y)$  is trivial.

Now let us compare  $\varphi$  with another gluing map  $S^2h : S^5 \rightarrow S^4$ , where  $h : S^3 \rightarrow S^2$  is the Hopf fibration. Let  $X = D^6 \cup_{S^2h} S^4 = S^2\mathbb{C}P^2$ . If  $S^2h \cong \varphi \in [S^5, S^4]$ , then the spaces  $X$  and  $Y$  are homotopy equivalent (as  $Y = \text{Cyl}(\varphi) \cup_{S^5} D_6$  and  $X = \text{Cyl}(S^2h) \cup_{S^5} D_6$ ).

If we could tell  $X$  and  $Y$  apart homotopically, that would mean  $[\varphi] \neq [S^2h] \in \pi_5(S^4) \cong \mathbb{Z}_2$ ; this together with  $[S^2h] \neq 0$  would give  $[\varphi] = 0$  and conclude our

theorem. Unfortunately, the homology and cohomology of the two spaces is the same, as both are CW complexes of only 3 cells, one in dimensions 0, 4 and 6 each. But using the concept of *Steenrod squares*, we can indeed tell the difference:

In  $H^4(X; Z_2) = H^4(S^2\mathbb{C}P^2; Z_2) \cong Z_2$ , the generator is  $S^2x$ , where  $x$  is the generator in  $H^2(\mathbb{C}P^2, Z_2)$ . The mod 2 Poincaré dual of  $x$  is  $[\mathbb{C}P^1] \hookrightarrow \mathbb{C}P^2$ , which has a self-intersection  $[\mathbb{C}P^0]$ , a single point; therefore,  $x^2 = D(D(x) \cap D(x)) = D(\{*\}) \neq 0 \in H^4(\mathbb{C}P^2, Z_2)$ . As the Steenrod square is stable, this gives  $Sq^2(S^2x) = S^2(Sq^2(x)) = S^2x^2 \neq 0 \in H^6(X; Z_2)$ .

This also implies that  $[S^2h] \neq 0 \in [S^5, S^4]$ , because if it were null-homotopic,  $X \cong X' = S^4 \vee S^6$  would follow. But for  $p : X' \rightarrow X'/S^6 = S^4$  and  $y \in H^4(X'; Z_2) = p^*H^4(S^4; Z_2)$ , we have  $Sq^2y \in p^*H^6(S^4; Z_2) = 0$ , from the naturality of the Steenrod square.

In  $Y$ , however,  $Sq^2$  sends the generator  $y \in H^4(Y; Z_2)$  to 0. To prove this, let us first look at  $\mathbb{C}P^3$ ; let  $z$  be the generator  $z \in H^4(\mathbb{C}P^3; Z_2)$ . Then,  $z = u^2$ , where  $u \in H^2(\mathbb{C}P^3; Z_2)$ , again trivially by mod 2 Poincaré duality. By Cartan's relations, this gives

$$Sq^2z = Sq^2(u \cup u) = Sq^2u \cup u + Sq^1u \cup Sq^1u + u \cup Sq^2u = 2(Sq^2u \cup u) + 0 \cup 0 = 0.$$

From the exact cohomology sequence of the pair  $(\mathbb{C}P^3, \mathbb{C}P^1)$ , we get that  $H^4(Y; Z_2) = H^4(\mathbb{C}P^3; Z_2)$  (where  $y$  goes to  $z$ ), and  $H^6(Y; Z_2) = H^6(\mathbb{C}P^3; Z_2)$ . By the naturality of Steenrod squares, the following diagram is commutative:

$$\begin{array}{ccc} H^4(Y; Z_2) & \xrightarrow[\approx]{} & H^4(\mathbb{C}P^3; Z_2) \\ Sq^2 \downarrow & & \downarrow Sq^2 \\ H^6(Y; Z_2) & \xrightarrow[\approx]{} & H^6(\mathbb{C}P^3; Z_2) \end{array} \quad \begin{array}{c} y \\ z \\ Sq^2y=0 \end{array}$$

