Lévy Market Models with Applications to Interest Rate Derivative Valuation

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Abstract

Herein we will develop three time inhomogeneous Lévy process driven market models, the Lévy Forward Rate Model, the Lévy Forward Process Model and the Lévy LIBOR Model. We will derive formulas for prices of various financial instruments using Fourier transform.

The way of presenting is aimed for those who have less mathematical background, e.g. for university students, so most of the results from mathematics and finance are proved, or at least motivated and referenced.

Contents

1 Introduction

The aim of this thesis is to introduce some Lévy models on forward rates, LIBOR rates and forward prices which use time-inhomogeneous Lévy-processes as driving processes in stead of classical Bronian motion drivers. We will concentrate on the Lévy Forward Rate Model, in which we will compute prices of bond options and swaptions. Moreover, we will prove the completeness of the model. Also we will provide an example of the Lévy Forward Rate Model, where the driving process is a normal inverse Gaussian (NIG) process.

Apart from the Lévy Forward Rate Model, we will introduce the Forward Process Model, and the Lévy LIBOR Model. We will prove that under some mild conditions the Forward Process Model can be embedded into the Forward Rate Model.

The organization of the thesis is as follows. We note some basic definitions from finance in Section 1. Section 2 contains the mathematical tools required by the models: the definition, some properties and construction of time-inhomogeneous Lévy processes are included in Section 2. We formulate the models in Section 3. We develop pricing formulas in Section 4. We finish the thesis with some remarks an conclusions in Section 5. We collect some additional tools for changing measures, and the proof of the completeness of the Lévy Forward Rate Model in the Appendix.

1.1 Definition of financial instruments

On the financial markets, the most commonly traded financial instruments are put/call options, caps, floors, swaps and swaptions. Our goal is to price these financial instruments. In this chapter, we will define them deduce their payoffs, and give a general formula for their prices as an expectation under a risk neutral measure and under a forward measure.

We will work on a finite time horizon, $[0, T^*]$, and on filtered probability space $(\Omega, (\mathcal{F}_t)_{0 \leq t \leq T^*}, \mathcal{F}, \mathbb{P})$ which satisfies the usual conditions, namely the filtration is right-continuous and complete, that is $\mathcal{F}_t = \bigcap_{T^* \geq s > t} \mathcal{F}_s$ and $\forall t \in [0, T^*], \forall A \in \mathcal{F}_t$ if $B \subseteq A$, and $\mathbb{P}() = 0$ then $B \in \mathcal{F}_t$.

We start with the most fundamental financial instrument:

Definition 1.1. ***Zero-coupon bond*** *or simply* ***zero bond*** *is a financial instrument which pays* 1 *unit of cash for the holder at time* T*, where* T *is called the maturity date. The price of it at time* $t \in [0, T^*]$ *is denoted by* $B(t, T)$*. That is the process*

$$t \to B(t, T) \ t \in [0, T]$$

is a non-negative process adapted to the filtration $(\mathcal{F}_t)_{T\geq t\geq 0}$ *with* $B(T,T) = 1$ *almost surely.*

$B(t,T)$ can also be interpreted as the value at time t of a future unit capital payed at T. We can think about the sigma-algebra $\mathcal{F}_t$ as the information gathered up to time t, thus we assume that $B(t,T)$ is $\mathcal{F}_t$-measurable, i.e the bond price process $B(.,T)$ is adapted.

We shall assume that there exists a zero-coupon bond for every maturity date $T \in [0,T^*]$ in order to define the forward and short rates as follows:

Definition 1.2. *The* ***forward rate*** *is denoted by* $f_{t,T}$*, and defined by*

$$f_{t,T} = -\frac{\partial}{\partial T}\log B(t,T),$$

if the derivative exists.

Remark We will usually assume that $B(t,T) = \exp(-\int_t^T f_{t,u}du)$ and the forward rates $f_{t,u}$ exist for $u \in [t,T^*]$.

Definition 1.3. ***Short rate:*** $r_t = f_{t,t}$.

If think about the short rate as the return of a risk free investment, and so we define the savings account as follows:

Definition 1.4. *The value of the* ***savings account*** *at time t is given by the formula*

$$B_t = \exp\int_0^t r_u du \text{ for } 0 \leq t \leq T^*$$

if the short rates exist.

Now we define one of the most traded derivatives, the European options. The European call option is a contract between two parties, the buyer and the seller of the call option. The buyer has the *right but not the obligation* to buy the underlying financial instrument at the strike price at maturity date. Clearly, the owner of the call option will only exercise his right, if at maturity date the price at the market of the underlying is higher than the strike price. In the thesis we will restrict ourselves to European options, see [Musiela & Rutkowski, 2005] for types of options.

Definition 1.5. ***(European) Call option*** *on a financial instrument with price* X_T *at maturity date* T *with strike price* K*, is a contingent claim with maturity date* T *with payoff* $C = (X_T - K)^+$.

The price of the call option is the price the buyer pays to the seller to enter the contract. We will denote the price at time $0 \le t \le T$ by C_t.

The pair of the call option is the put option, where the buyer of the put option has the *right, but not the obligation* to sell the underlying financial instrument at strike price at maturity date:

Definition 1.6. ***(European) Put option*** *on a financial instrument with price X_T at maturity date T with strike price K, is a contingent claim with maturity date T with payoff $P = (K - X_T)^+$.*

We will denote the price at time $0 \le t \le T$ by P_t.

Our next aim is to define LIBOR rates in terms of forward prices. Start with forward processes:

Definition 1.7. *The* ***forward process*** *is defined by the quotient $F(t,T,U) = \frac{B(t,T)}{B(t,U)}$, where $0 \le t \le T \le U \le T^*$.*

Notice that $F(t,T,U)$ is the value at time t of a unit capital paid at time T expressed in future capital payed at time U.

Now, we can define the LIBOR rates:

Definition 1.8. *The forward* ***LIBOR rate*** *$L_\delta(t,T)$ is defined by the equation*

$$1 + \delta L_\delta(t,T) = F(t,T,T+\delta).$$

The abbreviation LIBOR stands for London Interbank Offered Rate, which is settled at time T, thus the LIBOR rate is $L_\delta(T,T)$ The forward word emphasizes that $L_\delta(t,T)$ is the LIBOR rate foreseen at time t.

We can define options on LIBOR rates:

Definition 1.9. *A* ***caplet*** *is a call option on the LIBOR rate, with maturity $T+\delta$, with payoff $Cpt = (L_\delta(T,T+\delta) - \kappa)^+$, where κ is the strike rate.*

Definition 1.10. *A* ***floorlet*** *is a put option on the LIBOR rate, with maturity $T+\delta$, with payoff $Flt = (\kappa - L_\delta(T,T+\delta))^+$, where κ is the strike rate.*

Notice that the maturity date of a caplet/floorlet is $T+\delta$, and not T, since the LIBOR rates are settled at time T.

We can make new financial instruments by combining caplets and floorlets with different maturity dates:

Definition 1.11. ***Cap*** *is a sequence of caplets, more precisely there are starting and ending dates $0 \le T_1 < \ldots < T_n \le T^*$ and $0 \le S_1 < \ldots < S_n \le T^*$, $\delta_i = S_i - T_i$ for $1 \le i \le n$, the ith caplet pays $\alpha_i(L_{\delta_i}(T_i,T_i) - \kappa_i)^+$ for the holder of the cap at time S_i, where α_i is a positive constant which we call the ith notional.*

Definition 1.12. ***Floor*** *is a sequence of floorlets, more precisely there are starting and ending dates* $0 \leq T_1 < \ldots < T_n$ *and* $0 \leq S_1 < \ldots < S_n$, $\delta_i = S_i - T_i$ *for* $1 \leq i \leq n$, *the ith floorlet pays* $\alpha_i(\kappa_i - L_{\delta_i}(T_i, T_i))^+$ *for the holder of the floor at time* S_i, *where* $\alpha_i > 0$ *is the ith notional.*

Notice that the value of a cap/floor at time t is the sum of the values of caplets/floorlets at time t, thus it is enough to price the caplets/floorlets.

The next financial instrument we are interested in are swaps. A swap is an exchange of interest rate payments. The two parties in a swap contract are the receiver and the payer. The receiver is paying LIBOR rate to the payer, and gets fixed interest rate payments from the payer.

Definition 1.13. *A* ***swap*** *is a sequence of n interest rate exchanges, more precisely there are starting and ending dates* $0 \leq T_1 < T_2 < \ldots < T_{n+1} \leq T^*$, $\delta_i = T_{i+1} - T_i$ *for* $1 \leq i \leq n$. *In the ith exchange, the payer gets the interest rate payment* $\delta_i(L_{\delta_i}(T_i, T_i) - K)$ *at time* T_{i+1}, *while the receiver gets* $\delta_i(K - L_{\delta_i}(T_i, T_i))$ *at time* T_{i+1}.

From the perspective of the receiver, the value of a swap agreement at time t:

$$\begin{aligned}\sum_{i=1}^{n} \delta_i(K - L_{\delta_i}(t, T_i))B(t, T_{i+1}) &= K\sum_{i=1}^{n} \delta_i B(t, T_{i+1}) - \sum_{i=1}^{n}\left(\frac{B(t, T_i)}{B(t, T_{i+1})} - 1\right) B(t, T_{i+1}) \\ &= K\sum_{i=1}^{n} \delta_i B(t, T_{i+1}) + B(t, T_{n+1}) - B(t, T_1). \qquad (1)\end{aligned}$$

Now we can make options on swaps.

Definition 1.14. *The* ***receiver swaption*** *is a contract between the two parties the buyer and the seller. At time* T_1 *the buyer of the swaption has the right but not the obligation to enter a swap contract introduced in Definition 1.13 as a receiver.*

The pair of the receiver swaption is the payer swaption:

Definition 1.15. *The* ***payer swaption*** *is a contract between the two parties the buyer and the seller. At time* T_1 *the buyer of the swaption has the right but not the obligation to enter a swap contract introduced in Definition 1.13 as a payer.*

Usually at time T_1, the buyer does not enter the swap contract, in stead the buyer gets the present value of the future interest rate exchanges if it is positive. Thus the payoff of a receiver swaption at time T_1 is by formula 1:

$$(K\sum_{i=1}^{n} \delta_i B(T_1, T_{i+1}) + B(T_1, T_{n+1}) - 1)^+. \qquad (2)$$

Hence the payoff of a receiver swaption can be written in the form

$$(\sum_{i=1}^{n} c_i B(T_1, T_{i+1}) - 1)^+,$$

where c_i are positive constants. We will use this formula when we will evaluate the price of swaption in Section 4.

Remark A similar argument shows the payoff of a payer swaption can be written in the form

$$(1 - \sum_{i=1}^{n} c_i B(T_1, T_{i+1}))^+,$$

for c_i are positive constants.

1.2 Risk-neutral valuation

After defining the basic financial instruments, we will model the market of bonds, and derive formulas for their prices. We will use the risk neutral pricing principle, whereby the price of an option can be obtained by taking the expected value of its discounted payoff under a risk neutral measure. We need the definition of risk neutral measure:

Definition 1.16. *A probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F})$ is a* ***risk neutral measure*** *(RNM) or* ***equivalent martingale measure*** *(EMM), if $\mathbb{P}^* \sim \mathbb{P}$ and the the discounted bond price process*

$$t \mapsto \frac{B(t,T)}{B_t} \textit{ is a martingale for } 0 \leq t \leq T \leq T^*.$$

Remark Usually, we will use the equivalent condition

$$\frac{B(t,T)}{B_t} = \mathbb{E}_{\mathbb{P}^*}\left(\frac{1}{B_T}\middle| \mathcal{F}_t\right) \text{ for } 0 \leq t \leq T \leq T^*,$$

where $\mathbb{E}_{\mathbb{P}^*}$ denotes the expectation taken under $\mathbb{P}^*$.

Remark A RNM is an 'artificial' measure, in the sense that it is a mathematical tool for deriving option prices, and has nothing to do with the real probabilities derived from the behavior of the financial instruments. However, most of the time we will think about RNM as a probability measure.

Our market is infinite, in the sense that there are infinitely many assets, namely the zero coupon bonds. However, on finite markets, the Fundamental Theorem of Asset Pricing is valid (FTAP) (see [Delbaen & Schachermayer, 2006] for further details) which tells that there exists a RNM if and only if the market is arbitrage-free. That is to say there is no trading strategy with which one could get a non-negative payoff at maturity and with positive probability, gaining positive payoff with investing 0 capital at starting time. Loosely speaking 'gaining money without the chance of loosing from 0 investment'.

Following the FTAP for finite markets, we say:

Definition 1.17. *The market is **arbitrage-free**, if there exist an RNM.*

I do not want to clearly define trading strategies (also the definition of trading strategies in an infinite market is ambiguous), and arbitrage in our infinite market, since it would involve too many new definitions which we would not use later on in the thesis. (If the reader is interested in FTAP for infinite markets, [Tomas Björk, 1997] is a good starting point.)

Further on, we will always assume that the market is arbitrage free, moreover that $\mathbb{P}$ is an RNM, and $\mathbb{E}$ denotes the expected value taken under $\mathbb{P}$.

In order to be able to speak about options, in general, we define contingent claims as follows.

Definition 1.18. *A contingent claim is a financial instrument which at maturity date T has a payoff X_T, where X_t is $\mathcal{F}_T$-measurable random variable which is bounded below, i.e there is a constant $C \in \mathbb{R}$ such that $X_T \geq C$ almost surely.*

Remark The boundedness condition is needed to ensure that the expected value of X_T exists.

Th risk-neutral pricing principle says that the value of a contingent claim at time t, denoted by X_t is

$$X_t = B_t \mathbb{E}\left(\frac{X_T}{B_T} \middle| \mathcal{F}_t \right). \tag{3}$$

Remark The reason for the formula (3) is that if we extend the market with a financial instrument which price process is X_t, then the market will remain arbitrage-free, since $\mathbb{P}$ is still arbitrage-free.

The problem with the risk-neutral evaluation is that the price of an option (contingent claim) is not necessarily unique, since the RNM might not be unique, that is the reason we like complete markets:

Definition 1.19. *A market is **complete** if the RNM is unique.*

Remark The term complete comes from the theory of finite markets. An other fundamental theorem (often referred as the second FTAP,) tells that for any contingent claim with finite risk-neutral price there is a (replicating) trading strategy which has the same payoff at maturity date if and only if the RNM is unique. That is to say, we cannot extend the market by adding extra contingent claims. For more details on completeness of finite markets see [Delbaen & Schachermayer, 2006], and [Tomas Björk, 1997] for completeness of bond markets.

Remark Notice that the definition of completeness depends on the choice of the sigma-algebra $\mathcal{F}$. This remark will play key role in Appendix A.3.

Notice that call/put options, caps/floors and swaptions are all contingent claims, or linear combinations of contingent claims. Hence we can use formula (3) to price them.

We start with caplets. By the definition of caplet and LIBOR rate, we have:

$$\begin{aligned}(L_\delta(T,T)-\kappa)^+ &= \frac{1}{\delta}\left(\frac{1}{B(T,T+\delta)}-1-\kappa\delta\right)^+ \\ &= \frac{1+\kappa\delta}{\delta B(T,T+\delta)}\left(\frac{1}{1+\kappa\delta}-B(T,T+\delta)\right)^+.\end{aligned}$$

Thus the price of a caplet at time t is

$$\mathbb{E}\left(\frac{1+\kappa\delta}{\delta B_{T+\delta}B(T,T+\delta)}\left(\frac{1}{1+\kappa\delta}-B(T,T+\delta)\right)^+|\mathcal{F}_t\right).$$

Since $B(T,T+\delta)$ is $\mathcal{F}_T$-measurable, thus by conditioning on $\mathcal{F}_T$, we get:

$$\mathbb{E}\left(\mathbb{E}\left(\frac{1}{B_{T+\delta}}|\mathcal{F}_T\right)\frac{1+\kappa\delta}{\delta B(T,T+\delta)}\left(\frac{1}{1+\kappa\delta}-B(T,T+\delta)\right)^+|\mathcal{F}_t\right)=$$

$$=\mathbb{E}\left(\frac{1+\kappa\delta}{\delta B_T}\left(\frac{1}{1+\kappa\delta}-B(T,T+\delta)\right)^+|\mathcal{F}_t\right).$$

A similar argument applies for floorlets, hence to price caplets/floorlets, it is enough to price options on bonds. Since caps/floors are linear combinations of caplets/floorlets. Caplets/Floorlets are options on bonds, after pricing options on bonds, it is easy to price caps/floors.

The formulas (2) and (3) give that the price of a payer swaption at time t is

$$\mathbb{E}\left(\frac{1}{B_{T_1}}(K\sum_{i=1}^{n}\delta_i B(T_1,T_{i+1})+B(T_1,T_{n+1})-1)^+.\middle|\mathcal{F}_t\right).$$

1.3 Forward measure

In this section our aim is to further simplify the pricing formulas.

Examine the quotient

$$\frac{X_T}{B_T}.$$

This quotient tells us that how much money do we have to deposit to a bank account, to receive X_T at time T, or in other words, how much does X_T worth in terms of the bank account. So the bank account was our reference financial instrument, the so called numeraire.

We want to change the numeraire to the zero coupon bond with maturity date T. This will be handy, since the price of this bond at time T is 1. Then we have to change from the RNM $\mathbb{P}$ to a new measure $\mathbb{P}_T$ under which the process $t \mapsto \frac{X_t}{B(t,T)}$ is a martingale, where X_t is as defined by (3). This can be achieved if we define $\mathbb{P}_T$ through the Radon-Nikodym derivative

$$\nu = \frac{\mathsf{d}\mathbb{P}_T}{\mathsf{d}\mathbb{P}} = \frac{1}{B(0,T)B_T}.$$

We will denote the expectation under $\mathbb{P}_T$ by $\mathbb{E}_T$. We can check the claim with the Bayes Theorem:

$$\mathbb{E}_T(X_T|\mathcal{F}_t) = \frac{\mathbb{E}(\nu X_T|\mathcal{F}_t)}{\mathbb{E}(\nu|\mathcal{F}_t)} = \frac{B_t}{B(t,T)}\mathbb{E}\left(\frac{X_T}{B_T}\middle| \mathcal{F}_t\right) = \frac{X_t}{B(t,T)}.$$

Definition 1.20. *We call the measure $\mathbb{P}_T$ the **forward measure** associated to the maturity date $T \in [0,T^*]$ which is given by the formula*

$$\frac{d\mathbb{P}_T}{d\mathbb{P}} = \frac{1}{B(0,T)B_T}.$$

We sum up what we just proved in a proposition:

Proposition 1.1. *Under the forward measure $\mathbb{P}_T$ the price at time t of a contingent claim with maturity date T is given by*

$$X_t = B(t,T)\mathbb{E}_T(X_T|\mathcal{F}_t).$$

Remark Proposition 1.1 gives a simpler formula for prices of claims, since now we only need the distribution of X_T under $\mathbb{P}^T$, in comparison to the formula under the RNM, where we needed the joint distribution of X_T and B_T.

We have the following corollaries:

Corollary 1.1. *The price of a call option on the bond $B(T,U)$ with strike price K where $0 \le T \le U \le T^*$ is given by the formula*

$$B(t,T)\mathbb{E}_T(B(T,U)-K)^+.$$

Corollary 1.2. *The price of a payer swaption defined in 1.15 with strike rate K is given by the formula*

$$B(t,T)\mathbb{E}_{T_1}\left(\left(K\sum_{i=1}^{n}\delta_i B(T_1,T_{i+1}) + B(T_1,T_{n+1}) - 1\right)^+ \middle| \mathcal{F}_t\right).$$

We end this section with the following technical proposition:

Proposition 1.2. *Under $\mathbb{P}_T$, the forward process $F(.,U,T)$ is a martingale, for $0 \leq U \leq T \leq T^*$.*

Proof It is enough to prove that $F(t,U,T) = \mathbb{E}_T(F(U,U,T)|\mathcal{F}_t)$. Proposition 1.1 with $X_U = \frac{1}{B(U,T)}$ gives:

$$
\begin{aligned}
\mathbb{E}_T(F(U,U,T)|\mathcal{F}_t) &= \mathbb{E}_{\mathbb{P}_T}\left(\frac{1}{B(U,T)}\middle|\mathcal{F}_t\right) \\
&= \frac{B_t}{B(t,T)}\mathbb{E}\left(\frac{1}{B_T B(U,T)}\middle|\mathcal{F}_t\right) \\
&= \frac{B_t}{B(t,T)}\mathbb{E}\left(\mathbb{E}\left(\frac{1}{B_T \mathbb{E}\left(\frac{B_U}{B_T}\middle|\mathcal{F}_U\right)}\middle|\mathcal{F}_U\right)\middle|\mathcal{F}_t\right) \\
&= \frac{B_t}{B(t,T)}\mathbb{E}\left(\frac{1}{B_U}\mathbb{E}\frac{\left(\frac{1}{B_T}\middle|\mathcal{F}_U\right)}{\mathbb{E}\left(\frac{1}{B_T}\middle|\mathcal{F}_U\right)}\middle|\mathcal{F}_t\right) \\
&= \frac{B_t}{B(t,T)}\mathbb{E}\left(\frac{1}{B_U}\middle|\mathcal{F}_t\right) \\
&= \frac{B(t,U)}{B(t,T)}. \quad \square
\end{aligned}
$$

2 Time-inhomogeneous Lévy processes

In this section, we will introduce the time-inhomogeneous Lévy processes. We will construct these processes as a sum of two processes, first one is an Ito integral, having continuous trajectories almost surely, the second one is a process with cádlág stepfunction sample paths. We will concentrate on the construction of the second process, since Ito integrals are usually treated in basic stochastic calculus courses (see [Revuz & Yor, 2005] for more details). During the construction we will also derive the canonical representation of time-inhomogeneous Lévy processes, with its semi-martingale characteristics. These characteristics (see section 2.2.1 for further details) are the most important objects of this chapter, since they give a formalism with which we can easily keep track of the behavior of the process after a change of measure. We conclude this chapter with some handy formulas which we will use in the later parts of the thesis, when we will compute some option prices.

The reason for using time-inhomogeneous Lévy processes in stead of the homogeneous ones, is the fact that in the presented models the change of measure is essential. However if we start with a homogeneous Lévy processes, after a change measure we will get an inhomogeneous Lévy process.

We will work on a stochastic basis $(\Omega, (\mathcal{F}_s)_{0\leq s\leq T^*}, \mathcal{F}, \mathbb{P})$, where $(\mathcal{F}_s)_{0\leq s\leq T^*}$ is a filtration and we set $\mathcal{F} = \mathcal{F}_{T^*}$.