This shows that  $Sq^2y = 0$ , so indeed,  $X$  and  $Y$  are homotopically different. Thus  $[\varphi] = 0$ , which, in turn, gives our theorem (that  $j_* : \pi_6(S^5) \rightarrow \pi_6(T\xi)$  is an isomorphism).  $\square$

So we have essentially proved

**Theorem 3.32.** *The sequence*

$$Imm^{SO}(4, 2) \xrightarrow{\iota} Imm^{SO}(4, 3) \xrightarrow{[\Sigma]} Z_2 \longrightarrow 0$$

*is exact.*

*Proof.* Trivially,  $[\Sigma(\iota(f))] = 0$ : if there is no singular point at all, the invariant we defined using  $\Sigma \hookrightarrow \widetilde{\Delta}_2$  is zero. By *Whitney's strong immersion theorem*, for all classes  $[M] \in \Omega_4$  (even for all  $M$ ), there is an immersion  $g : M \looparrowright \mathbb{R}^7$ , but by Corollary 3.23 we know that  $[\Sigma(\bar{\pi}_7 \circ g)] = \sigma(M) \bmod 2$ , so  $[\Sigma]$  is an epimorphism to  $Z_2$ .

The only non-trivial part is the one that we have just proved: in the commutative diagram

$$\begin{array}{ccc} Emb^{SO}(4, 3) & \xrightarrow{[\Sigma]} & \pi_6(S^5) = Z_2 \\ \downarrow i & & \downarrow j_* \\ Imm^{SO}(4, 2) & \rightarrow Imm^{SO}(4, 3) \xrightarrow{[\theta]} & \pi_6(T\xi) \end{array}$$

the lower row is exact, and also  $[\Sigma] = [\theta]$  if we identify  $\pi_6(S^5) \cong \pi_6(T\xi)$  by the isomorphism  $j_*$ .  $\square$

Now we can finally prove our main result:

**Theorem 3.33.** *The sequence*

$$Imm^{SO}(4, 2) \rightarrow \Omega_4 \xrightarrow{\sigma \bmod 2} Z_2 \rightarrow 0$$

*is exact; that is, a cobordism class  $[M] \in \Omega_4$  has a representative  $M' \sim M$  that can be immersed in  $\mathbb{R}^6$  if and only if  $\sigma(M)$  is even.*

*Proof.* The exactness in  $Z_2$  follows from the already mentioned fact that  $\sigma : \Omega_4 \rightarrow Z$  is an isomorphism. As for the exactness in  $\Omega_4$ :

$$\begin{array}{ccccccc} Imm^{SO}(4, 2) & \xrightarrow{\iota} & Imm^{SO}(4, 3) & \xrightarrow{[\Sigma]} & Z_2 & \longrightarrow & 0 \\ & \searrow \alpha & \downarrow \beta & \nearrow \sigma \bmod 2 & & & \\ & & \Omega_4 & & & & \end{array}$$

By Corollary 3.23, the diagram above is commutative. By Whitney's strong immersion theorem, the map  $\beta : Imm^{SO}(4, 3) \rightarrow \Omega_4$  is an epimorphism. Finally, by Theorem 3.32, the upper row is exact. We need to check  $(\sigma \bmod 2) \circ \alpha = 0$ :

$$(\sigma \bmod 2) \circ \alpha = (\sigma \bmod 2) \circ \beta \circ \iota = [\Sigma] \circ \iota = 0$$

by the exactness of the upper row. Conversely,  $\ker(\sigma \bmod 2) \subset \text{im } \alpha$ : let

$$[M] \in \ker(\sigma \bmod 2)$$

We know that  $\beta$  is an epimorphism:

$$\exists [g] \in Imm^{SO}(4, 3) : \beta([g]) = [M]$$

$$[\Sigma]([g]) = (\sigma \bmod 2)(\beta([g])) = (\sigma \bmod 2)([M]) = 0$$

By exactness of the upper row,

$$\exists [f] \in Imm^{SO}(4, 3) : \iota([f]) = [g]$$

$$\alpha([f]) = \beta([g]) = [M]$$

Thus we proved that the sequence is indeed exact.  $\square$

### 3.4 Elimination of singularities by a geometric constuction

We now present an alternative proof for Theorem 3.32 that circumvents the use of classifying spaces, at the price of using a complicated geometric construction.

The non-trivial part is to prove that if  $f = \bar{\pi}_7 \circ g$ , where  $g : M \looparrowright \mathbb{R}^7$  is an immersion, and  $[\Sigma(f)] = 0$ , then  $\exists g' : M' \looparrowright \mathbb{R}^7$  such that  $\bar{\pi}_7 \circ g'$  is non-singular and  $[g] = [g'] \in Imm^{SO}(4, 3)$  are cobordant via immersions. The idea is to perform a surgery to eliminate a tubular neighborhood of the singularity curves.

We still have rely on Lemma 3.28, which may be proved in an elementary way by collating the standard neighborhoods along  $\Sigma$  carefully.

$f(\Sigma)$  is a closed 1-manifold, so it is a collection of embedded circles. In dimension 6, 1-manifolds do not have any linking invariants, so we can move them into any position we like by isotopy. Now, since  $[\Sigma(f)] = [f(\Sigma); u_1, \dots, u_5] = 0$  and the framed cobordism class is additive on the components of  $f(\Sigma)$ , we are finished proving the theorem if we can eliminate, by a prim cobordism of  $f$ ,

(I) a single component  $C \cong S^1$  with  $[C; u_1, \dots, u_5] = 0$  and

(II) a pair of components  $C = C_1 \cup C_2$ :  $C_1 \cong C_2 \cong S^1$  with  $[C_i; u_1, \dots, u_5] = 1$ ,  $i = 1, 2$ .

*Case I.*

Let us apply an isotopy on  $\mathbb{R}^6$  that takes  $C$  to the standard circle

$$S^1 = \{(x_1, x_2, 0, \dots, 0) : x_1^2 + x_2^2 = 1\} \hookrightarrow \mathbb{R}^6.$$

We have a framing on  $C = S^1$ , namely  $[u_1, \dots, u_5]$ , with  $[C; u_1, \dots, u_5] = 0$ . By Lemma 3.7, comparing  $\mathbf{u}$  with the standard framing  $v_1(p) = p, v_i(p) = \mathbf{e}_{i+1}$  ( $p \in S^1, 2 \leq i \leq 5$ ), the resulting  $A(p) = \mathbf{u}(p)\mathbf{v}(p)^{-1}$  is null-homotopic as a loop:  $[A] = 0 \in \pi_1(GL^+(5))$ . This means that we can extend  $A$  continuously to the disc that  $C$  surrounds. We raise the disc out to  $(\mathbb{R}^7)^+$  so that its tangent space be vertical ( $\mathbf{e}_7 \in T_x H$ ) exactly in  $C$ ; say,

$$H = D^2 = \{(x_1, x_2, 0, \dots, 0, t) : x_1^2 + x_2^2 + 4t^2 = 1\} \hookrightarrow \mathbb{R}^6 \times [0, 1/2]$$

Obviously, the standard framing  $\mathbf{v}$  can be extended to  $H$ , but we also extended  $A$ , which means  $\mathbf{u} = A\mathbf{v}$  can also be extended.  $\mathbf{u}$  is a basis in  $\nu(H)$ ; let us define a Riemannian metric in  $\nu(H)$  such that  $\mathbf{u}$  is orthonormal. Let  $\tau : D(\nu(H)) \rightarrow (\mathbb{R}^7)^+$  be an embedding that continues  $\varphi^{-1} : D(\nu(C)) = D^5 \times C \rightarrow \mathbb{R}^6$  in  $\partial H = C$ .