Recall the definition of Lévy processes (see Definition 1.1 of [Kyprianou, 2006]):

Definition 2.1. *An adapted stochastic process $(X_t)_{t\geq 0}$ is a Lévy process, if the following conditions hold:*

(i) The paths of X are $\mathbb{P}$-almost surely cádlág, that is right continuous with left limits.

(ii) $\mathbb{P}(X_0 = 0) = 1$.

(iii) X has stationary increments, that is $X_t - X_s$ is equal in distribution to X_{t-s} for $0 \leq s \leq t$.

(iv) X has independent increments, i.e. $X_t - X_s$ is independent of the σ-field $\mathcal{F}_s$ for all $s, t \in [0, T^*]$ $s < t$.

A well-know characterization of Lévy processes, the Lévy-Khinchine formula (see Theorem 1.6 of [Kyprianou, 2006]):

Theorem 2.1. *Let $b', c \in \mathbb{R}, c > 0$ and F a measure concentrated on $\mathbb{R}\backslash\{0\}$ with $\int_{\mathbb{R}} x^2 \wedge 1 F(dx) < \infty$. For this triple (b', c, F) define $\theta(iu)$ for $u \in \mathbb{R}$ as*

$$\theta_s(iu) = iub' - \frac{1}{2}uc^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux\boldsymbol{1}_{|x|\leq 1})F(dx). \tag{4}$$

Then there exists a Lévy process X with

$$\mathbb{E}\left(e^{iuX_t}\right) = \exp\left(t\theta(iu)\right) \text{ for } u \in \mathbb{R}, t \geq 0.$$

Conversely, if X is a Lévy process, then there exists a triplet (b', c, F) for which (4) holds.

We call the function θ of (4) as the cumulant of the Lévy process X. We can rewrite (2.1) as

$$\mathbb{E}\left(e^{iuX_t}\right) = \exp\left(\int_0^t \theta(iu)\mathsf{d}s\right) \text{ for } u \in \mathbb{R}, t \geq 0.$$

We define the time-inhomogeneous Lévy processes through the Lévy-Khinchine formula, that is we will allow time independence of the triplet (b', c, F). Also for greater generality, we define multidimensional ($\mathbb{R}^d$) valued time-inhomogeneous Lévy processes:

Definition 2.2. *An adapted stochastic process $L = (L_t)_{0\leq t\leq T^*}$ is a d dimensional time-inhomogeneous Lévy process, if the following conditions hold:*

(i) L has independent increments, i.e. $L_t - L_s$ is independent of the σ-field $\mathcal{F}_s$ for all $s, t \in [0, T^*]$ $s < t$.

(ii) For every $t \in [0, T^*]$, L_t possesses a characteristic function which has the following form. For all $u \in \mathbb{R}^d$:

$$\mathbb{E}(e^{i\langle u, L_t\rangle}) = \exp \int_0^t \theta_s(iu)\mathsf{d}s, \tag{5}$$

where

$$\theta_s(iu) = i\,\langle u, b'\rangle - \frac{1}{2}\,\langle u, c_s u\rangle + \int_{\mathbb{R}^d} (e^{i\langle u,x\rangle} - 1 - i\,\langle u, x\rangle\, \mathbf{1}_{|x|\leq 1})F_s(\mathsf{d}x) \text{ for } u \in \mathbb{R}^d.$$

Here θ_s is called the cumulant at time s, $b'_s \in \mathbb{R}^d$, c_s is a positive semidefinite symmetric $d \times d$ matrix, and F_s is a measure on $\mathbb{R}^d$ which integrates $1 \wedge |x|^2$, and $F_s(\{0\}) = 0$. The coefficients $(b', c, F) = (b'_s, c_s, F_s)_{0\leq s\leq T^*}$ are called the characteristics of L. $\langle ., .\rangle$ is the Euclidean scalar product on $\mathbb{R}^d$, and $|.|$ is the Euclidean norm. We need extra conditions on the characteristics:

$$\int_0^{T^*} \left(|b'_s| + \|c_s\| + \int_{\mathbb{R}^d} (|x|^2 \wedge 1)F_s(\mathsf{d}x) \right) \mathsf{d}s < \infty \tag{6}$$

where $\|.\|$ is an arbitrary norm on the $d \times d$ real matrices.

The notation b'_s might seem inconvenient for the first sight, but later we will use an other drift term closely associated with b'_s, which we will call b_s.

From Definition 2.2, it is not obvious why should such a process exist. As noted before, our firs aim is to construct a time-inhomogeneous Lévy process. Notice that in Definition 2.2 we did not require that L has almost surely cádlág paths, but we will see that the conditions already imply this property. First we show, that a time-inhomogeneous Lévy process L, is an additive process in law, that is:

Definition 2.3. *A stochastic process $X = \{X_t, t \in [0, T^*]\}$ is an additive process in law, if satisfies the following conditions:*

(i) For any choice of $0 \leq t_1 \leq \ldots \leq t_n$, the random variables $X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.

(ii) $X_0 = 0$ almost surely.

(iii) It is stochastically continuous, which means that $\lim_{s\to t} \mathbb{P}(|X_s - X_t| > \varepsilon) = 0$ for every $t \in [0, T^*]$.

Proposition 2.1. *A time-inhomogeneous Lévy process is an additive process in law.*

Proof Let L be a time-inhomogeneous Lévy process. By definition, L has independent increments. By independence of increments, for $0 \leq v \leq t \leq T^*$:

$$\mathbb{E}(e^{i\langle u, L_t\rangle}) = \mathbb{E}(e^{i\langle u, L_t - L_v\rangle})\mathbb{E}(e^{i\langle u, L_v\rangle}).$$

Thus by (5)

$$\mathbb{E}(e^{i\langle u, L_t - L_v\rangle}) = \exp \int_v^t \theta_s(iu)ds.$$

Now as $v \to t-$, we can see that $\mathbb{E}(e^{i\langle u, L_t - L_v\rangle}) \to 1$ for any fixed $u \in \mathbb{R}^d$, by using the Dominated Convergence Theorem since we had an integrability condition on the characteristics in (6). Since the characteristic function of $L_t - L_v$ converges point wise to 1, thus $L_t - L_v \to 0$ in distribution (by the Continuity theorem of characteristic functions), thus $\lim_{v \to t-} \mathbb{P}(|L_t - L_v| > \varepsilon) = 0$ for all $\varepsilon > 0$. By a similar argument one can show that $\lim_{v \to t+} \mathbb{P}(|L_t - L_v| > \varepsilon) = 0$, thus L is stochastically continuous. Also the characteristic function of L_0 is identically 1, thus L starts from 0 almost surely. So L satisfies all the conditions in the definition of additive processes in law. □

Corollary 2.1. **??***L has a cádlág version.*

Proof L is an additive process in law, Theorem 11.5 in [Sato, 1999] gives that L has cádlág version. □

Corollary **??** shows that an extra cádlág sample path condition in Definition 2.2 would not be a strict restriction, hence it is enough to give a construction of time-inhomogeneous Lévy processes with cádlág sample paths.

2.1 Construction of time-inhomogeneous Lévy processes

In this section, we will construct a time-inhomogeneous Lévy process, starting from constructing its jump measure, which is a Poisson random measure on $[0, T^*] \times \mathbb{R}^d$.

Definition 2.4. *The measure associated to the jumps of a cádlág $\mathbb{R}^d$ valued process X is a measure μ on $[0, T^*] \times \mathbb{R}^d$ such that $\mu = \sum_{t \in [0,T^*]} \delta_{(t, \Delta X_t)}$, where $\Delta X_t = X_t - \lim_{s \uparrow t} X_t$, and $\delta_{s,x}$ is the Dirac-measure concentrated on the point (s, x).*

For the construction, we need Poisson random measures.

Definition 2.5. *(Poisson random measure). Let $(S, \mathcal{S}, \hat{\nu})$ be an arbitrary σ-finite measure space. Let $\hat{\mu} : S \to \{0, 1, 2, \ldots\} \cup \{\infty\}$ in such a way that the family $\{\hat{\mu}(A) : A \in \mathcal{S}\}$ are random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then μ is called a Poisson random measure on $(S, \mathcal{S}, \hat{\nu})$ (or sometimes a Poisson random measure on S with intensity $\hat{\nu}$) if* *(i) for mutually disjoint $A_1, \ldots, A_n$ in $\mathcal{S}$, the variables $\hat{\mu}(A_1), \ldots, \hat{\mu}(A_n)$ are independent,*

(ii) for each $A \in \mathcal{S}$, $\hat{\mu}(A)$ is Poisson distributed with parameter $\hat{\nu}(A)$ (here we allow $0 \leq \hat{\nu}(A) \leq \infty$),

(iii) $\mathbb{P}$-almost surely $\hat{\mu}$ is a measure.

With the definitions above, we can state the following theorem which is the goal of this section:

Theorem 2.2. *The process L defined by*

$$L_t := L_t^1 + L_t^2 + L_t^3$$

for $0 \leq t \leq T^*$ is a time-inhomogeneous Lévy process, where

$$\begin{aligned} L_t^1 &= \int_0^t b_s' ds + \int_0^t \sqrt{c_s} dW_s \\ L_t^2 &= \int_0^t \int_{\{|x|>1\}} x\mu(ds, dx) \\ L_t^3 &= \int_0^t \int_{\{|x|\leq 1\}} x(\mu - \nu)(ds, dx), \end{aligned}$$

where b_s' and c_s are the same as in Definition 2.2, $\sqrt{c_s}$ is the measurable square root of c_s, ν is a measure on $[0, T^*] \times \mathbb{R}^d$, defined by $\nu(ds, dx) := F_s(dx)ds$, μ is a Poisson random measure on $[0, T^*] \times \mathbb{R}^d$ with intensity measure ν. L^1 corresponds to the continuous martingale part of L, L^2 can be interpreted as the 'big' jumps of L, whereas L^3 is the 'small' jumps of L, which is a purely discontinuous $\mathcal{L}^2$ martingale.

Remark Notice that a time-inhomogeneous Lévy process L has independent increments, its distribution is characterized by the distributions of the random variables L_t for $t \in [0, T^*]$. This shows that if there is a time-inhomogeneous Lévy process L', than there is a constructed process L (with the same characteristics), such that L and L' have the same distribution.

The remark gives that there is no loss of generality, if we assume that we are working with the process constructed above.

2.1.1 Construction of L^1

Define L^1 as an Itô integral with respect to a Brownian motion W

$$L_t^1 = \int_0^t b_s' \mathrm{d}s + \int_0^t \sqrt{c_s} \mathrm{d}W_s \text{ for } t \in [0, T^*].$$

We prove that the characteristic function of L^1 is

$$\mathbb{E}(\exp(i\left\langle u, L^1_t\right\rangle)) = \exp(\int_0^t \langle u, b'_s\rangle\,\mathsf{d}s - \int_0^t \langle u, c_s u\rangle\,\mathsf{d}s). \tag{7}$$

Take $u \in \mathbb{R}^d$. Define the process

$$X_t := \int_0^t iu\sqrt{c_s}\mathsf{d}W_s$$

the quadratic variation of X is $[X]_t = -\int_0^t \langle u, c_s u\rangle\,\mathsf{d}s$, therefore

$$\left|-\int_0^t \langle u, c_s u\rangle\,\mathsf{d}s\right| \leq |u|^2 \int_0^t \|c_s\|\,\mathsf{d}s < \infty$$

by the condition (6). Hence $\mathbb{E}(e^{\frac{1}{2}[X]})$ is finite, by Novikov's criterion (which can be found in [Revuz & Yor, 2005] Chapter VIII, Proposition 1.15), we get that $\exp(X - \frac{1}{2}[X])$ is a martingale. $\exp(X - \frac{1}{2}[X])$ starts from 1, so

$$\mathbb{E}(\exp(\int_0^t iu\sqrt{c_s}\mathsf{d}W_s\mathsf{d}s + \int_0^t \langle u, c_s u\rangle\,\mathsf{d}s)) = 1$$

from which the desired formula follows.

2.1.2 Construction of the jump measure μ

We need the following theorems for the construction:

Proposition 2.2. *(Theorem 2.4 in [Kyprianou, 2006]) There exists a Poisson random measure $\hat{\mu}$, as in Definition 2.5*

Proposition 2.3. *(Corollary 2.4 in [Kyprianou, 2006]) Suppose that $\hat{\mu}$ is a Poisson random measure on $(S, \mathcal{S}, \hat{\nu})$, then the support of $\hat{\mu}$ is $\mathbb{P}$-almost surely countable. If in addition, $\hat{\nu}$ is a finite measure, then the support is $\mathbb{P}$-almost surely finite.*

Proposition 2.4. *Suppose that $\hat{\mu}$ is a Poisson random measure on $(S, \mathcal{S}, \hat{\nu})$. Let $f : S \to \mathbb{R}^d$ be a measurable function.*

(i) Then

$$X = \int f(x)\hat{\mu}(\mathsf{d}x)$$

is almost surely absolutely convergent if and only if

$$\int_S (1 \wedge |f(x)|)\hat{\nu}(\mathsf{d}x) < \infty. \tag{8}$$

(ii) When condition (8) holds, then

$$\mathbb{E}(e^{i\langle u, X\rangle}) = \exp\left(\int_S (e^{i\langle u, f(x)\rangle} - 1)\hat{\nu}(\mathsf{d}x)\right) \tag{9}$$

for all $u \in \mathbb{R}^d$.

(iii) Further

$$\mathbb{E}(X) = \int_S f(x)\hat{\nu}(dx) \text{ when } \int_S |f(x)|\,\hat{\nu}(dx) < \infty, \tag{10}$$

and

$$\mathbb{E}(|X|^2) = \int_S |f(x)|^2\,\hat{\nu}(dx) + \left|\int_S f(x)\hat{\nu}(dx)\right|^2 \text{ when } \int_S |f(x)|^2\,\hat{\nu}(dx) < \infty. \tag{11}$$

Proof This theorem is a multidimensional version of Theorem 2.7 in [Kyprianou, 2006], where the function f is real valued. Using this theorem in [Kyprianou, 2006] for the coordinate functions of f, we get the desired equalities. □

The construction of μ starts with constructing its intensity measure. Let us define

$$\nu(\mathsf{d}s, \mathsf{d}x) = F_s(\mathsf{d}x)\mathsf{d}s$$

where F_s is the measure in the definition of time-inhomogeneous Lévy process. ν is σ-finite, since the sets $\left\{x \in \mathbb{R}^d : \frac{1}{n} \geq |x| \geq \frac{1}{n+1}\right\}$ have finite measure for $n \in \mathbb{N}$. Hence the conditions of Proposition 2.2 are satisfied, thus there is a Poisson random measure μ on $([0, T^*] \times \mathbb{R}^d, \mathcal{B}([0, T^*] \times \mathbb{R}^d))$ with intensity measure ν. Also, we can take μ such that μ and W are independent.

I will prove that μ can be considered as a jump measure of some cádlág process. In order to do this, we need to show that $\mathbb{P}$ almost surely, at every time point there is at most one jump:

Proposition 2.5. *We have*

$$\mathbb{P}(\forall t \in [0, T^*], \mu(\{t\} \times \mathbb{R}^d) \leq 1) = 1.$$

Proof Since $\nu[0, T^*] \times \{\mathbf{0}\} = 0$, the complement of the event above can be written in the form:

$$\left\{\exists t \in [0, T^*], \mu(\{t\} \times \mathbb{R}^d) \geq 2\right\} = \bigcup_{n=1}^{\infty} \left\{\exists t \in [0, T^*], \mu(\{t\} \times \{x : |x| \geq 1/n\}) \geq 2\right\}.$$

So it is enough to prove that for each n, $\mathbb{P}(\exists t \in [0, T^*], \mu(\{t\} \times \{x : |x| \geq 1/n\}) \geq 2) = 0$.

Now fix n. Although $\mu(\{t\} \times \{x : |x| \geq 1/n\})$ is almost surely 0, since it has Poisson distribution with parameter 0, it is not trivial that

$$\mu(\{t\} \times \{x : |x| \geq 1/n\}) = 0 \text{ for all } t \in [0, T^*]$$

with probability 1, since there are continuum number of t-s in [0,T*].

To handle this problem, we use the idea that if there is a t for which at time t there are at least two jumps, than if we partition the time horizon $[0, T^*]$ into finitely many sets, than there will be a set on which there was at least two jumps.

Define the function $h : [0, T^*] \to \mathbb{R}$ by

$$h(t) := \nu([0, t] \times \{x : |x| \geq 1/n\}).$$

By (6), we have

$$\int_{[0,T^*]} \int_{\mathbb{R}^d} (|x|^2 \wedge 1) F_s(\mathrm{d}x) < \infty,$$

which means that $h(t) = \nu([0, t] \times \{x : |x| \geq 1/n\}) < \infty$ for all $n \in \mathbb{N}$. By the definition of h, we can see that h is increasing and continuous, thus for each (fixed) $k \in \mathbb{N}$, there are $0 = t_0 < t_1 < t_2 < \ldots < t_k = T^*$, such that $h(t_i) - h(t_{i-1}) = h(T^*)/k$ for $i = 1, 2, \ldots, k$.

Define

$$H_i := [h(t_i), h(t_{i-1})] \times (\{x : |x| \geq 1/n\}) \text{ for } i = 1, \ldots k.$$

Then

$$\bigcup_{i=1}^{k} H_i = [0, T^*] \times \{x : |x| \geq 1/n\}$$

holds, thus

$$\{\exists t \in [0, T^*], \mu(\{t\} \times (\{x : |x| \geq 1/n\}) \geq 2\} \subset \bigcup_{l=1}^{k} \{\mu(H_i) \geq 2\}.$$

We know, that the distribution of $\mu(H_i)$ is Poisson with parameter $\nu(H_i)$, thus

$$\mathbb{P}(\mu(H_i) \geq 2) = 1 - e^{-\nu(H_i)}(1 + \nu(H_i)) = e^{-\nu(H_i)}(e^{\nu(H_i)} - 1 - \nu(H_i)) \leq e^{\nu(H_i)} - 1 - \nu(H_i).$$

Now take k large enough (s.t. $h(T^*)/k < 1$) then we have

$$e^{\nu(H_i)} - 1 - \nu(H_i) \leq \nu(H_i)^2 = h(T^*)^2/k^2,$$

thus

$$\mathbb{P}(\bigcup_{l=1}^{k} \{\mu(H_i) \geq 2\}) \leq \sum_{i=1}^{k} \mathbb{P}(\mu(H_i) \geq 2) \leq k h(T^*)^2/k^2 = h(T^*)^2/k \to 0 \text{ as } k \to \infty.$$

So we have that

$$\mathbb{P}(\exists t \in [0, T^*] \ \mu(\{t\} \times \{x : |x| \geq 1/n\} \geq 2) = 0$$

for all $n \in \mathbb{N}$, thus

$$\mathbb{P}(\exists t \in [0, T^*], \mu(\{t\} \times \mathbb{R}^d \geq 2) = 0,$$

what we wanted. $\square$

2.1.3 Construction of L^2

L^2 corresponds to the jumps greater than 1.

$$L_t^2 := \int_0^t \int_{\{|x|>1\}} x\mu(\mathsf{d}x).$$

By Proposition 2.4 part (ii), the characteristic function of L^2 is

$$\begin{aligned}\mathbb{E}(\exp(i\left\langle u, L_t^2\right\rangle)) &= \exp\int_0^t \int_{\{|x|>1\}} (e^{i\langle u,x\rangle}-1)\nu(\mathsf{d}s,\mathsf{d}x) \\ &= \exp\int_0^t \int_{\{|x|>1\}} (e^{i\langle u,x\rangle}-1)F_s(\mathsf{d}x)\mathsf{d}s. \qquad (12)\end{aligned}$$

2.1.4 Preliminaries for the construction of L^3

We argue along the lines of [Kyprianou, 2006] page 47-51.

Define the filtration $\tilde{\mathcal{F}}_t = \sigma(\mu(A) : A \in \mathcal{B}([0,t]) \times \mathcal{B}(\mathbb{R}^d))$ for $T \in [0, T^*]$. Let $\mathcal{M}_2$ be the set of all square integrable cádlág martingales adapted to the filtration $\tilde{\mathcal{F}}$. We can define a scalar product on this vector-space, let us denote it by $\langle .,. \rangle_M$, defined by

$$\langle X, Y\rangle_M := \mathbb{E}(X_{T^*}Y_{T^*}) \text{ for } X, Y \in \mathcal{M}_2.$$

Now we show that $\mathcal{M}_2$ is a Hilbert space. First I show that $\langle .,. \rangle_M$ is really a scalar product. It is clear, that $\langle .,. \rangle_M$ is a symmetric bilinear function. The only property left to show is: if$\langle X, X\rangle_M = 0$ then $X = 0$, this is a direct consequence of Doob's inequality:

$$\mathbb{E}(\sup_{0\le t\le T^*} X_t^2) \le 4\mathbb{E}(X_{T^*}^2) = 4\langle X, X\rangle_M = 0.$$

To prove that $\mathcal{M}_2$ is a Hilbert space, we need that it is complete. Let X_n be a Cauchy sequence in $\mathcal{M}_2$:

$$\langle X_n - X_m, X_n - X_m\rangle_M \to 0 \text{ as } n, m \to \infty,$$

which means $\mathbb{E}((X_{n,T^*} - X_{m,T^*})^2) \to 0$, thus X_{n,T^*} is a Cauchy sequence in the $\mathcal{L}_2$ space on Ω. $\mathcal{L}_2$ is complete, thus there is a square integrable random variable X_{T^*} such that $X_{n,T^*} \overset{\mathcal{L}_2}{\to} X_T^*$. Define

$$X_t := \mathbb{E}(X_{T^*}|\mathcal{F}_t),$$

then the process X is in $\mathcal{M}_2$, and $X^n \to X$ in the topology constructed on $\mathcal{M}_2$, where $X = (X_t)_{0\le t\le T^*}$.

2.1.5 Construction of L^3

Let

$$L_t^{\varepsilon,3} := \int_0^t \int_{\{x\in\mathbb{R}^d:1\geq|x|\geq\varepsilon\}} x(\mu-\nu)(\mathsf{d}s,\mathsf{d}x) \text{ for } t\in[0,T^*],$$

where $\int_s^t$ means that we are integrating on the set $(s,t]$ for $0\leq s\leq t\leq T^*$. By the definition, one can see that $L^{\varepsilon,3}$ is a cádlág process, and its jump measure is $\mu\big|_{[0,T^*]\times\{1\geq|x|\geq\varepsilon\}}$, since the correction term $-\int_0^\bullet\int_{\{1\geq|x|\geq\varepsilon\}}\nu(\mathsf{d}s,\mathsf{d}x)$ is a continuous function (and is finite by (6)).