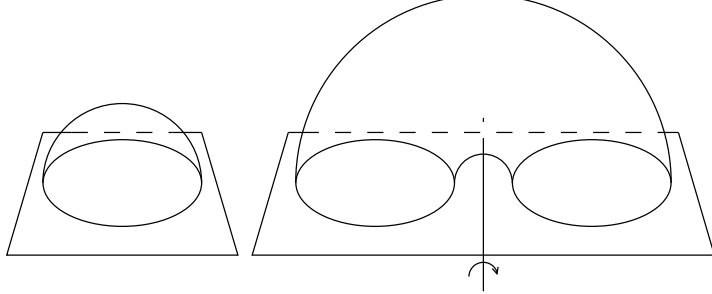


Figure 8: *The cobordism  $H \hookrightarrow (\mathbb{R}^7)^+$  over  $C \hookrightarrow \mathbb{R}^6$  in Case I and II, respectively.*

### *Case II.*

Now, we have two singular circles  $C_1 \cong C_2 \cong S^1 \hookrightarrow \mathbb{R}^6$ , with a framing  $[u_1, \dots, u_5]$  that admits the standard form of  $f$ , just like in *Case I*. Let us connect  $C_1$  and  $C_2$  with an embedded cylinder  $H \cong S^1 \times [0, 1] \hookrightarrow \mathbb{R}^6 \times [0, 1/2]$ , such that the tangent space of  $H$  is vertical exactly in  $C$ . (This can be done, for instance, if we send  $C_1$  and  $C_2$  to a pair of unit circles by isotopy, then rotate them around an axis  $\mathbb{R}^5 \subset \mathbb{R}^6 \times \{0\}$ , and finally squeeze down the height by an affine map.)

Once again, using Lemma 3.7, we get that  $\mathbf{u}$  can be extended to be a framing of the normal bundle of  $H$ . Let us take a Riemannian metric in  $\nu(H)$  so that  $\mathbf{u}$  is orthonormal, and let  $\tau : D(\nu(H)) \hookrightarrow (\mathbb{R}^7)^+$  be a map that extends  $\varphi^{-1} : D^5 \times C = D(\nu(C)) \hookrightarrow \mathbb{R}^6$  in  $\partial H = C = C_1 \cup C_2$  and maps to a tubular neighborhood of  $H$ .

So in either case, we have  $C \hookrightarrow \mathbb{R}^6$  with a nullcobordism  $H \hookrightarrow \mathbb{R}^6 \times [0, 1/2]$ , a framing  $\mathbf{u}$  over  $H$  for which  $\mathbf{u}|_C$  is, in the sense of Lemma 3.28, consistent with the standard form  $wh_2 \times \text{id}_{\mathbb{R}}$  of  $f$  over the singular stratum; and we have an embedding  $\tau : D(\nu(H)) \hookrightarrow (\mathbb{R}^7)^+$  that continues the map  $\varphi^{-1} : D^5 \times C = D(\nu(C)) \hookrightarrow \mathbb{R}^6$ .

Now we try to perform a surgery of the prim map  $f : M^4 \rightarrow \mathbb{R}^6$  in order to get rid of the singularity curve(s)  $C$ . For this, we use the *bridge* construction (see Figure 9): we try glue a strip  $D(H)$  to the neighborhood  $U(C) = \tau(D(\nu(C)))$  of  $C$ , and attach vertical "bars" to the edge of the strip ( $\partial^{rel} D(H)$ ), as well as to  $M \setminus U(C)$ . This already gives a bordism if we map the height identically. This bordism starts in our prim map  $f$  and ends in a map that does not even have the points  $f^{-1}(C)$  in its source  $M'$ . However, we still have some work to do in order to make the manifold smooth and the map prim.