Notice that for all $\varepsilon>0$, the process $L^{\varepsilon,3}$ is a time-inhomogeneous Lévy process; indeed it has stationary increments because it is given as an integral with respect to a Poisson random measure, and its characteristic function is:

$$\mathbb{E}(e^{i\langle u,L_t^{\varepsilon,3}\rangle}) = \exp\int_0^t\int_{\{1\geq|x|\geq\varepsilon\}}(e^{i\langle u,x\rangle}-1-i\langle u,x\rangle)F_s(\mathsf{d}x)ds,$$

by Proposition 2.4 (ii).

We can also see, that $L^{\varepsilon,3}$ is a martingale, since it has independent increments, and its expected value is constant 0, by (10), and also starts from 0. $L^{\varepsilon,3}$ is adapted to the filtration $\mathcal{F}$.

Now consider the sequence $L^{\varepsilon_n,3}$, where ε_n is some positive decreasing sequence converging to 0. I show, that $L^{\varepsilon_n,3}$ is a Cauchy sequence in $\mathcal{M}_2$. Let $n\leq m$ be two positive integers, then

$$\begin{aligned}\left\langle L^{\varepsilon_n,3}-L^{\varepsilon_m,3},L^{\varepsilon_n,3}-L^{\varepsilon_m,3}\right\rangle_M &= \mathbb{E}((L_{T^*}^{\varepsilon_n,3}-L_{T^*}^{\varepsilon_m,3})^2)\\ &= \mathbb{E}\left(\left(\int_0^{T^*}\int_{\varepsilon_n\geq|x|\geq\varepsilon_m}x(\mu-\nu)(\mathsf{d}s,\mathsf{d}x)\right)^2\right)\end{aligned} \tag{13}$$

Using (11), and the fact that

$$\mathbb{E}(\int_0^{T^*}\int_{\varepsilon_n\geq|x|\geq\varepsilon_m}x(\mu-\nu)(\mathsf{d}s,\mathsf{d}x))=0,$$

the value of the expectation 13 is:

$$\mathbb{E}(\int_0^{T^*}\int_{\varepsilon_n\geq|x|\geq\varepsilon_m}|x|^2\,\nu(\mathsf{d}s,\mathsf{d}x))\to 0 \text{ as } n,m\to\infty.$$

So we have that the sequence $L^{\varepsilon_n,3}$ is a Cauchy sequence in $\mathcal{M}_2$, thus there is a process L^3 s.t $L^{\varepsilon_n,3}\to L^3$ as $n\to\infty$. Notice that the limit exists for all positive decreasing sequences ε_n, thus the limit is independent of the sequence, thus we can conclude that $L^{\varepsilon,3}\to L^3$ as $\varepsilon\downarrow 0$. Thus

$$L_t^3 := \int_0^t\int_{n\mathbb{R}^d}x(\mu-\nu)(\mathsf{d}s,\mathsf{d}x)\in\mathcal{M}_2.$$

It remains to show the fact that the characteristic function of L^3 is

$$\mathbb{E}(e^{i\left\langle u,L_t^3\right\rangle}) = \exp\int_0^t\int_{\{|x|\leq 1\}}(e^{i\langle u,x\rangle}-1-i\langle u,x\rangle)F_s(\mathsf{d}x)\mathsf{d}s. \tag{14}$$

Proof:
Since $\left|e^{i\left\langle u,L_t^3\right\rangle}-e^{i\left\langle u,L_t^{\varepsilon,3}\right\rangle}\right|\leq 2$, thus by the Dominated Convergence Theorem,

$$\mathbb{E}(e^{i\left\langle u,L_t^{\varepsilon,3}\right\rangle})\to\mathbb{E}(e^{i\left\langle u,L_t^3\right\rangle}) \text{ as } \varepsilon\to 0.$$

On the other hand,

$$\int_0^t\int_{\{\varepsilon\leq|x|\leq 1\}}(e^{i\langle u,x\rangle}-1-i\langle u,x\rangle)F_s(\mathsf{d}x)\mathsf{d}s\to\int_0^t\int_{\{|x|\leq 1\}}(e^{i\langle u,x\rangle}-1-i\langle u,x\rangle)F_s(\mathsf{d}x)\mathsf{d}s,$$

by the Dominated Convergence Theorem, since $\int_0^{T^*}\int_{\mathbb{R}^d}(|x|^2\wedge 1)F_s(\mathsf{d}x)\mathsf{d}s<\infty$ and we have that $\left|e^{i\langle u,x\rangle}-1-i\langle u,x\rangle\right|\leq C|x|^2$ for some positive constant C. So taking limit on both sides of

$$\mathbb{E}(e^{i\left\langle u,L_t^{\varepsilon,3}\right\rangle}) = \exp\int_0^t\int_{\{\varepsilon\leq|x|\leq 1\}}(e^{i\langle u,x\rangle}-1-i\langle u,x\rangle)F_s(\mathsf{d}x)\mathsf{d}s,$$

we get the desired equality.

2.1.6 Proof of Theorem 2.2

Recall that L^1 is an Itô-integral constructed in section 2.1.1:

$$L_t^1=\int_0^t b_s'\mathsf{d}s+\int_0^t\sqrt{c_s}\mathsf{d}W_s.$$

L^2 is a process associated with the jumps of L greater than 1 constructed in section 2.1.3:

$$L_t^2=\int_0^t\int_{\{|x|>1\}}x\mu(\mathsf{d}s,\mathsf{d}x).$$

L^3 is an $\mathcal{L}^2$ limit of martingales constructed in section 2.1.3:

$$L_t^3=\int_0^t\int_{\{|x|\leq 1\}}x(\mu-\nu)(\mathsf{d}s,\mathsf{d}x).$$

We construct L, by

$$L:=L^1+L^2+L^3.$$

It remains to show that the characteristic function of L_t satisfies

$$\mathbb{E}(e^{i\langle u,L_t\rangle})=\exp(\int_0^t\left(i\langle u,b_s'\rangle-\frac{1}{2}\langle u,c_su\rangle+\int_{\mathbb{R}^d}(e^{i\langle u,x\rangle}-1-i\mathbf{1}_{|x|\leq 1}\langle u,x\rangle)F_s(\mathsf{d}x)\right)\mathsf{d}s),$$

and L has independent increments.

By the construction, we have that L_1 is independent of L_2, L_3, and by the basic properties of Poisson random measures, we have that L^2 and L^3 are also independent, thus L_1, L_2 and L_3 are independent, and we have

$$L_t = \int_0^t b'_s \mathsf{d}s + \int_0^t \sqrt{c_s}\mathsf{d}W_s + \int_0^t \int_{\{|x|>1\}} x\mu(\mathsf{d}s, \mathsf{d}x) + \int_0^t \int_{\{|x|\leq 1\}} x(\mu - \nu)(\mathsf{d}s, \mathsf{d}x). \quad (15)$$

Since L^1, L^2 and L^3 are independent, by substitution the characteristic function of L is:

$$\begin{aligned}
\mathbb{E}(e^{i\langle u,L_t\rangle}) &= \mathbb{E}(e^{i\left\langle u,L_t^1\right\rangle})\mathbb{E}(e^{i\left\langle u,L_t^2\right\rangle})\mathbb{E}(e^{i\left\langle u,L_t^3\right\rangle}) \\
&= \exp(\int_0^t \left(i\left\langle u, b'_s\right\rangle - \frac{1}{2}\left\langle u, c_s u\right\rangle + \int_{\mathbb{R}^d} (e^{i\langle u,x\rangle} - 1 - i\mathbf{1}_{|x|\leq 1}\left\langle u, x\right\rangle)F_s(\mathsf{d}x)\right)\mathsf{d}s). \\
&= \exp\int_0^t \theta_s(iu)\mathsf{d}s
\end{aligned}$$

We only have to prove that L has independent increments.

Since c_s is deterministic, we have that L^1 has independent increments, and L^2 and L^3 have also independent increments by the basic properties of Poisson random measure, thus L has independent increments. □

2.2 Properties of time-inhomogeneous Lévy processes

This section is based on [Kluge, 2005]. The theorems, propositions and assumptions are from section 1.3 in [Kluge, 2005].

The connections to the terminology of [Jacod & Shiryaev, 2003] is shown in the next subsection.

2.2.1 The semi martingale characteristics of L

We want to find the semi martingale characteristics of a time-inhomogeneous Lévy process L. The definition of the characteristics is rather complicated, and relies on some other definitions and theorems of [Jacod & Shiryaev, 2003].

First we need the following definitions about measures:

Definition 2.6 (Definition 1.6 in I.§1a [Jacod & Shiryaev, 2003])**.** *A random measure μ on $[0,T^*]\times\mathbb{R}^d$ is optional (resp. predictable), if the process $t \mapsto \int_0^t \int_{\mathbb{R}^d} V(s,x)\mu(ds, dx)$ is optional (resp. predictable) for every optional (resp. predictable) function V.*

Definition 2.7. *The compensator ν of an optional measure μ is a predictable measure if the following condition holds:*

$$\mathbb{E}(\int_0^t \int_{\mathbb{R}^d} W(s,x)\mu(ds, dx)) = \mathbb{E}(\int_0^t \int_{\mathbb{R}^d} W(s,x)\nu(ds, dx))$$

for every non-negative $\mathcal{B}([0,T^*]) \times \mathcal{P}$-measurable function W.

Remark In this thesis, the reader can think about the compensator of μ as the 'expected value' of the measure μ, since in most cases, the we will use random measures which have deterministic compensators.

Remark By Theorem 1.8 in II.§1a [Jacod & Shiryaev, 2003] tells us that the compensator of an optional measure exists. Moreover, μ (the jump measure of L) is a Poisson random measure, hence Proposition 1.21. in II.§1c in [Jacod & Shiryaev, 2003], gives that the compensator of μ is its density measure ν.

Definition 2.8. *A function $h : \mathbb{R}^d \to \mathbb{R}^d$ is a **truncation function**, if h is bounded, has a compact support and there is a neighborhood of 0 on which $h(x) = x$.*

Remark Usually we will use the function $x \mapsto \mathbf{1}_{|x|\leq 1}x$ as a truncation function.

For a d-dimensional process X and h a truncation function, define the process $X(h)$ by

$$X(h)_t := X_t - \sum_{s\leq t}(\Delta X_s - h(\Delta X_s)).$$

Note that $\Delta X(h)_t = \Delta h(X)_t$, which gives that the size of the jumps of $X(h)$ is bounded by a constant K, since h has a compact support.

Moreover, if X is a semi martingale, then $X(h)$ is also a semi martingale, and its jumps are bounded, thus Lemma 4.24 I.§4c in [Jacod & Shiryaev, 2003] gives that $X(h)$ is a special semi martingale, which means, that $X(h)$ has the following decomposition:

$$X(h) := X_0 + M(h) + B(h) \tag{16}$$

where $M(h)$ is a local martingale starting from 0, $B(h)$ is a predictable process with finite variation.

Now we can define the characteristics of a semi martingale X:

Definition 2.9 (Definition 2.6 in II.§2a [Jacod & Shiryaev, 2003])**.** *The characteristics of a d-dimensional semi martingale X associated with the truncation function $h(x)$ is a triplet (B', C, ν), where*

(i) $B' = B'(h)$ is a d-dimensional predictable process with bounded variation, which is the process appearing in decomposition (16).

(ii) C is a $\mathbb{R}^{d\times d}$ valued predictable process with bounded variation, namely

$$C^{ij} = \left\langle X^{i,c}, X^{j,c}\right\rangle_p,$$

where $\langle ., . \rangle_p$ is the predictable quadratic covariation.

(iii) ν^X *is a predictable random measure, namely the compensator of the jump measure of* X *(see Definition 2.4).*

At this point, it is handy to define the following set of processes, which will be used in the Appendix:

Definition 2.10. *A* $\mathbb{R}^d$ *valued process* X *on a finite time horizon* $[0, T^*]$ *is a PIIAC (process with independent increments and absolutely continuous characteristics), if the following conditions hold*

(i) X *has independent increments*

(ii) there is a version of the semi martingale characteristics of X *associated with the truncation function* h *denoted by* (B', C, ν)*, for which*

- *the paths of the processes* B' *and* C *are absolutely continuous with respect to the Lebesgue measure on* $[0, T^*]$
- *the set of all fixed discontinuity times is empty:* $\forall t \in [0, T^*]$ $\nu(\{t\} \times \mathbb{R}^d) = 0$

The following theorem can be found in [Kluge, 2005] as Lemma 1.2

Theorem 2.3. *The semi martingale characteristics associated with the truncation function* $h(x) := x\mathbf{1}_{|x|\leq 1}$ *are:*

$$\begin{aligned} B'_t : &= \int_0^t b'_s \mathsf{d}s \\ C_t : &= \int_0^t c_s \mathsf{d}s \\ \nu(\mathsf{d}s, \mathsf{d}x) &= F_s(\mathsf{d}x)ds \end{aligned} \tag{17}$$

for $t \in [0, T^*]$.

Proof Examine the representation (7):

$$L_t = \int_0^t b'_s \mathsf{d}s + \int_0^t \sqrt{c_s}\mathsf{d}W_s + \int_0^t \int_{\{|x|>1\}} x\mu(\mathsf{d}s, \mathsf{d}x) + \int_0^t \int_{\{|x|\leq 1\}} x(\mu - \nu)(\mathsf{d}s, \mathsf{d}x).$$

We can see that

$$L(h)_t = \int_0^t b'_s \mathsf{d}s + \int_0^t \sqrt{c_s}\mathsf{d}W_s + \int_0^t \int_{\{|x|\leq 1\}} x(\mu - \nu)(\mathsf{d}s, \mathsf{d}x),$$

from which we can deduce that $B'_t := \int_0^t b'_s \mathsf{d}s$.

As for the second semi martingale characteristic C, we can see that the continuous martingale part of L is $L^c = \int_0^\bullet \sqrt{c_s}\mathsf{d}W_s$, which is an Itô integral, thus its predictable quadratic variation is the same as its quadratic variation, which is $C_t = \int_0^t c_s \mathsf{d}s$.

For the third semi martingale characteristic, we have to find the compensator of μ. μ is a Poisson random measure, which is by Proposition 1.21. in II.§1c in [Jacod & Shiryaev, 2003], is the intensity measure of μ, namely ν, which can be written as $\nu(\mathsf{d}s, \mathsf{d}x) = F_s(\mathsf{d}x)\mathsf{d}s$. $\square$

The consequence of Theorem 2.3:

Corollary 2.2. *A process is a time-inhomogeneous Lévy process if and only if it is a PIIAC.*

2.2.2 Under the Assumption $\mathbb{EM}$

In the proceeding, we will work with time-inhomogeneous Lévy processes which satisfy the assumption:

Assumption 1 ($\mathbb{EM}$)**.** *There are constants $M, \varepsilon > 0$ such that for every $u \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d$*

$$\int_0^{T^*} \int_{\{|x|>1\}} e^{\langle u,x \rangle} F_s(dx) ds < \infty.$$

Without loss of generality, we will also assume that $\int_{\{|x|>1\}} e^{\langle u,x \rangle} F_s(dx) < \infty$ for all $s \in [0, T^*]$, $u \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d$.

Later on it will turn out, that we have this condition on L to ensure that the moment generating function is well defined for a whole strip in $\mathbb{C}^d$ (here a strip is a set of the form $\{z \in \mathbb{C}^d : |\Re z^i| \leq C \text{ for } i = 1, \ldots, d\}$ where C is some positive constant).

In some cases, it is more handy to work with the following assumption:

Assumption 2 ($\mathbb{EM}'$)**.** *There are constants $M, \varepsilon > 0$ such that*

$$\int_0^{T^*} \int_{\{|x|>1\}} \exp\left((1+\varepsilon)M \sum_{i=1}^d |x_i| \right) F_s(dx) ds < \infty.$$

Without loss of generality, we will also assume that

$$\int_{\{|x|>1\}} \exp\left((1+\varepsilon)M \sum_{i=1}^d |x_i| \right) F_s(dx) < \infty$$

for all $s \in [0, T^*]$.

Proposition 2.6. *The assumptions $\mathbb{EM}$ and $\mathbb{EM}'$ are equivalent. More precisely L satisfies $\mathbb{EM}$ with M, ε if and only if L satisfies $\mathbb{EM}'$ with M, ε.*

Proof $\mathbb{EM}' \Rightarrow \mathbb{EM}$: If $u \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d$, then the point wise inequality

$$\langle u, x\rangle \leq (1+\varepsilon)M \sum_{i=1}^{d} |x_i|,$$

gives

$$\int_{\{|x|>1\}} \exp\left(\langle u, x\rangle\right) F_s(\mathsf{d}x)\mathsf{d}s \leq \int_{\{|x|>1\}} \exp\left((1+\varepsilon)M \sum_{i=1}^{d} |x_i|\right) F_s(\mathsf{d}x)\mathsf{d}s.$$

$\mathbb{EM} \Rightarrow \mathbb{EM}'$: For each $x \in \mathbb{R}^d$, there is a $v \in \{-1, 1\}^d$ such that

$$\langle (1+\varepsilon)Mv, x\rangle = (1+\varepsilon)M \sum_{i=1}^{d} |x_i|,$$

thus

$$\int_{\{|x|>1\}} \exp\left((1+\varepsilon)M \sum_{i=1}^{d} |x_i|\right) F_s(\mathsf{d}x)\mathsf{d}s \leq \sum_{v\in\{-1,1\}^d} \int_{\{|x|>1\}} \exp\left(\langle (1+\varepsilon)Mv, x\rangle\right) F_s(\mathsf{d}x)\mathsf{d}s.$$

From these two inequalities, the statement follows. □

We modify the drift term in order to get a simpler formula:

Proposition 2.7. *Under the assumption* $\mathbb{EM}$*, the process* L *can be written in the form:*

$$L_t = \int_0^t b_s ds + \int_0^t \sqrt{c_s} dW_s + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu)(ds, dx), \tag{18}$$

with the definition

$$b_s := b'_s + \int_{\{|x|>1\}} x F_s(\mathsf{d}x).$$

Proof Clearly from assumption $\mathbb{EM}$, we can deduce that

$$\int_0^{T^*} \int_{\{|x|>1\}} |x| F_s(\mathsf{d}x)\mathsf{d}s < \infty,$$

a straightforward computation gives desired form. □

Remark We will often change between the drift terms b' and b, since in [Jacod & Shiryaev, 2003] most of the theorems are stated with a truncation function.

Definition 2.11. *We call the representation in Proposition 2.7 of a time-inhomogeneous Lévy process* L*,* ***the canonical representation*** *of* L*.*

Recall Definition 2.9, where we introduced the semi martingale characteristics associated with a truncation function. For convenience, define the semi martingale characteristics (without the term 'associated with the truncation function'!) of a semi martingale:

Definition 2.12. *Let X be a d-dimensional semi martingale X which can be written in the form*

$$X_t = B_t + X_t^c + \int_0^t \int_{\mathbb{R}^d} x(\mu^X - \nu^X)(\mathsf{d}x, \mathsf{d}s),$$

where B is a d-dimensional predictable process B with bounded variation, X^c is the continuous martingale part of X, μ^X is the jump measure of X, and ν^X is the compensator of μ^X.

Then the semi martingale characteristics of X is the triplet (B, C, ν^X), where B and ν^X are as above, C is an $\mathbb{R}^{d\times d}$ valued predictable process with bounded variation, namely

$$C^{i,j} = \left\langle X^{i,c}, X^{j,c} \right\rangle_p,$$

where $\langle ., . \rangle_p$ is the predictable quadratic covariation.

Corollary 2.3. *Under the assumption $\mathbb{EM}$, the semi martingale characteristics of L are:*

$$\begin{aligned} B_t : &= \int_0^t b_s \mathsf{d}s \\ C_t : &= \int_0^t c_s \mathsf{d}s \\ \nu(\mathsf{d}s, \mathsf{d}x) &= F_s(\mathsf{d}x) ds \end{aligned} \tag{19}$$

for $t \in [0, T^*]$.

Proof Consequence of Theorem 2.3. □

Our next aim is to compute some expectations involving Lévy processes.

Recall the definition of cumulant in Definition 2.2. The cumulant was defined only for purely imaginary vectors in $\mathbb{C}^d$, but with the Assumption ($\mathbb{EM}$) and the condition (6), we can extend θ_s for the following subset of complex d-dimensional space $\left\{z \in \mathbb{C}^d : \Re z \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d\right\}$, and get

$$\theta_s(z) = \langle z, b'_s \rangle + \frac{1}{2} \langle z, c_s z \rangle + \int_{\mathbb{R}^d} (e^{\langle z,x \rangle} - 1 - \mathbf{1}_{|x|\leq 1} \langle z, x \rangle) F_s(\mathsf{d}x) \text{ for } u \in \mathbb{R}^d, s \in [0, T^*].$$

Further on, we will use the following form of the cumulant:

Lemma 2.1. *Under the Assumption $\mathbb{EM}$, the cumulant θ_s can be written in the form*

$$\theta_s(z) = \langle z, b_s \rangle + \frac{1}{2} \langle z, c_s z \rangle + \int_{\mathbb{R}^d} (e^{\langle z,x \rangle} - 1 - \langle z, x \rangle) F_s(\mathsf{d}x) \text{ for } u \in \mathbb{R}^d, s \in [0, T^*].$$

Proof It is a direct consequence of Assumption $\mathbb{EM}$. □

Proposition 2.8. *Let L be a time-inhomogeneous Lévy process satisfying* ($\mathbb{EM}$) *with constants M and ε, then for any $z \in \mathbb{C}^d$ with $|\Re z| \leq M$ and $t \in [0, T^*]$, we have*

$$\mathbb{E}(e^{\langle z, L_t \rangle}) = \exp(\int_0^t \theta_s(z) \mathsf{d}s) \text{ for } t \in [0, T^*]. \tag{20}$$

Proof The proof can be found in [Kluge, 2005] as Lemma 1.8. This Lemma is an extension to time-inhomogeneous Lévy processes of Theorem 25.17 in [Sato, 1999]. The idea is to make a time-homogeneous Lévy process X, such that $X_1 = L_t$, then use Theorem 25.17 in [Sato, 1999] to get the desired statement.

The the distribution of L_t is infinitely divisible with Lévy-Khinchine triplet $(B_t, C_t, \nu([0,t], dx))$, where B, C, ν is as defined in (18). We have that the characteristic function is

$$\mathbb{E}(e^{i\langle u, L_t \rangle}) = \exp(\int_0^t \left(i \langle u, b_s \rangle - \frac{1}{2} \langle u, c_s u \rangle + \int_{\mathbb{R}^d} (e^{i\langle u,x \rangle} - 1 - i\mathbf{1}_{|x|\leq 1} \langle u, x \rangle) F_s(\mathsf{d}x) \right) \mathsf{d}s),$$

thus

$$\mathbb{E}(e^{i\langle u, L_t \rangle}) = \exp(i \langle u, B_t \rangle - \frac{1}{2} \langle u, C_s u \rangle + \int_{\mathbb{R}^d} (e^{i\langle u,x \rangle} - 1 - i\mathbf{1}_{|x|\leq 1} \langle u, x \rangle) \int_0^t (F_s(\mathsf{d}x)\mathsf{d}s)).$$

The last equality is valid, since the value of the double integral is finite, thus we can use Fubini's theorem to interchange the two integrals.