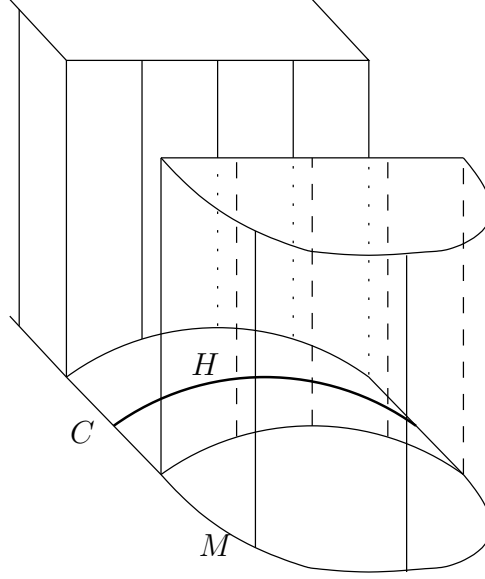


Figure 9:  $D(H)$  forms a bridge grounded on the neighborhood  $U(C)$  of the curve  $C$ .

Let  $\Phi : H \rightarrow (0, 1]$  be a continuous function,  $C^1$  (once continuously differentiable) in  $\text{int}(H)$ , such that for any  $p \in C$ , its value is  $\Phi(p) = 1$ , and for any  $n \in \nu(S \hookrightarrow H)_p$  inner normal vector, the derivative inwards,  $\partial_n \Phi$ , is  $+\infty$ . This  $\Phi$  will roughly mean the width of the bridge at  $x \in H$ ; it will help us glue our strip to  $M$  in a differentiable way.

But there is one more difficulty: the vertical bars still form an edge with the strip  $D(H)$  (the "walkway" of the bridge). To handle this problem, we will curl up the edge of the walkway so that the transition to the bars be seamless. More precisely, let  $\Psi : [0, 1] \rightarrow [0, 1/4]$  be strictly growing, continuous function that is  $C^1$  on the left-closed, right-open interval  $[0, 1)$ ; moreover,  $\Psi(0) = \Psi'(0) = 0$ ,  $\Psi(1) = 1/4$  and  $\Psi'(1) = \infty$ .

$G \subset \mathbb{R}^+ \times \mathbb{R} = \text{graph}\Psi \cup \{1\} \times [1/4, \infty)$  is a non-compact  $C^1$  manifold. Rotating this by  $O(3)$ , we get the "U-shaped objects" over each point  $x \in H$ , that are the combination of the "walkway" and the "bars" of the bridge.

Now we compose the manifold  $W$  from three pieces:

$$W_1 = \{(x, \mathbf{v}, t) \in H \times \mathbb{R}^3 \times [0, 1] : (\|\mathbf{v}\|/\Phi(x), t - x_7 - \pi_7(\tau_x(\langle wh_2(\mathbf{v}), \tau_x^* \mathbf{u} \rangle))) \in G\},$$

(where  $\langle \mathbf{a}, \mathbf{b} \rangle = \sum_j a_j \cdot b_j$ ), the bridge and the vertical bars issuing from it;

$$W_2 = \{(p, \mathbf{v}, t) \in C \times \mathbb{R}^3 \times [0, 1] : t \leq \Psi(\|\mathbf{v}\|)\},$$

the "triangle-shaped" little remainder under the U-shaped object in a point  $p \in C = \partial H$ ; and finally

$$W_3 = (M \setminus \text{int } U(C)) \times [0, 1].$$

The gluing between the three parts is obvious from the geometric plan we gave, albeit hard to formally handle:

$$\begin{aligned}
(x, \mathbf{v}, t) \in W_1 &\sim (p, \mathbf{v}', t') \in W_2 && \text{if } && x = p; \mathbf{v} = \mathbf{v}'; t = t' \\
(x, \mathbf{v}, t) \in W_1 &\sim (q, t') \in W_3 && \text{if } && x = p \in C; q = \chi^{-1}(\langle \mathbf{v}, (\chi_{\bar{p}})_* \tilde{\mathbf{u}} \rangle, p) \in \partial U(C); t = t' \\
(p, \mathbf{v}, t) \in W_2 &\sim (q, t') \in W_3 && \text{if } && q = \chi^{-1}(\langle \mathbf{v}, (\chi_{\bar{p}})_* \tilde{\mathbf{u}} \rangle, p); t = t'
\end{aligned}$$