There is a Lévy process X, with $X_1 = L_t$ in distribution, since there is a one-to-one correspondence between infinitely divisible distributions and Lévy processes. For further details see [Sato, 1999] Theorem 7.10. Theorem 25.17 in [Sato, 1999] applied for X gives the desired equality. □

Now we can prove the following proposition using the previous one, which is the main goal of this section:

Proposition 2.9. *Let L be a time-inhomogeneous Lévy process satisfying* $\mathbb{EM}$ *with constants M and ε, and take $f : [0, T^*] \to \mathbb{C}^d$ a continuous function for which $|\Re f(s)| \leq M$ for $s \in [0, T^*]$ where M is a positive constant. Then*

$$\mathbb{E}(\exp \int_0^t f(s) \mathsf{d}L_s) = \exp\left(\int_0^t \theta_s(f(s)) \mathsf{d}s \right) \text{ for } t \in [0, T^*]. \tag{21}$$

Proof This is Proposition 1.9 in [Kluge, 2005]. Since f is continuous on $[0, T^*]$, thus it is bounded, so the stochastic integral $\int_0^t f(s) \mathsf{d}L_s$ is well defined, and it is the limit of

$$\sum_{j=0}^{n-1} \left\langle f(t_j^n), L_{t_{j+1}^n} - Lt_j^n \right\rangle \text{ as } n \to \infty,$$

where $t_j^n = jt/n$ by [Jacod & Shiryaev, 2003] §I Proposition 4.44 in measure.

By the independence of increments of L, we have that

$$\begin{aligned}\mathbb{E}\exp\left\langle f(t_j^n), L_{t_{j+1}^n}\right\rangle &= \mathbb{E}\left(\exp\left\langle f(t_j^n), L_{t_{j+1}^n} - L_{t_j^n}\right\rangle \exp\left\langle f(t_j^n), L_{t_j^n}\right\rangle \Big| \mathcal{F}_{t_j^n}\right) \\ &= \mathbb{E}\exp\left\langle f(t_j^n), L_{t_{j+1}^n} - L_{t_j^n}\right\rangle \mathbb{E}\exp\left\langle f(t_j^n), L_{t_j^n}\right\rangle .\end{aligned}$$

Hence

$$\begin{aligned}\mathbb{E}\exp\sum_{j=0}^{n-1}\left\langle f(t_j^n), L_{t_{j+1}^n} - L_{t_j^n}\right\rangle &= \prod_{j=0}^{n}\mathbb{E}\exp\left\langle f(t_j^n), L_{t_{j+1}^n} - L_{t_j^n}\right\rangle \\ &= \prod_{j=0}^{n}\frac{\mathbb{E}\exp\left\langle f(t_j^n), L_{t_{j+1}^n}\right\rangle}{\mathbb{E}\exp\left\langle f(t_j^n), L_{t_{j+1}^n}\right\rangle} \\ &= \exp\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\theta_s(f(t_j))\mathsf{d}s.\end{aligned}$$

Since the functions θ_s and f are continuous, thus $\sum_{j=0}^{n-1}\mathbf{1}_{s\in(t_j^n,t_{j+1}^n]}\theta_s(f(t_j)) \to \theta_s(f(s))$ point wise. Since f is continuous, we have that $|f(s)| \le K$ for some $K > 0$.

$$\begin{aligned}|\theta_s(u)| &\le |u|\,|b_s| + \frac{1}{2}\|c_s\|\,|u|^2 + \int_{\mathbb{R}^d}\left|e^{\langle u,x\rangle} - 1 - \langle u,x\rangle\right| F_s(\mathsf{d}x)\mathsf{d}s \qquad (22)\\ &\le K_1\left(|b_s| + \|c_s\| + \int_{|x|\le 1}|x|^2 F_s(\mathsf{d}x)\mathsf{d}s + \int_{|x|>1}\exp(M\sum_{j=1}^{d}|x_j|)F_s(\mathsf{d}x)\mathsf{d}s\right)\end{aligned}$$

for some positive constant K_1 for $|\Re u_j| \le M$ for $j = 1, \dots, d$ and $|u| \le K$ (such K_1 does exist, since the integrand in (22) is $O(|u|^2\,|x|^2)$ if $|x| \le 1$, and it is $O(\exp(M\sum_{j=1}^{d}|x_j|))$ if $|x| > 1$). Define $g(s)$ by the RHS of the last inequality. By Assumption $\mathbb{EM}'$, we can conclude that the function g is integrable, and $|\theta_s(f(u))| \le g(s)$ for $u \in [0, T^*]$, thus by the Dominated Convergence Theorem we can conclude that

$$\sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}}\theta_s(f(t_j))\mathsf{d}s \to \int_0^t\theta_s(f(s))\mathsf{d}s \text{ as } n\to\infty.$$

Now we examine

$$X_n := \exp\sum_{j=0}^{n-1}\left\langle f(t_j^n), L_{t_{j+1}^n} - L_{t_j^n}\right\rangle \to \exp\int_0^t f(s)\mathsf{d}L_s \text{ as } n\to\infty.$$

We know, that it converges to $\exp\int_0^t f(s)\mathsf{d}L_s$ in measure. This is not enough, since we want convergence in $\mathcal{L}_1$. To conclude that, it is enough to prove that the sequence of processes

$$\exp\sum_{j=0}^{n-1}\left\langle f(t_j^n), L_{t_{j+1}^n} - L_{t_j^n}\right\rangle \to \exp\int_0^t f(s)\mathsf{d}L_s$$

is uniformly integrable. We can use the following equivalent definition of uniform integrability:

The set of random variables $\mathcal{H}$ is uniformly integrable, if and only if $\sup_{X\in\mathcal{H}} \mathbb{E}\,|X| < \infty$, and for all $\varepsilon > 0$ there is a $\delta > 0$ for which if A is an event with $\mathbb{P}(A) < \delta$, then $\mathbb{E}\,|X|\,\mathbf{1}_A < \varepsilon$ for $X \in \mathcal{H}$.

In our case, the condition $\sup_{n\in\mathbb{N}} \mathbb{E}\,|X_n| < \infty$ is satisfied, e.g because the fact that $\mathbb{E}(X_n) \le \exp \int_0^t g(s)\mathsf{d}s$. By the Hölder inequality, we have

$$\mathbb{E}\,|X_n|\,\mathbf{1}_A \le \left(\mathbb{E}\,|X_n|^{1+\varepsilon}\right)^{1/(1+\varepsilon)} (\mathbb{E}\mathbf{1}_A)^{\frac{1}{q}} \le \left(\sup_{n\in\mathbb{N}} \mathbb{E}\,|X_n|^{1+\varepsilon}\right)^{1/(1+\varepsilon)} \delta^{1/q},$$

where $\frac{1}{1+\varepsilon} + \frac{1}{q} = 1$. Just as we have defined the function g, we have that

$$|\theta_s(u)| \le K_1' \left(|b_s| + \|c_s\| + \int_{|x|\le 1} |x|^2 F_s(\mathsf{d}x)\mathsf{d}s + \int_{|x|>1} \exp((1+\varepsilon)M \sum_{j=1}^{d} |x_j|)F_s(\mathsf{d}x)\mathsf{d}s\right),$$

with positive constant K_1', for $|\Re u| \le (1+\varepsilon)M$ and $|u| \le (1+\varepsilon)K$ from which we get $\sup_{n\in\mathbb{N}} \mathbb{E}\,|X|^{1+\varepsilon} < \infty$.

Taking limit as $n \to \infty$ in the equation

$$\mathbb{E}\exp \sum_{j=0}^{n-1} \left\langle f(t_j^n), L_{t_{j+1}^n} - L_{t_j^n} \right\rangle = \exp \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \theta_s(f(t_j))\mathsf{d}s,$$

gives the desired formula. □

3 Lévy models

After introducing the mathematical tools, we can carry on with formulating the models. The first model, we start with the Heath-Jarrow-Morton framework (for further details see Chapter 11 in [Musiela & Rutkowski, 2005]). We model the forward rates, we generalize the ordinary model with a Brownian motion driving process by allowing time-inhomogeneous Lévy processes as drivers. All the models presented below can be found in [Kluge, 2005] with more detail. I also took the notation from [Kluge, 2005].

3.1 HJM formulation: Lévy Forward Rate Model

We postulate that the forward rates evolve according to the following stochastic differential equation for all $T \in [0, T^*]$ and $0 \le t \le T$:

$$\mathsf{d}f_{t,T} = \alpha(t,T)\mathsf{d}t - \sigma(t,T)\mathsf{d}L_t, \tag{23}$$

where the coefficients α, σ are deterministic functions $\alpha : [0, T^*]^2 \to \mathbb{R}$ and $\sigma : [0, T^*]^2 \to \mathbb{R}^d$. Note that in the equation (23), we do not use the values $\alpha(t, T)$ and $\sigma(t, T)$ for $t > T$. Hence without loss of generality, we can define $\alpha(t, T) = 0$ and $\sigma(t, T) = 0$ for $0 \leq T < t \leq T^*$. We call α as the drift, and σ as the volatility.

Remark The sign of the stochastic term is chosen to be negative in order to make the formulas simpler.

We are working on a finite time horizon $[0, T^*]$, and on a stochastic basis $(\Omega, (\mathcal{F}_t)_{0 \leq t \leq T^*}, \mathcal{F}, \mathbb{P})$, where $(\mathcal{F}_t)_{0 \leq t \leq T^*}$ is a filtration and $\mathcal{F}_{T^*} = \mathcal{F}$, $\mathbb{P}$ is a probability measure. $L = (L_t)_{0 \leq t \leq T^*}$ is a d-dimensional time-inhomogeneous Lévy process.

Referring to Section 1.1 and 1.2 for the concepts, a crucial observation about bond prices in this setting, to be proved in Appendix A.1, is as follows:

Proposition 3.1. *The discounted bond price processes are given by the formula*

$$\frac{B(t, T)}{B_t} = B(0, T) \exp\left(-\int_0^t A(s, T) ds + \int_0^t \Sigma(s, T) dL_s \right), \tag{24}$$

where

$$A(s, T) := \int_{s \wedge T}^T \alpha(s, S) dS \text{ and } \Sigma(s, T) := \int_{s \wedge T}^T \sigma(s, S) dS.$$

The next step is to set the parameters of the model such that our starting probability measure $\mathbb{P}$ becomes a risk neutral measure, (i.e. the discounted bond price processes $t \mapsto B(t, T)/B_T$ are martingales for all $T \in [0, T^*]$ under the measure $\mathbb{P}$), hence the model admits no arbitrage.

It turns out, that we need extra assumptions for computations:

Assumptions:

- L satisfies ($\mathbb{EM}$) (see Section 2.2.2) with constants ε, M,
- With the constant M from ($\mathbb{EM}$) for $i = 1, \ldots, d$:

$$\left|\Sigma(s, T)^i\right| \leq M \text{ for } 0 \leq s \leq T \leq T^*.$$

Proposition 3.2. *Under the assumptions of the model, with the* ***drift condition***

$$A(s, T) = \theta_s(\Sigma(s, T)) \text{ for } 0 \leq s \leq T \leq T^*, \tag{25}$$

the model is arbitrage free, moreover $\mathbb{P}$ is an RNM.

Proof Define

$$X_t := \int_0^t \Sigma(s, T) \mathsf{d}L_s \text{ for } t \in [0, T^*].$$

Since L has independent increments, so does X. Thus for $0 \le s \le t \le T^*$:

$$\begin{aligned}\mathbb{E}(e^{X_t}) &= \mathbb{E}(\mathbb{E}(e^{X_t - X_s} e^{X_s} | \mathcal{F}_s)) \\ &= \mathbb{E}(e^{X_s}\mathbb{E}(e^{X_t - X_s} | \mathcal{F}_s)) \\ &= \mathbb{E}(e^{X_s}\mathbb{E}(e^{X_t - X_s})) \\ &= \mathbb{E}(e^{X_s})\mathbb{E}(e^{X_t - X_s}).\end{aligned}$$

Also, by the last equation, we get

$$\mathbb{E}(e^{X_t}|\mathcal{F}_s) = e^{X_s}\mathbb{E}(e^{X_t - X_s}|\mathcal{F}_s) = e^{X_s}\mathbb{E}(e^{X_t - X_s}) = e^{X_s}\frac{\mathbb{E}(e^{X_t})}{\mathbb{E}(e^{X_s})},$$

which shows that

$$t \mapsto \frac{e^{X_t}}{\mathbb{E}(e^{X_t})}, \ t \in [0, T^*]$$

is a martingale.

By the assumptions we made, the conditions of Proposition 2.9 are satisfied, hence $\mathbb{E}(e^{X_t}) = \exp\left(\int_0^t \theta_s(\Sigma(s,T))\mathsf{d}s\right)$, thus

$$t \mapsto \exp\left(-\int_0^t \theta_s(\Sigma(s,T))\mathsf{d}s + \int_0^t \Sigma(s,T)\mathsf{d}L_s\right), \ t \in [0, T^*]$$

is a martingale under $\mathbb{P}$. Thus take

$$A(s,T) = \theta_s(\Sigma(s,T)),$$

then $\frac{B(t,T)}{B_t}$ is a martingale, i.e. $\mathbb{P}$ is a RNM. □

Remark Note that, this is the only deterministic choice for A to assure that $\frac{B(t,T)}{B_t}$ is a martingale.

I finish this section with theorem on the completeness of the Forward Rate Model, the proof is moved to the Appendix since its complexity.

Theorem 3.1. *Under the drift condition and the assumptions of the model if*

- *the dimension of the driving process L_t is* 1*, or*
- *the dimension of the vector space span$(\Sigma(t,T) : t \le T)$ is at most* 1 *for almost all T,*

then the model is complete.

Proof See Appendix A.3. □

3.1.1 Under the forward measure

As we saw in Section 1.3, it is easier to price options under the appropriate forward measure. By this reason, we will examine the distribution of the driving process L under the forward measure. This is to be done with the canonical representation of Lévy processes described in Section 2.2.2.

Recall from Proposition 2.7:

$$L_t = \int_0^t b_s \mathsf{d}s + \int_0^t \sqrt{c_s} \mathsf{d}W_s + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu)(\mathsf{d}s, \mathsf{d}x),$$

where b_s is a deterministic vector in $\mathbb{R}^d$, c_s is a deterministic non-negative definite symmetric $d \times d$ matrix, $\sqrt{c_s}$ is the measurable version of the square root of c_s, μ is the random measure of the jumps of L, and ν is the compensator of μ. Proposition 3.1 combined with the drift condition (25) gives

$$\frac{\mathsf{d}\mathbb{P}_T}{\mathsf{d}\mathbb{P}} = \frac{1}{B_T B(0,T)} = \exp\left(- \int_0^T \theta_s(\Sigma(s,T))\mathsf{d}s + \int_0^T \Sigma(s,T)\mathsf{d}L_s \right).$$

Proposition 3.3. *With the drift condition, under the forward measure $\mathbb{P}_T$, the semi-martingale characteristics of L are given by*

$$\begin{aligned} B_s^T &= B_s + \int_0^s c_u \Sigma(u,T) du + \int_0^s \int_{\mathbb{R}^d} x(e^{\langle \Sigma(u,T), x\rangle} - 1)\nu(ds, dx) \\ C_s^T &= C_s \\ \nu^T(ds,dx) &= e^{\langle \Sigma(s,T), x\rangle}\nu(ds,dx). \end{aligned} \tag{26}$$

Moreover the $\mathbb{P}$ Brownian motion W under $\mathbb{P}_T$ becomes $W_t = W_t^T + \int \sqrt{c_s}\Sigma(s,T^*)ds$, where W^T is a Brownian motion under $\mathbb{P}^T$.

Proof See Appendix A.2. □

The immediate consequence is:

Corollary 3.1. *With the drift condition, the characteristics of L under $\mathbb{P}_T$ are:*

$$\begin{aligned} b_s^T &= b_s + c_s\Sigma(s,T) + \int_{\mathbb{R}^d} x(e^{\langle \Sigma(s,T), x\rangle} - 1)\nu(\{s\}, dx) \\ c_s^T &= c_s \\ F_s^T(dx) &= e^{\langle \Sigma(s,T), x\rangle} F_s(dx). \end{aligned} \tag{27}$$

Remark From the corollary, we know, that L under the forward measure has deterministic characteristics, moreover, they are absolutely continuous and L still has independent increments. Hence L under the forward measure is a PIIAC, which is an other way of saying that L is a time-inhomogeneous Lévy process by Corollary 2.2.

So after changing to the forward measure, the driving process becomes an other time-inhomogeneous Lévy processes, which is not true if L is a Lévy process.

3.2 Forward Process Model

In the upcoming model, we assume that on the market there are only finitely many bonds. We will model the forward processes in stead of forward rates.

We have n bonds with maturity dates $0 = T_0 < T_1 < T_2 < \ldots < T_n = T^*$. For convenient notation define $T_i^* = T_{n-i}$ for $i = 0, \ldots, n$. The forward process is:

$$F(t, T_k, T_{k+1}) = \frac{B(t, T_k)}{B(t, T_{k+1})} \text{ for } k = 1, \ldots, n-1 \text{ and } t \in [0, T_k].$$

Remark Note that if we model these $n-1$ forward processes, then we will know all bond prices, except $B(t, T^*)$ for $t \in (T_1^*, T^*]$.

3.2.1 Outline of the model

We start with a d-dimensional time-inhomogeneous Lévy process L^{T^*}, as a driving process under the forward measure $\mathbb{P}_{T^*}$ associated with maturity date T^*. We will start with postulating:

$$F(t, T_1^*, T^*) = F(0, T_1^*, T^*) \exp(\int_0^t \lambda(s, T_1^*) \mathsf{d} L_s^{T^*}).$$

Then we derive a condition under which $F(., T_1^*, T^*)$ is a martingale in the form as in the previous model, then we change to a forward measure with maturity date T_1^*. We postulate that

$$F(t, T_2^*, T_1^*) = F(0, T_2^*, T_1^*) \exp(\int_0^t \lambda(s, T_2^*) \mathsf{d} L_s^{T_1^*}),$$

where $L^{T_1^*}$ is the new driving process, which differs from L^{T^*} by a deterministic drift term, which ensures that $F(., T_2^*, T_1^*)$ is a martingale under $\mathbb{P}_{T_1^*}$. Then we iterate the procedure.

For the precise computation, we need the following assumptions:

3.2.2 Assumptions

(i) L^{T^*} satisfies $\mathbb{EM}$ (see Section 2.2.2) with constants M and ε.

(ii) The bond prices at time 0 are given, moreover, $B(0, T_k)$ is a strictly positive non-increasing sequence.

(iii) For each maturity date T_k $(k = 1, \ldots, n-1)$, there are deterministic functions $\lambda(., T_k) : [0, T^*] \to \mathbb{R}^d$ which satisfy:

$$\left| \sum_{i=1}^{k} \lambda^j(s, T_i^*) \right| \leq M \text{ for } s \in [0, T_k^*] \text{ and } j = 1, \ldots, d, \; k = 1, \ldots, n-1,$$

where M is the constant from $\mathbb{EM}$. Without loss of generality, we further specify that $\lambda(s, T_i) = 0$ for $s > T_i$.

3.2.3 Construction of the model

The process L^{T^*} has canonical representation under the forward measure $\mathbb{P}_{T^*}$:

$$L_t^{T^*} = \int_0^t b_s^{T^*} \mathsf{d}s + \int_0^t \sqrt{c_s} \mathsf{d}W_s^{T^*} + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu^{T^*})(\mathsf{d}s, \mathsf{d}x).$$

We start with modeling the forward process with maturity date T^*:

$$F(t, T_1^*, T^*) = F(0, T_1^*, T^*) \exp(\int_0^t \lambda(s, T_1^*) \mathsf{d}L_s^{T^*}).$$

As we saw in Proposition 1.2, we have to ensure that $F(., T_1^*, T^*)$ is a martingale, thus we want, that $\exp(\int_0^t \lambda(s, T_1^*) \mathsf{d}L_s^{T^*})$ is a martingale. Recall that in the description of the last model, in Proposition 3.2, we gave a condition on the parameters to achieve this. Hence take $\Sigma(s, T) = \lambda(s, T_i^*)$ then we have that

$$\exp(-\int_0^{\bullet} \theta_s(\lambda(s, T_1^*)) \mathsf{d}s + \int_0^{\bullet} \lambda(s, T_1^*) \mathsf{d}L_s^{T^*})$$

is a martingale. We want $\exp(\int_0^t \lambda(s, T_1^*) \mathsf{d}L_s^{T^*})$ to be a martingale, which is true if and only if $\theta_s(\lambda(s, T_1^*)) = 0$. This equation gives the following necessary and sufficient condition on $b_s^{T^*}$:

$$\left\langle \lambda(s, T_1^*), b_s^{T^*} \right\rangle = -\frac{1}{2} \left\langle \lambda(s, T_1^*), c_s^{T^*} \lambda(s, T_1^*) \right\rangle - \int_{\mathbb{R}^d} \left(e^{\langle \lambda(s, T_1^*), x \rangle} - 1 - \langle \lambda(s, T_1^*), x \rangle \right) F_s^{T^*}(\mathsf{d}x),$$

for which we will refer to as *the drift condition in the Forward Rate Model.* Note that we can always find such b^{T*}, since if $\lambda(s, T_1^*) \neq 0$, then clearly we can find such b, otherwise the condition vacuously true.

Now we can change to the forward measure associated with date T_1^*, and see, how the process L^{T^*} changes. Since

$$\frac{\mathsf{d}\mathbb{P}_{T_1^*}}{\mathsf{d}\mathbb{P}} = \frac{1}{B(0, T_1^*) B_{T_1^*}} = \frac{F(0, T_1^*, T_1^*)}{B_{T_1^*}},$$

and

$$\frac{\mathsf{d}\,\mathbb{P}_{T^*}|_{\mathcal{F}_{T_1^*}}}{\mathsf{d}\,\mathbb{P}|_{\mathcal{F}_{T_1^*}}} = \frac{B(T_1^*, T^*)}{B(0, T^*) B_{T_1^*}} = \frac{F(T_1^*, T_1^*, T^*)}{B_{T_1^*}},$$

and

$$\frac{\mathsf{d}\mathbb{P}_{T_1^*}}{\mathsf{d}\mathbb{P}_{T^*}} = \frac{\mathsf{d}\mathbb{P}_{T_1^*}}{\mathsf{d}\mathbb{P}} / \frac{\mathsf{d}\,\mathbb{P}_{T^*}|_{\mathcal{F}_{T_1^*}}}{\mathsf{d}\,\mathbb{P}|_{\mathcal{F}_{T_1^*}}},$$

we have

$$\frac{\mathsf{d}\mathbb{P}_{T_1^*}}{\mathsf{d}\mathbb{P}_{T^*}} = \frac{F(T_1^*, T_1^*, T^*)}{F(0, T_1^*, T^*)} = \exp(\int_0^t \lambda(s, T_1^*) \mathsf{d}L_s^{T^*}).$$

Notice that the density process has the same form as in the Forward Rate Model, and the drift condition was derived in the same way, so we can use Proposition 3.3, and get that under $\mathbb{P}_{T_1^*}$:

$$L_t^{T^*} = \int_0^t \hat{b}_s \mathsf{d}s + \int_0^t \sqrt{c_s} \mathsf{d}W_s^{T_1^*} + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu^{T_1^*})(\mathsf{d}s, \mathsf{d}x),$$

where $W_t^{T_1^*} = W_t^{T^*} - \int_0^t \sqrt{c_s}\lambda(s, T_1^*)\mathsf{d}s$,

$$\nu^{T_1^*}(\mathsf{d}s, dx) = e^{\left\langle \lambda(s,T_1^*),x \right\rangle F_s(\mathsf{d}x)}$$

and $\hat{b}$ is some drift term. Now, we want to model the next forward rate by

$$F(t, T_2^*, T_1^*) = F(0, T_2^*, T_1^*) \exp(\int_0^t \lambda(s, T_2^*) \mathsf{d}L_s^{T_1^*}),$$

where $L^{T_1^*}$ is some time-inhomogeneous Lévy process. We want that the the new driving process $L^{T_1^*}$ to be as similar to L^{T^*} as possible. Although we cannot take $L^{T_1^*} = L^{T^*}$, since the drift condition might not be satisfied. The idea is to change the drift term to some $b^{T_1^*}$, with which $F(., T_2^*, T_1^*)$ becomes a martingale. Define $b^{T_1^*}$ by the following equation:

$$\left\langle \lambda(s, T_2^*), b_s^{T_1^*} \right\rangle = -\frac{1}{2} \left\langle \lambda(s, T_2^*), c_s^{T^*} \lambda(s, T_2^*) \right\rangle - \int_{\mathbb{R}^d} \left(e^{\left\langle \lambda(s,T_2^*),x \right\rangle} - 1 - \langle \lambda(s, T_2^*), x \rangle \right) F_s^{T_1^*}(\mathsf{d}x).$$

Then we are in the same setup as we started the model. We can repeat the procedure described above, and get the following:

3.2.4 Results

For $i = 0, \ldots, n-1$:

$$F(t, T_{i+1}^*, T_i^*) = F(0, T_{i+1}^*, T_i^*) \exp(\int_0^t \lambda(s, T_{i+1}^*) \mathsf{d}L_s^{T_i^*}).$$

Under $\mathbb{P}_{T_i^*}$:

$$L_t^{T_i^*} = \int_0^t b_s^{T_i^*} \mathsf{d}s + \int_0^t \sqrt{c_s} \mathsf{d}W_s^{T_i^*} + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu^{T_i^*})(\mathsf{d}s, \mathsf{d}x),$$

where

$$W_t^{T_i^*} = W_t^{T^*} - \int_0^t \sqrt{c_s} \sum_{j=1}^{i} \lambda(s, T_j^*) \mathsf{d}s,$$

$$\nu^{T_i^*}(\mathsf{d}s, \mathsf{d}x) = \exp\left(\left\langle \sum_{j=1}^{i} \lambda(s, T_j^*), x \right\rangle \right) \nu^{T^*},$$

$b^{T_i^*}$ is defined by the following equation:

$$\left\langle \lambda(s, T_{i+1}^*), b_s^{T_i^*} \right\rangle = -\frac{1}{2} \left\langle \lambda(s, T_{i+1}^*), c_s^{T^*} \lambda(s, T_{i+1}^*) \right\rangle - \int_0^t \int_{\mathbb{R}^d} \left(e^{\left\langle \lambda(s, T_{i+1}^*), x \right\rangle} - 1 - \left\langle \lambda(s, T_{i+1}^*), x \right\rangle \right) F_s^{T_i^*}(\mathsf{d}x),$$

and

$$F_s^{T_i^*}(\mathsf{d}x) = \exp\left(\left\langle \sum_{j=1}^{i} \lambda(s, T_j^*), x \right\rangle \right) F_s(\mathsf{d}x).$$

Up to now, we have not checked that the computations are valid. There is a hidden condition on the volatility structure which we are using. This condition comes when we are using the Proposition 3.3, namely the volatility structure has to satisfy some boundedness condition:

$$\int_0^{T_i^*} \int_{\{|x|>1\}} |x| \exp\left(\left\langle \sum_{j=1}^{i} \lambda(s, T_j^*), x \right\rangle \right) F_s(\mathsf{d}x)\mathsf{d}s < \infty.$$

This condition is satisfied, if we have that $\left| \sum_{j=1}^{i} \lambda(s, T_j^*)^k \right| \leq M$ for $k = 1, \ldots, d$, which is exactly condition (iii) of the model.

3.3 Embedding the Forward Process Model into the Forward Rate Model

Take the Forward Process Model as above, with the same notation. We will extend it by constructing a Forward Rate Model, in which the forward processes will coincide with the forward processes in the Forward Process Model. We will denote all the newly constructed variables, measures, processes with a ˜ mark above them. In this section, we will prove the following theorem:

Theorem 3.2. *Under the following stronger assumptions of the Forward Process Model:*

- *the driving process L^{T^*} satisfies assumption $\mathbb{EM}$ (see Section 2.2.2) with constants M, ε*
- *the volatility structure satisfies*

  $$\left| \sum_{j=1}^{k} \lambda(s, T_j^*)^i \right| \leq M/3 \text{ for } k = 1, \ldots, n-1 \text{ and } i = 1, \ldots, d,$$

  where M is the constant in assumption $\mathbb{EM}$.

there is a Forward Rate Model, which is an extension of the Forward Process Model.

Remark Note that the second condition is a modification of the assumption of the Forward Rate Model.

Proof We have to define the parameters of the Forward Rate Model, which will be the extension of the Forward Process Model.

(a) Choice of the volatility structure in the Forward Rate Model.

We will choose $\tilde{\sigma}$ such that $\tilde{\Sigma}(s, T^*_{i+1}) - \tilde{\Sigma}(s, T^*_i) = \lambda(s, T^*_i)$ is true, the convenient choice is

$$\tilde{\sigma}(s,t) = -\sum_{j=1}^{n-1} \mathbf{1}_{t \in [T^*_i, T^*_{i+1})} \frac{1}{T^*_{i-1} - T^*_i} \lambda(s, T^*_i) \text{ for } .$$

(b) Choice of the driving process of the Forward Rate Model.

We have the time-inhomogeneous Lévy process L^{T^*} with characteristics $b^{T^*}, c^{T^*}, \nu^{T^*}$ with jump measure μ and Brownian motion W^{T^*}. Define the process $\tilde{L}$ by the following equation:

$$\tilde{L}_t := \int_0^t \tilde{b}_s \mathsf{d}s + \int_0^t \sqrt{\tilde{c}_s} \mathsf{d}\tilde{W}_s + \int_0^t \int_{\mathbb{R}^d} x(\tilde{\mu} - \tilde{\nu})(\mathsf{d}s, \mathsf{d}x),$$

where

$$\begin{aligned} \tilde{c_s} &:= c_s^{T^*},\ \tilde{\mu} := \mu, \\ \tilde{W}_t &:= W^{T^*} - \int_0^t \sqrt{c_s} \tilde{\Sigma} s, T^* \mathsf{d}s \\ \tilde{\nu}(\mathsf{d}s, \mathsf{d}x) &= e^{-\langle \tilde{\Sigma}(s,T^*), x \rangle} \nu^{T^*}(\mathsf{d}s, \mathsf{d}x) \\ \tilde{b}_s &:= b_s^{T^*} - c_s \tilde{\Sigma}(s, T^*) - \int_{\mathbb{R}^d} x(e^{\langle \tilde{\Sigma}(s,T^*), x \rangle} - 1) \nu^{T^*}(\mathsf{d}s, \mathsf{d}x). \end{aligned} \tag{28}$$

Define $\tilde{A}$ by the drift condition, i.e. $\tilde{A}(s,t) := \tilde{\theta}_s(\tilde{\Sigma}(s,t))$ where $\tilde{\theta}$ is the cumulant of $\tilde{L}$.

(c) Construction of the probability measure $\tilde{\mathbb{P}}$ in the Forward Rate Model, which will be the risk neutral measure in the model.

We define $\tilde{\mathbb{P}}$ by its density function with respect to the forward measure given from the Forward Process Model $\mathbb{P}_{T^*}$:

$$\frac{\mathsf{d}\mathbb{P}}{\mathsf{d}\mathbb{P}_{T^*}} := \exp\left(\int_0^{T^*} \tilde{\theta}_s(s, T^*) \mathsf{d}s - \int_0^{T^*} \tilde{\Sigma}(s, T^*) \mathsf{d}\tilde{L}_s \right).$$

By the construction, we have that under $\mathbb{P}_{T^*}$, $L^{T^*} = \tilde{L}$, by the construction of the characteristics of $\tilde{L}$ and Proposition 3.3, and under $\mathbb{P}_{T^*}$ the process $\tilde{W}$ is a Brownian motion under $\tilde{\mathbb{P}}$.

Now we have constructed Forward Rate Model.

(d) Proof of $\tilde{F}(., T_1^*, T^*) = F(., T_1^*, T^*)$ under $\mathbb{P}_{T^*}$.

By the construction and Corollary 3.1, we have that under $\mathbb{P}_{T^*}$, since the driving processes L^{T*} and $\tilde{L}$ are the same under $\mathbb{P}_{T^*}$. By bond price formula (Proposition

3.1) and the drift condition (Proposition 3.2) in the Forward Rate Model, we can deduce that

$$\frac{\tilde{F}(t,T_1^*,T^*)}{\tilde{F}(0,T_1^*,T^*)} = \exp(-\int_0^t (\tilde{\theta}_s(\tilde{\Sigma}(s,T_1^*)) - \tilde{\theta}_s(\tilde{\Sigma}(s,T^*)))\mathsf{d}s + \int_0^t (\tilde{\Sigma}(s,T^*) - \tilde{\Sigma}(s,T_1^*))\mathsf{d}\tilde{L}_s).$$

We have chosen $\tilde{\sigma}$ in such way, that $\tilde{\Sigma}(s,T^*) - \tilde{\Sigma}(s,T_1^*) = \lambda(s,T_1^*)$, thus we can see that apart from a deterministic term in the exponent, the two forward processes $\tilde{F}(.,T_1^*,T^*)$ and $F(.,T_1^*,T^*)$ are the same under the measure $\mathbb{P}_{T^*}$. The two forward processes are martingales under $\mathbb{P}_{T^*}$, thus the deterministic terms has to be the same.

(e) Proof of $\tilde{F}(.,T_{i+1}^*,T_i^*) = F(.,T_{i+1}^*,T_i^*)$ and $\tilde{\mathbb{P}}_{T_i^*} = \mathbb{P}_{T_i^*}$ for $i = 0,\ldots,n-1$.

Induction on i. For $i = 0$, $\tilde{F}(.,T_{i+1}^*,T_i^*) = F(.,T_{i+1}^*,T_i^*)$ is true by (d). By construction, $\tilde{\mathbb{P}}_{T^*} = \mathbb{P}_{T^*}$.

Assume that the statement is true for some $n-2 \geq i \geq 0$, we will prove, that it is also true for $i+1$.

By the inductive hypothesis, the forward processes $\tilde{F}(.,T_{i+1}^*,T_i^*)$ and $F(.,T_{i+1}^*,T_i^*)$ are the same. The density function is given by

$$\frac{\mathsf{d}\mathbb{P}_{T_{i+1}^*}}{\mathsf{d}\tilde{\mathbb{P}}_{T_i^*}} = \frac{F(T_{i+1}^*,T_{i+1}^*,T_i^*)}{F(0,T_{i+1}^*,T_i^*)} = \frac{\mathsf{d}\tilde{\mathbb{P}}_{T_{i+1}^*}}{\mathsf{d}\mathbb{P}_{T_i^*}},$$

hence we can get the measures $\tilde{\mathbb{P}}_{T_{i+1}^*}$ and $\mathbb{P}_{T_{i+1}^*}$ in the same way from the measures $\tilde{\mathbb{P}}_{T_i^*}$ and $\mathbb{P}_{T_i^*}$. The inductive hypothesis also gives that $\tilde{\mathbb{P}}_{T_i^*} = \mathbb{P}_{T_i^*}$, hence we can conclude that $\tilde{\mathbb{P}}_{T_{i+1}^*} = \mathbb{P}_{T_{i+1}^*}$.

Recall that in the Forward Process Model, the process $L^{T_{i+1}^*}$ apart from its drift term, it is the same process as L^{T^*}, and its drift is chosen in such way, that $F(.,T_{i+2}^*,T_{i+1}^*)$ is a martingale under $\mathbb{P}_{T_{i+1}^*}$.

As in step (d), we have that

$$\frac{\tilde{F}(t,T_{i+1}^*,T_i^*)}{\tilde{F}(0,T_{i+1}^*,T_i^*)} = \exp\left(\int_0^t (\tilde{\theta}_s(s,T_i^*) - \tilde{\theta}_s(s,T_{i+1}^*))\mathsf{d}s + \int_0^t (\tilde{\Sigma}(s,T_i^*) - \tilde{\Sigma}(s,T_{i+1}^*))\mathsf{d}\tilde{L}_s\right).$$

By the definition of $\tilde{\sigma}$, we have $\tilde{\Sigma}(s,T_{i+2}^*) - \tilde{\Sigma}(s,T_{i+1}^*) = \lambda(s,T_{i+1}^*)$, thus

$$\frac{\tilde{F}(t,T_{i+2}^*,T_{i+1}^*)}{\tilde{F}(0,T_{i+2}^*,T_{i+1}^*)} = \exp\left(\int_0^t (\tilde{\theta}_s(s,T_{i+1}^*) - \tilde{\theta}_s(s,T_{i+2}^*))\mathsf{d}s + \int_0^t \lambda(s,T_{i+1}^*)\mathsf{d}\tilde{L}_s\right),$$

thus the processes $\log \tilde{F}(.,T_{i+2}^*,T_{i+1}^*)$ and $\log F(.,T_{i+2}^*,T_{i+1}^*)$ are the same apart from their drift term. Since $\tilde{F}(.,T_{i+2}^*,T_{i+1}^*)$ and $F(.,T_{i+2}^*,T_{i+1}^*)$ are martingales under $\mathbb{P}_{T_{i+1}^*} = \tilde{\mathbb{P}}_{T_{i+1}^*}$, hence their drift terms have to be the same.

(f) Checking the conditions in Forward Process Model, we have the conditions:

- L^{T^*} satisfies $\mathbb{EM}$ with constants M, ε, i.e.

$$\int_0^{T^*} \int_{\{|x|>1\}} e^{\langle u,x\rangle} F_s^{T^*}(\mathsf{d}x)\mathsf{d}s < \infty \text{ for } u \in \mathbb{R}^d, |\Re(u_i)| \leq (1+\varepsilon)M \text{ for } i = 1, \dots, d.$$

- $$\left|\sum_{j=1}^{k} \lambda(s, T_j^*)^i\right| \leq M/3 \text{ for } k = 1, \dots, n-1 \text{ and } i = 1, \dots, d.$$

The second condition translates to

$$\left|\tilde{\Sigma}(s, T_k^*) - \tilde{\Sigma}(s, T^*)^i\right| \leq M/3 \text{ for } k = 1, \dots, n-1 \text{ and } i = 1, \dots, d.$$

Taking $k = n-1$, we get that $\tilde{\Sigma}(s, T_k^*) = 0$, thus $\left|\tilde{\Sigma}(s, T^*)^i\right| \leq M/3$ for $i = 1, \dots$. By the triangle inequality, we can conclude that

$$\left|\tilde{\Sigma}(s, T_k^*)\right| \leq \frac{2}{3}M \text{ for } i = 1, \dots, d.$$

As for the first condition

$$\int_0^{T^*} \int_{\{|x|>1\}} e^{\left\langle u - \tilde{\Sigma}(s,T^*), x\right\rangle} \tilde{F}_s(\mathsf{d}x)\mathsf{d}s < \infty \text{ for } u \in \mathbb{R}^d, |\Re(u_i)| \leq (1+\varepsilon)M \text{ for } i = 1, \dots, d.$$

We have the inequality $-\left\langle \tilde{\Sigma}(s, T^*), x\right\rangle \geq -\max_i \tilde{\Sigma}(s, T^*) \sum_{j=1}^d |x_j| \geq -\frac{M}{3}\sum_{j=1}^d |x_j|$, thus we have that

$$\int_0^{T^*} \int_{\{|x|>1\}} \exp\left(\langle u, x\rangle - \frac{M}{3}\sum_{j=1}^d |x_j|\right) \tilde{F}_s(\mathsf{d}x)\mathsf{d}s < \infty$$

for $u \in \mathbb{R}^d, |\Re(u_i)| \leq (1+\varepsilon)M$ where $i = 1, \dots, d$. Now we can use the same idea as in Proposition 2.6. For every $x \in \mathbb{R}^d$ there is (at least one) $v \in \{-1, 1\}^d$, such that $\langle (1+\varepsilon)Mv, x\rangle = (1+\varepsilon)M\sum_{j=1}^d |x_j|$. Thus if we sum up the above integrals by taking $u = (1+\varepsilon)Mv$ for some $v \in \{-1, 1\}^d$, we can conclude that

$$\int_0^{T^*} \int_{\{|x|>1\}} \exp\left(\left((1+\varepsilon)M - \frac{M}{3}\right)\sum_{j=1}^d |x_j|\right) \tilde{F}_s(\mathsf{d}x)\mathsf{d}s < \infty,$$

from which we can see that $\tilde{L}$ satisfies $\mathbb{EM}'$ with constants $\frac{2}{3}M$ and ε.

Take $\tilde{M} = \frac{2}{3}$ and $\tilde{\varepsilon} = \varepsilon$, then we have that $\tilde{L}$ satisfies $\mathbb{EM}$ with constants $\tilde{M}$ and $\tilde{\varepsilon}$.

By the definition of $\tilde{\sigma}$ we have that

$$\tilde{\Sigma}(s, t) = \frac{t - T_{i+1}^*}{T_i^* - T_{i+1}^*}\tilde{\Sigma}(s, T_{i+1}^*) + \frac{T_i^* - t}{T_i^* - T_{i+1}^*}\tilde{\Sigma}(s, T_i^*) \text{ for } t \in [T_{i+1}^*, T_i^*],$$

which is a convex combination, we can conclude that for all $t \in [0, T^*]$

$$\left|\tilde{\Sigma}(s, t)\right| \leq \frac{2}{3}M = \tilde{M} \text{ for } i = 1, \dots, d.$$

Which conditions are in the same form as in the Forward Rate Model. □

Remark If we think about the conditions in the models that their role is to ensure that the measure changes can be done, than we can conclude that the Forward Process Model can be embedded in the Forward Rate Model, since we could do all the required measure changes, since the forward measure $\mathbb{P}_{T_1}$ is $\tilde{\mathbb{P}}$, since the the process L^{T_1} and $\tilde{L}$ only differ by their drift term.

3.4 LIBOR Rate Model

Now we will model directly the LIBOR rates, with the same backwards induction as in the Forward Process Model. It turns out, that the driving process L with which we start with, will no longer be a time-inhomogeneous Lévy process under the new measure, since the characteristics become stochastic. Hence if we want to use the model, then we have to approximate the characteristics, which raises a couple of problems. The description of the model can be found in [Eberlein & Özkan, 2005] and in [Kluge, 2005].

3.4.1 Outline of the model

As in the Forward Process Model, we have a finite finitely many bonds with maturity dates $0 = T_0 < T_1 < T_2 < \ldots < T_n = T^*$. For convenient notation, define and $T_i^* = T_{n-i}$ for $i = 0, \ldots, n$. Also define $\delta_i^* = T_i^* - T_{i+1}^*$. The LIBOR rate is by Definition 1.8:

$$L_\delta(t,T) = \frac{F(t,T,T+\delta) - 1}{\delta} = \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right).$$

We use the shorthand notation

$$L(t,T_i^*) = L_{\delta_i^*}(t,T_{i+1}^*) \text{ for } i = 0,1,\ldots,n-1 \text{ and } 0 \le t \le T_{i+1}^*.$$

Start with a d-dimensional time-inhomogeneous Lévy process L^{T^*} with respect to the measure $\mathbb{P}_{T^*}$. We postulate that

$$L(t,T_1^*) = L(0,T_1^*)\exp\left(\int_0^t \lambda(s,T_1^*)\mathsf{d}L_s^{T^*}\right).$$

The next step is to derive a condition on the drift term of L^{T^*} under which $L(.,T_1^*)$ is a martingale. Then change to the forward measure $\mathbb{P}_{T_1^*}$ which is the forward measure associated with maturity date T_1^*, then postulate that

$$L(t,T_2^*) = L(0,T_2^*)\exp\left(\int_0^t \lambda(s,T_2^*)\mathsf{d}L_s^{T_1^*}\right),$$

where $L^{T_1^*}$ is a process which differs from L^{T^*} by its drift term. The (usually non-deterministic) drift term is chosen such way that $L(.,T_2^*)$ is a martingale under $\mathbb{P}_{T_1^*}$. We can repeat these steps, by induction, we will construct the processes $L^{T_i^*}$ and the processes $L(.,T_i^*)$. We will need the following assumptions for the precise computation:

3.4.2 Assumptions

We will need the following assumptions:

(i) The process L^{T^*} satisfies $\mathbb{EM}$ (see Section 2.2.2) with constants ε, M.

(ii) The initial LIBOR rates are positive, or equivalently, the the sequence of initial bond prices $B(0,T_i)$ for $i = 1, 2, \dots n$ is a strictly decreasing sequence.

(iii) For each maturity date T_k, there is a function $\lambda(.,T_k) : [0,T^*] \to \mathbb{R}^d$ which represents the volatility structure of $L(.,T_i)$, and

$$\sum_{i=1}^{n-1} \left|\lambda^j(t,T_i)\right| \leq M \text{ for all } t \in [0,T^*] \text{ and } j = 1,2,\dots,d,$$

where M is the constant from assumption $\mathbb{EM}$ and $\lambda(t,T_i) = 0$ for $T^* \geq t > T_i$.