$W_1$  and  $W_3$  can easily be equipped with an oriented  $C^1$ -differentiable manifold structure (they are 5-dimensional, with boundary). So can  $W_1 \cup W_2$ , as the choice of  $\Phi$  implies that the  $\cup$ -objects in  $W_1$  over inner points  $x \in \text{int } H$  osculate to those over the boundary point  $p \in C$  for  $x \rightarrow p$ . As  $H$  has a vertical tangent space in  $p \in C$ ,  $W_1 \cup W_2$  is  $C^1$  even in the "pole" of the  $\cup$ -object,  $(p, \mathbf{0}, 0)$ .

The gluing of  $W_3$  to  $W_1 \cup W_2$  occurs along the  $C^1$  boundary manifold  $\partial U(S) \times [0, 1]$ . So we defined  $W = W_1 \cup W_2 \cup W_3$  as a  $C^1$  manifold.

Now let us give the  $C^1$ -map  $h : W \rightarrow \mathbb{R}^6 \times [0, 1]$ . We define

$$h_1 : W_1 \rightarrow \mathbb{R}^7 \times [0, 1]; h_1(x, \mathbf{v}, t) = (\bar{\pi}_7(x + \tau_x(\langle wh_2(\mathbf{v}), \tau^* \mathbf{u} \rangle), t))$$

$$h_2 : W_2 \rightarrow \mathbb{R}^7 \times [0, 1]; h_2(p, \mathbf{v}, t) = (\bar{\pi}_7(p + \tau_p(\langle wh_2(\mathbf{v}), \tau^* \mathbf{u} \rangle), t))$$

$$h_3 : W_3 \rightarrow \mathbb{R}^7 \times [0, 1]; h_3(q, t) = (f(q), t)$$

$$h = h_1 \cup h_2 \cup h_3.$$

It is straightforward to verify that  $h$  is defined consistently along the boundaries and is  $C^1$  on each component. The gluings between  $W_1$ ,  $W_2$  and  $W_3$  were defined exactly so that in a boundary point  $w$ , derivatives  $T_w h$  agree by the different definitions of  $h$ . (It is at this point that constructing in  $C^1$  is easier than in the smooth case ( $C^\infty$ ) – if the derivative  $T_w h_i$  is independent on  $i$  for all ambiguous (glued) points  $w$ , then we already know  $h$  to be  $C^1$ .)

Using the fact that  $\tau$  is a diffeomorphism and that  $wh_2$  is an immersion except in 0, we get that  $h$  only has singularities in the points  $H = (x, \mathbf{0}, x_7) \in W_2$  and on the cylinders  $(\Sigma(f) \setminus C) \times [0, 1] \subset W_3$  over the other singular curves. In both cases, the singularities are Whitney umbrellas.

To make  $h$  a prim, we only need to give an orientation on the line bundle  $\ker Th$  (or, which is the same for a 1-bundle, a nonzero section). On  $(\Sigma(f) \setminus C) \times [0, 1]$ , we can raise the orientation given by  $g$  on  $(\Sigma(f) \setminus C)$ . In  $x \in H$ ,  $\tau_*(\partial/\partial_1)$  (which is a vector in the direction that goes  $y \rightarrow y^2$ ) is the good choice that continues the orientation given by  $g$  on  $\ker Tf|_C$ . Thus we made  $h$  a prim cobordism.

Denote  $\partial W = \partial_0 W \cup \partial_1 W = M \cup M'$ ; the upper boundary  $M'$  is an oriented  $C^1$ -class 4-manifold. The prim cobordism  $h$  connects the prim maps  $h_0 = f : M \rightarrow \mathbb{R}^6$

and  $h_1 = f' : M' \rightarrow \mathbb{R}^6$ .  $f'$  has no singular points in  $(W_1 \cup W_2) \cap (\mathbb{R}^6 \times \{1\})$ ; this means that we have got rid of the singular set  $C$ .