3.4.3 Construction of the model

By assumption, the process L^{T^*} satisfies $\mathbb{EM}$, by Proposition 2.7 has canonical representation under the forward measure $\mathbb{P}_{T^*}$:

$$L_t^{T^*} = \int_0^t b_s^{T^*} \mathsf{d}s + \int_0^t \sqrt{c_s} \mathsf{d}W_s^{T^*} + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu^{T^*})(\mathsf{d}s, \mathsf{d}x).$$

We start with modeling the LIBOR rate for the longest maturity date T^*:

As we saw in Proposition 1.2, we have to ensure that $F(.,T_1^*,T^*)$ is a martingale, or equivalently, that the process $L(.,T_1^*)$ is a martingale. Just as in the Forward Process Model, we have the condition on the drift term:

$$\left\langle \lambda(s,T_1^*), b_s^{T^*} \right\rangle = -\frac{1}{2} \left\langle \lambda(s,T_1^*), c_s^{T^*} \lambda(s,T_1^*) \right\rangle - \int_{\mathbb{R}^d} \left(e^{\langle \lambda(s,T_1^*),x \rangle} - 1 - \langle \lambda(s,T_1^*), x \rangle \right) F_s^{T^*}(\mathsf{d}x).$$

We have that

$$L(t,T_1^*) = L(0,T_1^*) \exp(\int_0^t \lambda(s,T_1^*) \mathsf{d}L_s).$$

We can use use Lemma A.1 with $u(s) = \lambda(s,T_1^*)$ and $Y(s,x) = \exp(\langle \lambda(s,T_1^*), x \rangle)$ by Lemma A.2. Combining Lemma A.1 with the drift condition, we get

$$\frac{L(t,T_1^*)}{L(0,T_1^*)} = \mathcal{E}(H^1)_t,$$

where $\mathcal{E}(H^1)$ is the Doléans-Dade exponential of H^1, where

$$H_t^1 = \int_0^t \lambda(s, T_1^*)\sqrt{c_s}\mathsf{d}W_s + \int_0^t \int_{\mathbb{R}^d} \left(e^{\left\langle \lambda(s,T_1^*),x\right\rangle} - 1\right)(\mu - \nu^{T^*})(\mathsf{d}s, \mathsf{d}x).$$

Hence $\mathsf{d}L(t, T_1^*) = L(t-, T_1^*)\mathsf{d}H^1$; combining it with the fact that $\mathsf{d}F(t, T_1^*, T^*) = \delta_1 \mathsf{d}L(t, T_1^*)$ we have that

$$\mathsf{d}F(t, T_1^*, T^*) = F(t-, T_1^*, T^*)\left(\frac{\delta_1 L(t-, T_1^*)}{1 + \delta_1 L(t-, T_1^*)}\lambda(s, T_1^*)\sqrt{c_s}\mathsf{d}W_s + \int_{\mathbb{R}^d} \frac{\delta_1 L(t-, T_1^*)}{1 + \delta_1 L(t-, T_1^*)}\left(e^{\left\langle \lambda(s,T_1^*),x\right\rangle} - 1\right)(\mu - \nu^{T^*})(\mathsf{d}s, \mathsf{d}x)\right)$$

Define

$$\alpha(t, T_1^*) := \frac{\delta_1 L(t-, T_1^*)}{1 + \delta_1 L(t-, T_1^*)}\lambda(s, T_1^*) \text{ and } \beta(t, x, T_1^*) := \frac{\delta_1 L(t-, T_1^*)}{1 + \delta_1 L(t-, T_1^*)}\left(e^{\left\langle \lambda(s,T_1^*),x\right\rangle} - 1\right) + 1.$$

With this notation, we have

$$\mathsf{d}F(t, T_1^*, T^*) = F(t-, T_1^*, T^*)\left(\alpha(s, T_1^*)\sqrt{c_s}\mathsf{d}W_s + \int_{\mathbb{R}^d} (\beta(t, x, T_1^*) - 1)(\mu - \nu^{T^*})(\mathsf{d}s, \mathsf{d}x)\right).$$

Which means, that $F(t, T_1^*, T^*) = F(0, T_1^*, T^*)\mathcal{E}(M^1)$, where

$$M_t^1 = \int_0^t \alpha(s, T_1^*)\sqrt{c_s}\mathsf{d}W_s + \int_0^t \int_{\mathbb{R}^d} (\beta(t, x, T_1^*) - 1)(\mu - \nu^{T^*})(\mathsf{d}s, \mathsf{d}x).$$

Just as in the Forward Process Model, we have that the density process is

$$\frac{\mathsf{d}\mathbb{P}_{T_1^*}\big|_{\mathcal{F}_t}}{\mathsf{d}\mathbb{P}_{T^*}\big|\mathcal{F}_t} = \frac{F(t, T_1^*, T^*)}{F(0, T_1^*, T^*)} = \mathcal{E}(M^1)_t.$$

By Lemma A.2 we can choose $u(s) = \alpha(s, T_1^*)$ and $Y(s, x) = \beta(s, x, T_1^*)$ in Proposition A.2, and get that under $\mathbb{P}_{T_1^*}$,

$$L_t^{T^*} = \int_0^t \tilde{b}_s \mathsf{d}s + \int_0^t \sqrt{c_s}\mathsf{d}W_s^{T_1^*} + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu^{T_1^*})(\mathsf{d}s, \mathsf{d}x),$$

where

$$\begin{aligned} \nu^{T_1^*}(\mathsf{d}s, \mathsf{d}x) &= \beta(s, x, T_1^*)\nu^{T^*}(\mathsf{d}s, \mathsf{d}x) \\ W_t^{T_1^*} &= W_t^{T^*} - \int_0^t \alpha(s, T_1^*)\sqrt{c_s}\mathsf{d}s, \end{aligned}$$

and the bounded variation term is some $\tilde{b}$. Now we change $\tilde{b}$ to get a new process $L^{T_1^*}$. I postulate that

$$L(t, T_2^*) = L(0, T_2^*)\exp\left(\int_0^t \lambda(s, T_2^*)\mathsf{d}L_s^{T_1^*}\right).$$

Note that now we cannot use Proposition 2.9, since the process $L^{T_1^*}$ usually is not a time-inhomogeneous Lévy process, since its characteristics are no longer deterministic. To avoid this problem, we will choose $b^{T_1^*}$ in such way, that $\exp(\int_0^t \lambda(s,T_2^*)\mathsf{d}L_s^{T_1^*})$ becomes a Doléans-Dade exponential H^2, which is defined by

$$H_t^2 = \int_0^t \lambda(s,T_2^*)\sqrt{c_s}\mathsf{d}W_s + \int_0^t\int_{\mathbb{R}^d}\left(e^{\langle\lambda(s,T_2^*),x\rangle}-1\right)(\mu-\nu^{T_1^*})(\mathsf{d}s,\mathsf{d}x).$$

We have to compute $\mathcal{E}(H^2)$, which can be done by using Lemma A.1, since the conditions are satisfied by Lemma A.2 if we take $u(s) = \lambda(s,T_2^*)$ and $Y(s,x) = e^{\langle\lambda(s,T_2^*),x\rangle}$. Lemma A.1 gives the desired condition on $b^{T_1^*}$:

$$\left\langle\lambda(s,T_2^*),b_s^{T_1^*}\right\rangle = -\frac{1}{2}\left\langle\lambda(s,T_2^*),c_s^{T_1^*}\lambda(s,T_2^*)\right\rangle - \int_{\mathbb{R}^d}\left(e^{\langle\lambda(s,T_2^*),x\rangle}-1-\langle\lambda(s,T_2^*),x\rangle\right)F_s^{T_1^*}(\mathsf{d}x),$$

where $F_s^{T_1^*}(\mathsf{d}x) = \beta(s,x,T_1^*)F_s^{T^*}(\mathsf{d}x)$. From this point, the induction works: compute $\mathsf{d}F(t,T_2^*,T_1^*)$, define $\alpha(.,T_2^*)$ and $\beta(.,T_2^*)$, define the process M^2, use Proposition A.2, define $L^{T_2^*}$, derive a drift condition. After the induction, we have the following:

3.4.4 Results

For $i = 0,\dots,n-1$ and $t \in [0,T_{i+1}^*]$:

$$F(t,T_{i+1}^*,T_i^*) = F(0,T_{i+1}^*,T_i^*)\exp(\int_0^t \lambda(s,T_{i+1}^*)\mathsf{d}L_s^{T_i^*}).$$

Under $\mathbb{P}_{T_i^*}$:

$$L_t^{T_i^*} = \int_0^t b_s^{T_i^*}\mathsf{d}s + \int_0^t \sqrt{c_s}\mathsf{d}W_s^{T_i^*} + \int_0^t\int_{\mathbb{R}^d} x(\mu-\nu^{T_i^*})(\mathsf{d}s,\mathsf{d}x),$$

where

$$\begin{aligned} W_t^{T_i^*} &= W_t^{T^*} - \int_0^t \sqrt{c_s}\sum_{j=1}^i \alpha(s,T_j^*)\mathsf{d}s, \\ \nu^{T_i^*}(\mathsf{d}s,\mathsf{d}x) &= \prod_{j=1}^i \beta(s,T_j^*)\nu^{T^*}(\mathsf{d}s,\mathsf{d}x) \end{aligned}$$

with

$$\begin{aligned} \alpha(s,T_i^*) &= \frac{\delta_i L(t-,T_i^*)}{1+\delta_i L(t-,T_i^*)}\lambda(t,T_i^*) \\ \beta(s,x,T_i^*) &= \frac{\delta_i L(t-,T_i^*)}{1+\delta_i L(t-,T_i^*)}\left(e^{\langle\lambda(t,T_i^*),x\rangle}-1\right)+1. \end{aligned}$$

$b^{T_i^*}$ is defined by the following equation:

$$\begin{aligned}\left\langle \lambda(s,T_{i+1}^*), b_s^{T_i^*} \right\rangle &= -\frac{1}{2}\left\langle \lambda(s,T_{i+1}^*), c_s^{T^*}\lambda(s,T_{i+1}^*)\right\rangle - \\ &\int_0^t \int_{\mathbb{R}^d} \left(e^{\left\langle \lambda(s,T_{i+1}^*),x\right\rangle} - 1 - \left\langle \lambda(s,T_{i+1}^*), x\right\rangle \right) F_s^{T_i^*}(\mathsf{d}x),\end{aligned}$$

where $F_s^{T_i^*}(\mathsf{d}x) = \prod_{j=1}^{i}\beta(s,T_j^*)F_s(\mathsf{d}x)$.

4 Pricing of derivatives

Finally we can price the basic financial instruments, call and put options, and swaptions. We will use the Lévy Forward Rate Model. The methods and the material discussed in this section can be found in [Kluge, 2005] with more details, and there can be found formulas for pricing other derivatives.

We will derive formulas involving convolutions and Fourier transforms. So before stating the valuation, I recall the fundamental properties of Fourier transformation in $\mathcal{L}_1(\mathbb{R})$. The Fourier transform of $f \in \mathcal{L}_1$ is $\mathfrak{F}(f)(s) := \int_{\mathbb{R}} e^{isx} f(x)\mathsf{d}x$.

The following theorem summarizes the properties of convolution and Fourier transformation we will use:

Theorem 4.1. *If $f, g \in \mathcal{L}_1(\mathbb{R})$, then*

(i) The convolution $f * g$ defined by

$$(f*g)(x) := \int_{\mathbb{R}} f(x-z)g(y)\mathsf{d}y$$

is also in $\mathcal{L}_1$, furthermore if g is bounded, than the function $f*g$ is a continuous.

(ii) The Fourier transforms satisfy the equation: $\mathfrak{F}(f*g) = \mathfrak{F}(f)\mathfrak{F}(g)$.

(iii) If f is continuous in the point x. If the limit

$$\lim_{S\to\infty} \frac{1}{2\pi}\int_{-S}^{S} e^{-isx}\mathfrak{F}(f)(s)\mathsf{d}u$$

exists, then the the value of the limit is $f(x)$.

Proof It is a direct consequence of Theorem 2.6 and 2.7 in [Kluge, 2005]. □

Note that by the virtue of the second section, to price caps and floors it is enough to price options on bonds. We will start where we finished in Section 2:

4.1 Pricing call options

Corollary 1.1 gives that the price of a call option on a bond at time 0 is given by the formula:

$$C_0(T,U,K) := B(0,T)\mathbb{E}_T((B(T,U)-K)^+).$$

The first step is to write the expectation $\mathbb{E}_T((B(T,U)-K)^+$ as a convolution. Proposition 3.1 gives the following formula for bond prices:

$$\frac{B(T,U)}{B_T} = B(0,U)\exp\left(-\int_0^T \theta_s(\Sigma(s,U))\mathsf{d}s + \int_0^T \Sigma(s,U)\mathsf{d}L_s\right),$$

taking $U=T$ we have:

$$\frac{B(T,T)}{B_T} = B(0,T)\exp\left(-\int_0^T \theta_s(\Sigma(s,T))\mathsf{d}s + \int_0^T \Sigma(s,T)\mathsf{d}L_s\right),$$

from which we can conclude that

$$B(T,U) = \frac{B(0,U)}{B(0,T)}\exp\left(\int_0^T (\theta_s(\Sigma(s,T)) - \theta_s(\Sigma(s,U)))\mathsf{d}s + \int_0^T (\Sigma(s,U)-\Sigma(s,T))\mathsf{d}L_s\right).$$

Define

$$D := \frac{B(0,U)}{B(0,T)}\exp\left(\int_0^T (\theta_s(\Sigma(s,T)) - \theta_s(\Sigma(s,U)))\mathsf{d}s\right),$$

and

$$X := \int_0^T (\Sigma(s,U)-\Sigma(s,T))\mathsf{d}L_s. \tag{29}$$

With these definitions, we have that

$$B(T,U) = De^X \text{ and } C_0(T,U,K) = B(0,T)\mathbb{E}_{\mathbb{P}_T}(De^X - K)^+.$$

Suppose that the distribution function of X is continuous with respect to the forward measure $\mathbb{P}_T$, i.e. $\mathbb{P}_T^X$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. ($\mathbb{P}_T^X$ is defined by $\mathbb{P}_T^X(A) := \mathbb{P}_T(X\in A)$ for Lebesgue-measurable set $A\subset\mathbb{R}$.)

Proposition 4.1. *Suppose that the distribution of X possesses a density function $\varphi = \frac{d\mathbb{P}_T^X}{d\lambda}$ (λ is the Lebesgue measure). Then*

$$C_0(T,U,K) = B(0,T)\mathbb{E}(De^X - K)^+ = B(0,T)\int_{\mathbb{R}}(De^x - K)^+\varphi(x)dx = B(0,T)(v*\varphi)(0),$$

where $v(x) = (De^{-x}-K)^+$.

We also need extra conditions to ensure that there is such density function φ:

Proposition 4.2 (Proposition 2.8 in [Kluge, 2005])**.** *Assume that $\Sigma(s,T)\neq\Sigma(s,U)$ for $s\in[0,T]$. Then each of the following conditions implies that $\mathbb{P}_T^X$ is absolutely continuous with respect to the Lebesgue-measure on $\mathbb{R}$:*

- *There is a Borel set $S \subset [0,T]$ with positive Lebesgue-measure, such that c_s is positive definite on the set S.*

- *There are constants $C, \gamma, \eta > 0$ for which*

$$|\exp(\theta_s(iu))| \leq C \exp(-\gamma\, |u|^{\eta}) \text{ for all } s \in [0,t].$$

Proof The proof is based on the fact that if the characteristic function of X is integrable, then we can use the inversion formula for Fourier transformation, and we will get the density function $\frac{\mathsf{d}\mathbb{P}_T^X}{\mathsf{d}\lambda}$. $\square$

Our next aim is to prove the following formula for call option prices:

Theorem 4.2. *Suppose that the distribution of X possesses a Lebesgue-density. Choose an $R > 1$ for which $M_T^X(R) < \infty$. Then we have*

$$C_0(T,U,K) = \frac{1}{2\pi i} K B(0,T) \int_{\Re z = R} e^{-z\xi} \frac{1}{z(z-1)} M_T^X(z) \mathsf{d}z,$$

where

$$\xi := \log \frac{B(0,T)}{B(0,U)} - \int_0^T (\theta_s(\Sigma(s,T)) - \theta(\Sigma(s,U)))\mathsf{d}s + \log K.$$

Proof We have that

$$C_0(T,U,K) = B(0,T)K\mathbb{E}_{\mathbb{P}_T}(DK^{-1}e^x - 1)^+ = B(0,T)K \int_{-\infty}^{\infty} (e^{-(\xi - x)} - 1)^+ \varphi(x)\mathsf{d}x.$$

The problem is that $(e^{-x} - 1)^+$ is not integrable, to avoid this problem, take an $R > 1$ for which $M_T^X(R) < \infty$. Then

$$C_0(T,U,K) = B(0,T)Ke^{-R\xi} \int_{-\infty}^{\infty} e^{R(\xi - x)}(e^{-(\xi-x)} - 1)^+ e^{Rx}\varphi(x)\mathsf{d}x = B(0,T)Ke^{-R\xi}(h * e^{Rx}\varphi(x))(\xi),$$

where $h(x) = e^{Rx}(e^{-x} - 1)^+$ The function h is integrable, since

$$\int_{\mathbb{R}} h(x)\mathsf{d}x = \int_{-\infty}^0 e^{Rx}(e^{-x} - 1)\mathsf{d}x = \int_{-\infty}^0 e^{(R-1)x} - e^{Rx}\mathsf{d}x = \frac{1}{R-1} - \frac{1}{R}.$$

The Fourier transform of h is

$$\begin{aligned} \mathfrak{F}(h)(u) &= \int_{\mathbb{R}} e^{iux} h(x)\mathsf{d}x = \int_{-\infty}^0 e^{(R+iu)x}(e^{-x} - 1)\mathsf{d}x = \int_{-\infty}^0 e^{(R-1+iu)x} - e^{(R+iu)x}\mathsf{d}x \\ &= \frac{1}{R + iu - 1} - \frac{1}{R + iu} = \frac{1}{(R+iu)(R+iu-1)}, \end{aligned}$$

from which we can see that $\mathfrak{F}(h)$ is integrable, since $\mathfrak{F}(h)(u) = O(u^2)$ as $u \to \infty$. The Fourier transform of $e^{Rx}\varphi(x)$ is

$$\mathfrak{F}(e^{Rx}\varphi(x))(u) = \int_{\mathbb{R}} e^{(R+iu)x}\varphi(x)\mathsf{d}x = \mathbb{E}_{\mathbb{P}_T}(e^{(R+iu)X}) = M_T^X(R+iu).$$

By the basic properties of Fourier transform, we have that

$$\mathfrak{F}(h * e^{Rx}\varphi(x)) = \mathfrak{F}(h)\mathfrak{F}(e^{Rx}\varphi(x)).$$

Since $\mathfrak{F}(e^{Rx}\varphi(x))(u) = M_T^X(R+iu)$ is bounded, since $\left|M_T^X(R+iu)\right| \leq M_T^X(R)$, and $\mathfrak{F}(h)$ is integrable, we have that $\mathfrak{F}(h * e^{Rx}\varphi(x))$ is integrable, thus

$$(h * e^{Rx}\varphi(x))(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\xi ui}\frac{1}{(R+iu)(R+iu-1)}M_T^X(R+iu)\mathsf{d}u,$$

from which the statement follows. □

Now, I show, under the assumptions we made, there is an $R > 1$ for which $M_T^X(R) < \infty$ and give a formula for M_T^X.

Proposition 4.3. *Under the assumptions of the Lévy Forward Rate Model, for $\Re z \in (-\varepsilon/2, 1+\varepsilon/2)$, where ε is taken as in assumption $\mathbb{EM}$, $M_T^X(z) < \infty$ and is given by the formula:*

$$M_T^X(z) = \exp\int_0^T (\theta_s(z\Sigma(s,U) + (1-z)\Sigma(s,T)) - \theta_s(\Sigma(s,T)))\,\mathsf{d}s.$$

Proof

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}_T} e^{zX} &= \mathbb{E}\left(\frac{\mathsf{d}\mathbb{P}_T}{\mathsf{d}\mathbb{P}}e^{zX}\right)\\
&= \exp\left(-\int_0^T \theta_s(\Sigma(s,T))\mathsf{d}s\right)\mathbb{E}\exp\left(\int_0^T z(\Sigma(s,U)-\Sigma(s,T)) + \Sigma(s,T)\mathsf{d}s\right)
\end{aligned}$$

Since $R := \Re z \in [-\varepsilon/2, 1+\varepsilon/2]$, then

$$\begin{aligned}
|\Re(z\Sigma(s,T) + (1-z)\Sigma(s,t))| &= |R\Sigma(s,T) + (1-R)\Sigma(s,t)|\\
&\leq |R|\,|\Sigma(s,T)| + |1-R|\,|\Sigma(s,t)|\\
&\leq M(|R| + |1-R|)\\
&< (1+\varepsilon)M.
\end{aligned}$$

Hence can use Proposition 2.9, for the function $f(z) = z\Sigma(s,T) + (1-z)\Sigma(s,t)$, and we get the desired equality. □

We can also calculate the value of a put option:

Corollary 4.1. *The value of a put option is given by the formula:*

$$P_0(T,U,K) = \frac{1}{2\pi i}KB(0,T)\int_{\Re z = R} e^{-\xi z}\frac{1}{z(z-1)}M_T^X(z)\mathsf{d}z,$$

where

$$\xi := \log\frac{B(0,T)}{B(0,U)} - \int_0^T (\theta_s(\Sigma(s,T)) - \theta(\Sigma(s,U)))\mathsf{d}s + \log K,$$

and $R < 0$ such that $M_T^X(R) < \infty$.

Note that the only difference is that now we are integrating on a different path.

Proof We can do the same computation as in the proof of the call option price formula. An other proof can be given by using the Residue Theorem for the function $e^{-\xi z}\frac{1}{z(z-1)}M_T^X(z)$ and the call option price formula and the call-put parity. □

4.2 Pricing swaptions

Corollary (1.2) gives the following formula for the price of a receiver swaption:

$$PS_0 := B(0,T)\mathbb{E}_T\left(\sum_{j=1}^{n} c_j B(T,T_j) - 1\right)^+,$$

where C, c_j $j = 1, \ldots n$ are given positive constants.

Define

$$D_j := \frac{B(0,T_j)}{B(0,T)} \exp\left(\int_0^T (\theta_s(\Sigma(s,T)) - \theta_s(\Sigma(s,T_j)))\mathsf{d}s\right),$$

and

$$X_j := \int_0^T (\Sigma(s,T_j) - \Sigma(s,T))\mathsf{d}L_s$$

for $j = 1, \ldots, n$. With the new notation, we have

$$PS_0 = CB(0,T)\mathbb{E}_T\left(\sum_{j=1}^{n} c_j D_j e^{X_j} - 1\right)^+.$$

In order to be able to compute the expectation with the same convolution trick, we would require that for each i, there is a deterministic function f_j and a random variable X, for which $X_j = f_j(X)$. It would be even better, if the functions f_j would be simple, in order to make the computations easier. This can be guaranteed by choosing the volatility structure wisely. We will choose the volatility structure:

Assumption 3 ($\mathbb{VOL}$)**.** *There are functions $\sigma_1 : [0,T^*] \to \mathbb{R}^d$ and $\sigma_2 : [0,T^*] \to (0,\infty)$ such that for all $T, s \in [0,T^*]$, $s \leq T$*

$$\sigma(s,T) = \sigma_2(T)\sigma_1(s) \neq 0.$$

Under $\mathbb{VOL}$, we have $\Sigma(s,T) = \int_0^T \sigma(s,u)\mathsf{d}u = \int_0^T \sigma_2(u)\mathsf{d}u\sigma_1(s)$, thus

$$\Sigma(s,T_j) - \Sigma(s,T) = \int_T^{T_j} \sigma_2(u)\mathsf{d}u\sigma_1(s) = \frac{\int_T^{T_j}\sigma_2(u)\mathsf{d}u}{\int_T^{T_n}\sigma_2(u)\mathsf{d}u}\left(\Sigma(s,T_n) - \Sigma(s,T)\right).$$

Which means, that $X_j = B_j X$, where $X = X_n$, and

$$0 \leq B_j := \frac{\int_T^{T_j}\sigma_2(u)\mathsf{d}u}{\int_T^{T_n}\sigma_2(u)\mathsf{d}u} \leq 1.$$

Thus

$$PS_0 = B(0,T)\mathbb{E}_T\left(\sum_{i=j}^{n} c_j D_j e^{B_j X} - 1\right)^+ .$$

Our aim in this subsection, is to derive a the following pricing formula for swaptions:

Theorem 4.3. *Suppose that the conditions of the Lévy Forward Rate Model and the assumption* $\mathbb{VOL}$ *holds. Then if for some* $R > 1$ *and* $\delta > 0$ $M_T^X(R+\delta) < \infty$ *and* $M_T^X(R-\delta) < \infty$, *then the price of a receiver swaption is given by the formula*

$$PS_0 = B(0,T)\int_{\Re z = R} M_T^X(z) e^{zZ}\left(\frac{1}{z} - \sum_{j=1}^{n}\frac{e^{B_j Z}}{z - B_j}\right) du,$$

where the integral understood in Cauchy principal value.

Remark Note that

$$M_T^X(z) = \exp\int_0^T \left(\theta_s(z\Sigma(s,T_n) + (1-z)\Sigma(s,T)) - \theta_s(\Sigma(s,T))\right)\mathsf{d}s,$$

which follows from Proposition 4.3.

Proof Now we can use the method we used in the last subsection:

$$PS_0 = B(0,T)\int_{\mathbb{R}}\left(\sum_{j=1}^{n} c_j D_j e^{B_j x} - 1\right)^+ \varphi(x)\mathsf{d}x.$$

We have the same problem as before, the function $\left(\sum_{j=1}^{n} c_j D_j e^{B_j x} - 1\right)^+$ is not integrable, thus we do the same as before:

$$PS_0 = B(0,T)\int_{\mathbb{R}} e^{-Rx}\left(\sum_{i=1}^{n} c_j D_j e^{B_j x} - 1\right)^+ e^{Rx}\varphi(x)\mathsf{d}x = B(0,T)(h * e^{Rx}\varphi(x))(0),$$

where $h(x) = e^{Rx}\left(\sum_{j=1}^{n} c_j D_j e^{-B_i x} - 1\right)^+$ for some $R > 1$. Now we have to calculate the Fourier transform of h. Note that the function $\sum_{j=1}^{n} c_j D_j e^{-B_j x} - 1$ is strictly decreasing, since c_j, D_j, B_j are positive constants, and as $x \to \infty$, $\sum_{j=1}^{n} c_j D_j e^{-B_j x} \to 0$ and as $x \to -\infty$ $\sum_{j=1}^{n} c_j D_j e^{-B_j x} \to \infty$, thus it has a unique 0, call it Z. Thus

$$\begin{aligned}
\int_{\mathbb{R}} e^{(R+iu)x}\left(\sum_{j=1}^{n} c_j D_j e^{-B_j x} - 1\right)^+ \mathsf{d}x &= \int_{-\infty}^{Z} e^{(R+iu)x}\left(\sum_{j=1}^{n} c_j D_j e^{-B_j x} - 1\right)\mathsf{d}x \\
&= \sum_{j=1}^{n}\frac{-e^{(R-B_j+iu)Z}}{R - B_j + iu} + \frac{e^{(R+iu)Z}}{R+iu} \\
&= e^{(R+iu)Z}\left(\frac{1}{R+iu} - \sum_{j=1}^{n}\frac{e^{B_j Z}}{R - B_j + iu}\right).
\end{aligned}$$

In this case, $\mathfrak{F}(h)$ is usually not integrable. But we show, that the integral

$$\frac{1}{2\pi}\int_{-N}^{N} e^{(R+iu)Z}\left(\frac{1}{R+iu}-\sum_{j=1}^{n}\frac{e^{B_jZ}}{R-B_j+iu}\right)M_X^T(R+iu)\mathsf{d}u$$

converges as $N\to\infty$. To prove this, it is enough to show that $\int_{-N}^{N}\frac{e^{iuZ}}{r+iu}M_T^X(r+iu)\mathsf{d}u$ converges as $N\to\infty$ for $r>0$ for which $M_T^X(r)<\infty$.

$$\begin{aligned}\int_{-N}^{N}\frac{e^{iuZ}}{r+iu}M_T^X(r+iu)\mathsf{d}u &= \int_{-N}^{N} e^{iuZ}\frac{r-iu}{r^2+u^2}\left(\int_{\mathbb{R}} e^{(r+iu)x}\varphi(x)\mathsf{d}x\right)\mathsf{d}u\\ &= \int_{\mathbb{R}}\left(\int_{-N}^{N} e^{iu(Z+x)}\frac{r-iu}{r^2+u^2}\mathsf{d}u\right)e^{rx}\varphi(x)\mathsf{d}x\\ &= \int_{\mathbb{R}}\left(\int_{0}^{N} 2\Re(e^{iu(Z+x)}\frac{r-iu}{r^2+u^2})\mathsf{d}u\right)e^{rx}\varphi(x)\mathsf{d}x.\end{aligned}$$

Now consider the inner integral:

$$\begin{aligned}\int_{0}^{N}\Re(e^{iu(Z+x)}\frac{r-iu}{r^2+u^2})\mathsf{d}u &= \int_0^N \cos(u(Z+x))\frac{r}{r^2+u^2}+\sin(u(Z+x))\frac{u}{r^2+u^2}\mathsf{d}u.\\ &= \int_0^N \cos(u(Z+x))\frac{r}{r^2+u^2}+\frac{\sin(u(Z+x))}{u}+\\ &\quad \sin(u(Z+x))\frac{r^2}{u(r^2+u^2)}\mathsf{d}u.\end{aligned}$$

The first and second term converge as $N\to\infty$, and they are bounded, since

$$\left|\int_0^N \cos(u(Z+x))\frac{r}{r^2+u^2}\mathsf{d}u\right|\le\int_0^\infty\frac{r}{r^2+u^2}\mathsf{d}u<\infty,$$

and

$$\int_0^N\frac{\sin(u(Z+x))}{u}\mathsf{d}u=\int_0^{N/|Z+x|}\frac{\sin u}{u}\mathsf{d}u$$

which integral converges as $n\to\infty$, thus it is uniformly bounded in x. (If $Z+x=0$, then we are integrating constant 0 which does not cause any problem.)

As for third term, we have that if we divide it by $Z+x$, then it will converge, and it will be bounded, since

$$\left|\int_0^N\frac{\sin(u(Z+x))}{Z+x}\frac{r^2}{r^2+u^2}\mathsf{d}u\right|\le\int_0^\infty\frac{r^2}{r^2+u^2}\mathsf{d}u<\infty.$$

So we have that

$$\begin{aligned}\int_{-N}^{N}\frac{e^{iuZ}}{r+iu}M_T^X(r+iu)\mathsf{d}u &= 2\int_{\mathbb{R}}\left(\int_0^N\frac{\cos(u(Z+x)r)}{r^2+u^2}\mathsf{d}u+\int_0^{N/|Z+x|}\frac{\sin(u)}{u}\mathsf{d}u\right)e^{rx}\varphi(x)\mathsf{d}x\\ &\quad +2\int_{\mathbb{R}}\int_0^N\frac{\sin(u(Z+x))}{u(Z+x)}\frac{r^2}{r^2+u^2}\mathsf{d}u(Z+x)e^{rx}\varphi(x)\mathsf{d}x.\end{aligned}$$

Now we can use the Dominated Convergence theorem for the two integrals separately, since the inner integrals are bounded, and converge and $\int_{\mathbb{R}} e^{xr}\varphi(x)\mathsf{d}x<\infty$ by assumption and $\int_{\mathbb{R}}|x|\,e^{xr}\varphi(x)\mathsf{d}x<\infty$ if we also know, that for some $\delta>0$ $M_T^X(r+\delta)<\infty$ and $M_T^X(r-\delta)<\infty$. $\square$

4.3 Examples

In this subsection we will compute the price of a call on a bond in the Lévy Forward Rate Model, where the driving process L is an NIG process under the appropriate forward measure. We will use the Ho-Lee and the Vasicek 'volatility' structures (σ).

4.3.1 NIG process

The Normal Inverse Gaussian (NIG) process is a three parameter family of Lévy processes $NIG(\alpha, \beta, \delta)$ where the parameters α, β, δ satisfy $\alpha, \delta > 0$ and $\alpha > \beta > -\alpha$. Its Lévy-Khinchine triplet is $(\gamma, 0, \nu)$, with

$$\begin{aligned}\gamma &= \frac{2\delta\alpha}{\pi} \int_0^1 \sinh(\beta x) K_1(\alpha x), \mathsf{d}x \\ \nu(\mathsf{d}x) &= \frac{\delta\alpha}{\pi} \frac{\exp(\beta x) K_1(\alpha |x|)}{|x|} \mathsf{d}x\end{aligned}$$

where K_λ is the modified Bessel function of the third kind with index λ.

Let L be a one dimensional NIG process with parameters (α, β, δ). Then the moment generating function of L_T is given by the formula

$$\mathbb{E}(e^{L_T}) = M_T^L(z) = exp(\delta T(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + z)^2})), \tag{30}$$

which converges for $|\beta + \Re z| < \alpha$. Moreover, the density function of L_T is

$$f(x) = \frac{\alpha\delta T}{\pi} \frac{\exp\left(\delta T \sqrt{\alpha^2 - \beta^2 + \beta(x - \mu)}\right)}{\sqrt{(x-\mu)^2 + (\delta T)^2}} K_1(\alpha\sqrt{(x-\mu)^2 + (\delta T)^2}).$$

Note that since L is a time-homogeneous Lévy process, its cumulant is time-independent, hence (30) gives

$$\theta_t(z) = \theta(z) = \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + z)^2}) \text{ for } t \in [0, T^*].$$

For further properties of processes, see [Barndorff-Nielsen, 1997].

4.3.2 Pricing call options

We will use two volatility structures:

- Ho-Lee:

  $$\sigma(s, S) = \hat{\sigma},$$

  hence

  $$\Sigma(s, S) = \hat{\sigma}(S - s).$$

- Vasicek
$$\sigma(s,S)=\hat{\sigma}e^{-a(S-s)},$$
hence
$$\Sigma(s,S)=\frac{\hat{\sigma}}{a}\left(1-e^{-a(S-s)}\right).$$

Theorem 4.2 gives that the price of the call option is
$$C_0(T,U,K)=\frac{1}{2\pi i}KB(0,T)\int_{\Re z=R}e^{-z\xi}\frac{1}{z(z-1)}M_T^X(z)\mathsf{d}z,$$
if X possesses a density function, and if $R>1$ satisfies $M_T^X(R)<\infty$. Take R close to 1, then the condition is fulfilled if $\theta(\Sigma(0,T))$ and $\theta(\Sigma(0,U))$ are well defined.

In order to price options, have to check that X (defined in (29)) possesses a density function, and we have to find the moment generating function of X.

The distribution of X has a density function, since the conditions of Proposition 4.2 with $\gamma=\delta$, $\eta=1$ and $C=\exp(\delta\sqrt{\alpha^2-\beta^2})$.

From Proposition 4.3, the moment generating function of X is given by
$$M_T^X(z)=\exp\int_0^T\left(\theta_s(z\Sigma(s,U)+(1-z)\Sigma(s,T))-\theta_s(\Sigma(s,T))\right)\mathsf{d}s,$$
which is in our case
$$M_T^X(z)=\exp\delta\int_0^T\sqrt{\alpha^2-(\beta+\Sigma(s,T))^2}-\sqrt{\alpha^2-(\beta+z\Sigma(s,U)+(1-z)\Sigma(s,T))^2})\mathsf{d}s.$$

In order to compute the value of the call option, we also need the value of ξ in Theorem 4.2, which is given by
$$\xi:=\log\frac{B(0,T)}{B(0,U)}-\int_0^T(\theta_s(\Sigma(s,T))-\theta(\Sigma(s,U)))\mathsf{d}s+\log K.$$
In our case,
$$\xi:=\log\frac{B(0,T)}{B(0,U)}-\delta\int_0^T(\sqrt{\alpha^2-(\beta+\Sigma(s,U))^2}-\sqrt{\alpha^2-(\beta+\Sigma(s,T))^2})\mathsf{d}s+\log K.$$

So we have to compute integrals of the form
$$\int_0^T\sqrt{\alpha^2-(\beta+f(s))^2}\mathsf{d}s,$$
where $f(s)$ can have values $\Sigma(s,T),\Sigma(s,U)$ or $z\Sigma(s,U)+(1-z)\Sigma(s,T)$.

These integrals have different forms for different volatility structures:

- Ho-Lee case $\Sigma(s,S)=\hat{\sigma}(S-s)$, after the substitution $u=-f(s)$, in all three choices for f, we get
$$\int_0^T\sqrt{\alpha^2-(\beta+f(s))^2}\mathsf{d}s=\hat{\sigma}\int_{-f(0)}^{-f(T)}\sqrt{\alpha^2-(u-\beta)^2}\mathsf{d}u$$
$$=\frac{\hat{\sigma}}{2}\left[(u-\beta)\sqrt{\alpha^2-(u-\beta)^2}+\alpha^2\arctan\left(\frac{u-\beta}{\sqrt{\alpha^2+(u-\beta)^2}}\right)\right]_{u=-f(0)}^{u=-f(T)}.$$

- Vasicek case $\Sigma(s,S) = \frac{\hat{\sigma}}{a}\left(1 - e^{-a(S-s)}\right)$, after the substitution $u = \frac{\hat{\sigma}}{a} - f(s)$, for all three choices of f, we get

$$\int_0^T \sqrt{\alpha^2 - (\beta + f(s))^2}\mathsf{d}s = \frac{1}{a}\int_{\frac{\hat{\sigma}}{a}-f(0)}^{\frac{\hat{\sigma}}{a}-f(T)} \frac{1}{u}\sqrt{\alpha^2 - (\hat{\beta} - u)^2}\mathsf{d}u$$
$$= \frac{1}{a}\Bigg[\sqrt{\alpha^2 + (u-\hat{\beta})^2} + \hat{\beta}\arctan\left(\frac{u - \hat{\beta}}{\sqrt{\alpha^2 + (u-\hat{\beta})^2}}\right)$$
$$-\sqrt{\alpha^2 - \hat{\beta}^2}\log\left(\frac{\left(\sqrt{\alpha^2 - \hat{\beta}^2} + \sqrt{\alpha^2 - (\hat{\beta} - u)^2}\right)^2 - u^2}{u}\right)\Bigg]_{u=\frac{\hat{\sigma}}{a}-f(0)}^{u=\frac{\hat{\sigma}}{a}-f(T)}.$$

5 Conclusion, remarks

In this paper I presented three, essentially two models since the Forward Process Model can be embedded into the Forward Rate Model.

In the Forward Rate Model we had the bond price formula:

$$B(t,T) = B(0,T)\exp\left(-\int_0^t A(s,T)\mathsf{d}s + \int_0^t \Sigma(s,T)\mathsf{d}L_s\right).$$

In this formula, nothing assures that the bond price $B(t,T)$ will be less than or equal to 1 for $t < T$, which means that it can happen that the LIBOR calculated in this model is negative. On the other hand, the model is complete if the dimension of the driving process is 1, and as we saw in section the price of call options and swaptions are given by a closed formula, and can be computed quickly by fast Fourier transform.

In the LIBOR Rate Model the problem of negative rates is solved, since we postulated that

$$L(t,T_i^*) = L(0,T_i^*)\exp(\int_0^t \lambda(s,T_i^*)\mathsf{d}L_s^{T_{i-1}^*}).$$

An other obstacle occurred when we calculated the the characteristics of the driving process: they become no longer deterministic, which meant that $L^{T_{i-1}^*}$ is no longer a time-inhomogeneous Lévy process. We can deal with this problem by approximating the characteristics, for example

$$\begin{aligned}\alpha(t,T_i^*) &\approx \frac{\delta_i L(0,T_i^*)}{1 + \delta_i L(0,T_i^*)}\lambda(t,T_i^*)\\ \beta(t,x,T_i^*) &\approx \frac{\delta_i L(0,T_i^*)}{1 + \delta_i L(0,T_i^*)}\left(\langle\lambda(t,T_i^*),x\rangle - 1\right) + 1.\end{aligned}$$

After the approximation, the characteristics will be easily computable, and one can deduce formulas for option prices. See Chapter 3 in [Kluge, 2005] for details. The drawback of the approximation that the new approximated model might not be arbitrage free. The approximation also raises questions concerning the quality of the approximation, namely how close is the approximated option price value to the value given by the model?

A Appendix

In the appendix, I collected some of the proofs. The first is the proof of the bond price formula in the Forward Rate Model. Then I present a lemma about Doléans-Dade exponential of some specific processes, which is followed by a proposition on how a certain measure change modifies the semi martingale characteristic of some process. In the last section, we prove that the Lévy Forward Rate Model is complete under some further assumptions.

A.1 Proof of Proposition 3.1

Proposition A.1 (Proposition 3.1). *In the Forward rate model the bond prices are given by the formula*

$$\frac{B(t,T)}{B_t} = B(0,T)\exp\left(-\int_0^t A(s,T)ds + \int_0^t \Sigma(s,T)dL_s\right),$$

where

$$A(s,T) := \int_{s\wedge T}^T \alpha(s,S)dS \text{ and } \Sigma(s,T) := \int_{s\wedge T}^T \sigma(s,S)dS.$$

Proof By the (23) we have:

$$f_{t,S} = f_{0,S} + \int_0^t \alpha(s,S)\mathsf{d}s - \int_0^t \sigma(s,S)\mathsf{d}L_s. \tag{31}$$

Integrating (31) with respect to S:

$$\int_t^T f_{t,S}\mathsf{d}S = \int_t^T f_{0,S}\mathsf{d}S + \int_t^T\int_0^t \alpha(s,S)\mathsf{d}s\mathsf{d}S - \int_t^T\int_0^t \sigma(s,S)\mathsf{d}L_s\mathsf{d}S. \tag{32}$$

By taking $t=S$ in (31) and integrating with respect to S we also have:

$$\int_0^t f_{S,S}\mathsf{d}S = \int_0^t f_{0,S}\mathsf{d}S + \int_0^t\int_0^S \alpha(s,S)\mathsf{d}s\mathsf{d}S - \int_0^t\int_0^S \sigma(s,S)\mathsf{d}L_s\mathsf{d}S.$$

Since $\alpha(s,S) := 0$ and $\sigma(s,S) := 0$ for $s < S$, we can rewrite the last equation

$$\int_0^t f_{S,S}\mathsf{d}S = \int_0^t f_{0,S}\mathsf{d}S + \int_0^t\int_0^t \alpha(s,S)\mathsf{d}s\mathsf{d}S - \int_0^t\int_0^t \sigma(s,S)\mathsf{d}L_s\mathsf{d}S \tag{33}$$

Adding up (32) and (33) we get:

$$\int_t^T f_{t,S}\mathsf{d}S + \int_0^t f_{S,S}\mathsf{d}S = \int_0^T f_{0,S}\mathsf{d}S + \int_0^T \int_0^t \alpha(s,S)\mathsf{d}s\mathsf{d}S - \int_0^T \int_0^t \sigma(s,S)\mathsf{d}L_s\mathsf{d}S$$

By the virtue of the ordinary and stochastic version of Fubini's theorem, (see Theorem 65 in Chapter IV. [Potter, 2004]) we can switch the last order of integration in the last two double integrals, and get

$$\int_t^T f_{t,S}\mathsf{d}S + \int_0^t f_{S,S}\mathsf{d}S = \int_0^T f_{0,S}\mathsf{d}S + \int_0^t \int_0^T \alpha(s,S)\mathsf{d}S\mathsf{d}s - \int_0^t \int_0^T \sigma(s,S)\mathsf{d}S\mathsf{d}L_s$$

Notice that $B(t,T) = \exp(-\int_t^T f_{t,S}\mathsf{d}S)$ and $B_t = \exp(\int_0^t f_{S,S}\mathsf{d}S)$. With $A(s,T) := \int_{s\wedge T}^T \alpha(s,S)\mathsf{d}S$ and $\Sigma(s,T) := \int_{s\wedge T}^T \sigma(s,S)\mathsf{d}S$ the last equation becomes:

$$-\log B(t,T) + \log B_t = -\log B(0,T) + \int_0^t A(s,T)\mathsf{d}s - \int_0^t \Sigma(s,T)\mathsf{d}L_s,$$

from which the proposition follows. □

A.2 Tools for measure change

Let L be a time-inhomogeneous Lévy process L satisfying $\mathbb{EM}$, by Proposition 2.7 it can be written in the form

$$L_t = \int_0^t b_s\mathsf{d}s + \int_0^t \sqrt{c_s}\mathsf{d}W_s + \int_0^t \int_{\mathbb{R}^d} x(\mu-\nu)(\mathsf{d}s,\mathsf{d}x).$$

For calculating the semi-martingale characteristics of the driving process L, after a measure change, it will turn out, that the following set of pairs of processes is handy:

Let $\mathcal{Y}$ be the set of processes (u,Y), where

- $u : \Omega \times [0,T^*] \to \mathbb{R}^d$ is a predictable process, such that

$$\int_0^{T^*} \langle u_t, c_t u_t\rangle \,\mathsf{d}t < \infty \ \mathbb{P}\text{-a.s.} \tag{34}$$

- Y is a $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable positive valued function, such that

$$\int_0^{T^*} \int_{\mathbb{R}^d} (|Y(s,x)-1| \wedge |Y(s,x)-1|^2)F_t(\mathsf{d}x)\mathsf{d}s < \infty. \tag{35}$$

Definition A.1. *The process R is the* ***Doléans-Dade exponential*** *of the given semi martingale X, if it satisfies the equation*

$$R = 1 + R_- \cdot X.$$

We start with a lemma about calculating the Doléans-Dade exponential:

Lemma A.1. *Let* $(u, Y) \in \mathcal{Y}$*. Let* H *be the process defined by*

$$H_t = \int_0^t u(s)\sqrt{c_s} dW_s + \int_0^t \int_{\mathbb{R}^d} (Y(s,x) - 1)(\mu - \nu)(ds, dx)$$

where W_s *and* μ *and* ν *from the the representation of* L*. Then* H *is well defined, and if we have the further condition*

$$\int_0^t \int_{\mathbb{R}^d} (Y(s,x) - 1 - \log Y(s,x))\nu(ds, dx) < \infty,$$

then the Doléans-Dade exponential satisfies

$$\mathcal{E}(H)_t = \exp(-\frac{1}{2}\int_0^t \langle u(s), c_s u(s)\rangle\, ds - \int_0^t \int_{\mathbb{R}^d} (Y(s,x) - 1 - \log Y(s,x))\nu(ds, dx) +$$

$$\int_0^t u(s)\sqrt{c_s} dW_s + \int_0^t \int_{\mathbb{R}^d} \log Y(s,x)(\mu - \nu)(ds, dx)).$$

Proof The process H is well defined, since (34) assures that the Itô integral is well defined, by (35) and Theorem 1.33 II.§1d in [Jacod & Shiryaev, 2003] gives that the integral with respect to $(\mu - \nu)$ exists.