We constructed  $W$  and  $h$  only as  $C^1$ -differentiable, but we can introduce a  $C^\infty$  atlas on  $W$  that is compatible with the one already given on  $M$  and  $W_3$ , and approximate  $h$  with a smooth generic map to get a smooth prim cobordism. New singular curves will not appear, because the class  $\Sigma^1$  can be characterized with only the first derivative of  $h$  (as  $\Sigma^1(h)$  is the pullback  $(j_h^1)^{-1}(\Sigma^1(W, \mathbb{R}^8))$  from the *first* jet space), so new singularities do not appear after a  $C^1$ -small perturbation.

So we succeeded in cutting out the singular curve(s)  $C$  in both cases; iterating this process, we finally get a map  $g_K : M_K \looparrowright \mathbb{R}^7$  for which  $[g_K] = [g] \in Imm^{SO}(4, 3)$ , but also  $f_K = \bar{\pi}_7 \circ g_K : M' \looparrowright \mathbb{R}^6$  is an immersion. This concludes the proof.

### 3.5 Hughes's theorem

Now we present an interesting consequence of our main result, Theorem 3.33. Recall

**Theorem 3.34** (Banchoff). *Let  $f : N^2 \looparrowright \mathbb{R}^3$  be a self-transverse immersion ( $N$  may be unoriented). Then  $\#\{\Delta_3(f)\} \equiv \chi(N) \pmod{2}$ , where  $\chi(N)$  is the Euler characteristic of  $N$ . In particular, surfaces of odd Euler characteristic cannot be immersed in  $\mathbb{R}^3$  without a triple point.*

This theorem can be proved easily by cutting out standard neighborhoods of the triple points and replacing them with the triple point-free part of the *Boy surface*. (The theorem can be generalized to generic (singular) maps  $f : N^2 \rightarrow \mathbb{R}^3$ , using similar theory that we did in this work.)

Notice that Banchoff's theorem says that  $\#\{\Delta_3(f)\}$  is even in the case where  $N^2$  is oriented, since an oriented surface has even Euler characteristic. Theorem 3.33 yields a partial generalization of Banchoff's theorem in double dimensions:

**Theorem 3.35** (Hughes). *Let  $f : M^4 \looparrowright \mathbb{R}^6$  be a self-transverse immersion, where  $M$  is oriented. Then  $\#\{\Delta_3(f)\}$  must be even.*

*Proof.* Herbert's multiple point formula (Theorem 2.2) says:

$$m_{r+1} = f^*(n_r) + e \cup m_r$$

where  $m_r$  is the dual class of the  $r$ -tuple points in the source and  $n_r$  is that in the target;  $e = e(\nu(f)) \pmod{2}$  is the mod 2 normal Euler class. Now, as  $\mathbb{R}^6$  has trivial cohomology except in dimension 0,

$$m_3 = f^*(n_2) + e \cup m_2 = 0 + e \cup (f^*(n_1) + e \cup m_1) = e^2 \cup m_1 = e^2.$$



So the number of triple points is:

$$3\#\{\Delta_3\} \bmod 2 = \#\{\tilde{\Delta}_3\} \bmod 2 = m_3 \cap [M] = e^2[M] \bmod 2 = p_1[M] \bmod 2 = 3\sigma(M) \bmod 2.$$

But we proved that the very existence of  $f$  shows  $\sigma(M)$  to be even; this means  $\#\{\Delta_3\}$  is even, too.  $\square$

In fact, Hughes's theorem is equivalent with the slightly weaker statement

$$\text{im}(Imm^{SO}(4, 2) \rightarrow \Omega_4 \xrightarrow{\sigma} Z) \leq 2Z.$$

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