Theorem 4.61 in Chapter I. §4f in [Jacod & Shiryaev, 2003] gives that

$$\mathcal{E}(H)_t = e^{H_t - H_0 - \frac{1}{2}\langle H^c\rangle_{p,t}} \prod_{s \le t} (1 + \Delta H_s) e^{-\Delta H_s},$$

where H^c is the continuous martingale part of H, which is $\int_0^\bullet u(s)\sqrt{c_s} \mathrm{d}W_s$ in our case, thus

$$\langle H^c\rangle_{p,t} = [H^c]_t = \int_0^t \langle u(s), c_s u(s)\rangle\, \mathrm{d}s$$

(the bracket [] denotes the quadratic variation). Moreover

$$\Delta H_s = \int_{\mathbb{R}^d} (Y(s,x) - 1)\mu(\{s\}, \mathrm{d}x),$$

where the integral is 0 or just an integral with respect to a Dirac measure, by Proposition 2.5. Thus

$$(1 + \Delta H_s) = \int_{\mathbb{R}^d} Y(s,x)\mu(\{s\}, \mathrm{d}x) = \exp(\int_{\mathbb{R}^d} \log Y(s,x)\mu(\{s\}, \mathrm{d}x)),$$

hence

$$(1 + \Delta H_s)e^{-\Delta H_s} = \exp(\int_{\mathbb{R}^d} (1 - Y(s,x) + \log Y(s,x))\mu(\{s\}, \mathrm{d}x)),$$

from which we have

$$\begin{aligned}\mathcal{E}(H)_t = \exp(&\int_0^t u(s)\sqrt{c_s}\mathsf{d}W_s + \int_0^t \int_{\mathbb{R}^d} (Y(s,x)-1)(\mu-\nu)(\mathsf{d}s,\mathsf{d}x)\\ &-\frac{1}{2}\int_0^t \langle u(s), c_s u(s)\rangle\,\mathsf{d}s + \int_0^t \int_{\mathbb{R}^d} (1-Y(s,x)+\log Y(s,x))\mu(\mathsf{d}s,\mathsf{d}x))\\ = \exp(-\frac{1}{2}&\int_0^t \langle u(s), c_s u(s)\rangle\,\mathsf{d}s - \int_0^t \int_{\mathbb{R}^d} (Y(s,x)-1-\log Y(s,x))\nu(\mathsf{d}s,\mathsf{d}x)\\ &+\int_0^t u(s)\sqrt{c_s}\mathsf{d}W_s + \int_0^t \int_{\mathbb{R}^d} \log Y(s,x)(\mu-\nu)(\mathsf{d}s,\mathsf{d}x)).\square\end{aligned}$$

Proposition A.2. *Let $\hat{\mathbb{P}}$ be a probability measure on Ω such that $\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \mathcal{E}(H)_{T^*}$, where H is the process in Lemma A.1. With the additional condition*

$$\int_0^t \int_{\left\{x\in\mathbb{R}^d:|x|>1\right\}} |x|\, Y(s,x)\nu(ds,dx) < \infty,$$

the semi-martingale characteristics of L under the new measure $\hat{\mathbb{P}}$ are given by

$$\begin{aligned}\hat{B}_t &= B_t + \int_0^t c_s u(s)ds + \int_0^t \int_{\mathbb{R}^d} x(Y(s,x)-1)\nu(ds,dx)\\ \hat{C}_t &= C_t \qquad\qquad (36)\\ \hat{\nu}(ds,dx) &= Y(s,x)\nu(ds,dx),\end{aligned}$$

for $s\in[0,T]$. Moreover $\hat{W}_t := W_t - \int \sqrt{c_s}u(s)ds$ is a Brownian motion.

Remark Note that the proof of the Proposition A.2 and Lemma A.1 works also, if we do not assume that ν is deterministic, just the fact that it can be written in the form $\nu(\mathsf{d}s,\mathsf{d}x) = F_s(\mathsf{d}x)\mathsf{d}s$, where F_s is a (not necessarily deterministic) measure on $\mathbb{R}^d$, for which $\int_{\mathbb{R}^d}(|x|^2\wedge 1)F_s(\mathsf{d}x)$.

Remark Proposition A.2 was inspired by Proposition 2.3 [Kluge, 2005], and the proof uses the proof of Lemma 2.4 and Proposition 2.3 [Kluge, 2005].

Proof We will use Theorem 3.24 Chapter III, §3d in [Jacod & Shiryaev, 2003], which states that if Z is the density process, and if there is a $\hat{\mathcal{P}} := \mathcal{P}\times\mathcal{B}([0,T])$-measurable non-negative function Y, and a $\mathbb{R}^d$-valued predictable process $\beta = (\beta_t)_{0\le t\le T}$ ($\mathcal{P}$ is the predictable σ-field) such that

$$\begin{aligned}YZ_- &= M_\mu^{\mathbb{P}}(Z|\hat{\mathcal{P}}) &(37)\\ \left[Z^c, L^{c,i}\right]_t &= \int_0^t Z_{s-}(\beta c_s)^i \mathsf{d}t, &(38)\end{aligned}$$

then the semi-martingale characteristics of L relative to the new measure $(\mathbb{P}_T)$ with the truncation function $x\mathbf{1}_{|x|\leq 1}$ are:

$$
\begin{aligned}
\hat{B}'_t &= B'_t + \int_0^t c_s\beta_s \mathsf{d}s + \int_0^t \int_{\mathbb{R}^d} x\mathbf{1}_{|x|\leq 1}(Y_s(x)-1)\nu(\mathsf{d}s,\mathsf{d}x) \qquad (39)\\
\hat{C}_t &= C_t\\
\hat{\nu}(\mathsf{d}s,\mathsf{d}x) &= Y_s(x)\nu(\mathsf{d}s,\mathsf{d}x),
\end{aligned}
$$

Now we prove that the conditions (37) and (38) are satisfied with the choice $\beta_s = u(s)$ and $Y_s(x) = Y(s,x)$.

Proof of the validity of condition (37).

Recall that $M_\mu^{\mathbb{P}}$ is a measure on $(\Omega\times[0,T^*],\mathcal{F}\times\mathcal{B}([0,T^*])\times\mathcal{B}(\mathbb{R}^d))$, defined by

$$M_\mu^{\mathbb{P}}(X) = \mathbb{E}(\int_0^{T^*}\int_{\mathbb{R}^d} X_s(x)\mu(\mathsf{d}s,\mathsf{d}x)),$$

where X is a non-negative measurable function. Also recall, that $M_\mu^{\mathbb{P}}(|\hat{\mathcal{P}})$ is the 'conditional expectation' the measure $M_\mu^{\mathbb{P}}$. More precisely, for every non-negative measurable function U, $M_\mu^{\mathbb{P}}(U|\hat{\mathcal{P}})$ is the $M_\mu^{\mathbb{P}}$ almost everywhere unique $\hat{\mathcal{P}}$-measurable function such that

$$M_\mu^{\mathbb{P}}(XM_\mu^{\mathbb{P}}(U|\hat{\mathcal{P}})) = M_\mu^{\mathbb{P}}(XU) \text{ for all non-negative } \hat{\mathcal{P}}\text{-measurable } X.$$

In other words,

$$\mathbb{E}(\int_0^{T^*}\int_{\mathbb{R}^d} X_s(x)M_\mu^{\mathbb{P}}(U|\hat{\mathcal{P}})_s(x)\mu(\mathsf{d}s,\mathsf{d}x)) = \mathbb{E}(\int_0^{T^*}\int_{\mathbb{R}^d} X_s(x)U_s(x)\mu(\mathsf{d}s,\mathsf{d}x))$$

for all non-negative $\hat{\mathcal{P}}$-measurable X.

So we have to check that the following equality holds for all non-negative $\hat{\mathcal{P}}$-measurable X:

$$\mathbb{E}(\int_0^{T^*}\int_{\mathbb{R}^d} X_s(x)Y(s,x)Z_{s-}\mu(\mathsf{d}s,\mathsf{d}x)) = \mathbb{E}(\int_0^{T^*}\int_{\mathbb{R}^d} X_s(x)Z_s\mu(\mathsf{d}s,\mathsf{d}x)).$$

Lemma A.1 gives that

$$\frac{Z_s}{Z_{s-}}\mathbf{1}_{Z_{s-}\neq 0} = \exp(\int_{\mathbb{R}^d}\log Y(s,x)\mu(\{s\},\mathsf{d}x)) = \exp(\log Y(s,\Delta L_s)) = Y(s,\Delta L_s).$$

Hence

$$
\begin{aligned}
\mathbb{E}(\int_0^T\int_{\mathbb{R}^d} X_s(x)Y(s,x)Z_{s-}\mu(\mathsf{d}s,\mathsf{d}x)) &= \mathbb{E}(\sum_{s\leq T} X_s(\Delta L_s)Y(s,\Delta L_s)Z_{s-})\\
&= \mathbb{E}(\sum_{s\leq T} X_s(\Delta L_s)\frac{Z_s}{Z_{s-}}\mathbf{1}_{Z_{s-}\neq 0}Z_{s-})\\
&= \mathbb{E}(\sum_{s\leq T} X_s(\Delta L_s)Z_s)\\
&= \mathbb{E}(\int_0^T\int_{\mathbb{R}^d} X_s(x)Z_s\mu(\mathsf{d}s,\mathsf{d}x)).
\end{aligned}
$$

So (37) is satisfied.

Proof of the validity of condition (38).

By Lemma A.1, we have that $Z = 1 + Z_- \cdot H$, thus $Z^c = 1 + Z_- \cdot H^c$, thus

$$Z^c = Z_- \sqrt{c}\xi \cdot W = Z_- \sqrt{c}\beta \cdot W \text{ and } L^{i,c} = \sqrt{c}^i \cdot W,$$

where $\sqrt{c}^i$ is the i-th row of the matrix $\sqrt{c_s}$, so

$$[Z^c, L^{c,i}]_t = \int_0^t Z_{s-}\beta_s \sqrt{c_s} \mathsf{d}[W]_s \sqrt{c_s}^i = \int_0^t Z_{s-}(\beta_s c_s)^i \mathsf{d}s,$$

what we wanted.

Hence we can use Theorem 3.24 in [Jacod & Shiryaev, 2003], and get that

$$\begin{aligned} \hat{B}'_t &= B'_t + \int_0^t c_s u(s)\mathsf{d}s + \int_0^t \int_{\mathbb{R}^d} x \mathbf{1}_{|x|\leq 1}(Y(s,x)-1)\nu(\mathsf{d}s, \mathsf{d}x) \qquad (40) \\ \hat{C}_t &= C_t \\ \hat{\nu}(\mathsf{d}s, \mathsf{d}x) &= Y(s,x)\nu(\mathsf{d}s, \mathsf{d}x). \end{aligned}$$

All is left to show that the first semi-martingale characteristic changes during the measure change as desired. We had

$$B_t = B'_t + \int_0^t \int_{\{x\in\mathbb{R}^d : |x|>1\}} x\nu(\mathsf{d}s, \mathsf{d}x)$$

$$\begin{aligned} \hat{B}_t &= \hat{B}'_t + \int_0^t \int_{\{x\in\mathbb{R}^d : |x|>1\}} x\hat{\nu}(\mathsf{d}s, \mathsf{d}x) \\ &= \hat{B}'_t + \int_0^t \int_{\{x\in\mathbb{R}^d : |x|>1\}} xY(s,x)\nu(\mathsf{d}s, \mathsf{d}x). \end{aligned}$$

Combining with , we get

$$\hat{B}_t = B_t + \int_0^t c_s u(s)\mathsf{d}s + \int_0^t \int_{\mathbb{R}^d} x(Y(s,x)-1)\nu(\mathsf{d}s, \mathsf{d}x).$$

The computation was correct, since we have the condition

$$\int_0^t \int_{\{x\in\mathbb{R}^d : |x|>1\}} |x|\, Y(s,x)\nu(\mathsf{d}s, \mathsf{d}x) < \infty. \qquad (41)$$

Only left to show that $\hat{W}$ is a Brownian motion under $\hat{\mathbb{P}}$. This is done by applying the previous argument to W instead of L. In that case, the semi martingale characteristics are constant $(0, I, 0)$, where I is a $d \times d$ identity matrix.

Thus the semi martingale characteristics under the new measure become $(\int_0^\bullet c_s u(s)\mathsf{d}s, I, 0)$, hence $\hat{W} = W - \int_0^\bullet c_s u(s)\mathsf{d}s$ is a Brownian motion under the new measure since the semi martingale characteristics of $\hat{W}$ are $(0, I, 0)$. □

Corollary A.1. *Under the conditions of Proposition A.2 the process L can be written in the form*

$$L_t = \int_0^t \hat{b}_s ds + \int_0^t \sqrt{c_s} u(s) d\hat{W}_s + \int_0^t \int_{\mathbb{R}^d} x(\mu - \hat{\nu})(ds, dx),$$

where

$$\begin{aligned} \hat{b}_t &= b_t + \sqrt{c_t} u(t) + \int_{\mathbb{R}^d} x(Y(t,x) - 1) F_t(dx) \\ \hat{c}_t &= c_t \\ \hat{\nu}(ds, dx) &= Y(s,x)\nu(ds, dx) \\ \hat{W}_t &= W_t - \int_0^t \sqrt{c_s} u(s) ds. \end{aligned}$$

Now I collect the conditions which are needed for Lemma A.1 and Proposition A.2.

(i) For the existence of the process H, we need that for the Itô integral:

$$\int_0^{T^*} \langle u(s), c_s u(s) \rangle \, \mathrm{d}s < \infty \;\; \mathbb{P} \text{ almost surely.}$$

(ii) for the integral with respect to $(\mu - \nu)$ we have the condition by Theorem 1.33 II.§1.d in [Jacod & Shiryaev, 2003]:

$$\int_0^{T^*} \int_{\mathbb{R}^d} |Y(s,x) - 1| \wedge (Y(s,x) - 1)\nu(\mathrm{d}s, \mathrm{d}x) < \infty \, \mathbb{P} \text{ a.s.}$$

This condition is implied by

$$\int_0^{T^*} \int_{\mathbb{R}^d} \left(\mathbf{1}_{|x|>1} |Y(s,x) - 1| + \mathbf{1}_{|x|\le 1} |Y(s,x) - 1|^2 \right) \nu(\mathrm{d}s, \mathrm{d}x) < \infty \, \mathbb{P} \text{ a.s.}$$

(iii) The additional condition in Lemma A.1:

$$\int_0^t \int_{\mathbb{R}^d} (Y(s,x) - 1 - \log Y(s,x))\nu(\mathrm{d}s, \mathrm{d}x) < \infty \, \mathbb{P} \text{ a.s.}$$

(iv) The additional condition in Proposition A.2:

$$\int_0^t \int_{|x|>1} |x|(Y(s,x) \vee 1)\nu(\mathrm{d}s, \mathrm{d}x) < \infty \, \mathbb{P} \text{ a.s.}$$

Now I show that we can use the Lemma A.1 and Proposition A.2 in all of the models.

Lemma A.2.

- *In the Forward Rate Model $u(s) = \Sigma(s,T)$ and $Y(s,x) = \exp(\langle \Sigma(s,T), x \rangle)$ satisfy the conditions above.*

- *In the LIBOR Rate Model $u(s) = \lambda(s, T_i^*)$ and $Y(s,x) = \langle \lambda(s, T_i^*), x\rangle$ satisfy the conditions of Lemma A.1*

- *In the LIBOR Rate Model $u(s) = \alpha(s, T_i^*)$ and $Y(s,x) = \beta(s,x,T_i^*)$ satisfy the conditions above.*

Proof Notice that in all cases, the u processes are bounded by the conditions of the model, thus condition (i) is always satisfied since we have that $\int_0^{T^*} \|c_s\| \, \mathsf{d}s < \infty$.

Condition (ii) in the Forward Rate Model: we have that $\exp(\langle \Sigma(s,T), x\rangle) - 1 = O(|x|)$ for $|x| \leq 1$, thus

$$\int_0^{T^*} \int_{|x|\leq 1} |\exp \langle \Sigma(s,T), x\rangle - 1| \, \nu(\mathsf{d}s, \mathsf{d}x) < \infty,$$

since in the definition of time-inhomogeneous Lévy processes we had

$$\int_0^{T^*} \int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu(\mathsf{d}s, \mathsf{d}x) < \infty.$$

Also

$$|\exp \langle \Sigma(s,T), x\rangle - 1| \leq K \exp(M \sum_{j=1}^d |x_j|),$$

for $|x| > 1$ for some $K > 0$, thus by Assumption $\mathbb{EM}'$ we have that (ii) is satisfied.

Condition (iii) in the Forward Rate Model: $\exp \langle \Sigma(s,T), x\rangle - 1 - \langle \Sigma(s,T), x\rangle = O(|x|^2)$ for $|x| \leq 1$, and

$$\exp \langle \Sigma(s,T), x\rangle - 1 - \langle \Sigma(s,T), x\rangle = O(\exp(M \sum_{j=1}^d |x_j|) \text{ for } |x| > 1,$$

thus we have the desired inequality.

Condition (iv) in the Forward Rate Model:

$$|x| \, (\exp \langle \Sigma(s,T), x\rangle \vee 1) = O(\exp((1+\varepsilon) M \sum_{j=1}^d |x_j|) \text{ for } |x| > 1,$$

thus we have the desired inequality.

In the LIBOR Rate Model, the compensator becomes non-deterministic, more precisely

$$\nu^{T_{i-1}^*}(\mathsf{d}s, \mathsf{d}x) = \prod_{k=1}^i \beta(s,x,T_k^*) \nu(\mathsf{d}s, \mathsf{d}x),$$

where

$$\beta(s,x,T_i^*) = \frac{\delta_i L(s-, T_{i-1}^*)}{1 + \delta_i L(s-, T_{i-1}^*)} \left(e^{\langle \lambda(s, T_i^*), x\rangle} - 1 \right) + 1. \tag{42}$$

$L(s-, T^*_{i-1})$ is non-negative, thus we can conclude that

$$|\beta(s,x,T^*_i)| \le \exp\langle\lambda(s,T^*_i),x\rangle \vee 1 \text{ and } |\beta(s,x,T^*_i)-1| \le |\exp\langle\lambda(s,T^*_i),x\rangle - 1|\,.$$

With these upper bounds for β, we can easily check that the conditions (ii), (iii) and (iv) are valid for $Y(s,x) = \exp\langle\lambda(s,T^*_i),x\rangle$.

As for the part when $Y(s,T^*_i) = \beta(s,x,T^*_i)$ with the inequalities above we can easily conclude conditions (ii) and (iv). As for (iii), we need the inequality

$$\exp(-|\langle\lambda(s,T^*_i),x\rangle|) \le \beta(s,x,T^*_i) \le \exp(|\langle\lambda(s,T^*_i),x\rangle|),$$

which can be derived from (42). Thus we have that for $|x| > 1$, $|\log\beta(s,x,T^*_i)| \le K|x|$ for some constant $K > 0$. Hence it is easy to conclude that

$$\int_0^t \int_{|x|>1} (\beta(s,x,T^*_i) - 1 - \log\beta(s,x,T^*_i))\nu(\mathsf{d}s,\mathsf{d}x) < \infty \ \mathbb{P} \text{ a.s.}$$

One can easily check that $\exp(-M\sum_{j=1}^d |x_j|) - 1 \le \beta(s,x,T^*_i) - 1 \le \exp M\sum_{j=1}^d |x_j| - 1$.

Define $g(x) = x - \log(1+x)$, then the minimum of g is at 0, thus

$$g(\beta(s,x,T^*_i)-1) \le g(\exp(-M\sum_{j=1}^d |x_j|) - 1) \vee g(\exp M\sum_{j=1}^d |x_j| - 1),$$

thus we can conclude that for $|x| \le 1$, $\beta(s,x,T^*_i) - 1 - \log\beta(s,x,T^*_i) = O(|x|^2)$ so we are done. □

Corollary A.2 (Proposition 3.3)**.** *If L satisfies $\mathbb{EM}$ with M and ε, and $|\Sigma(t,T)^i| \le M$ for $t \in [0,T]$ and $i = 1,\cdots,d$, b satisfies the drift condition, then under the forward measure $\mathbb{P}_T$, the semi-martingale characteristics of L are given by*

$$\begin{aligned} B^T_s &= B_s + \int_0^s c_u\Sigma(u,T)du + \int_0^s\int_{\mathbb{R}^d} x(e^{\langle\Sigma(u,T),x\rangle} - 1)\nu(ds,dx) \\ C^T_s &= C_s \\ \nu^T(ds,dx) &= e^{\langle\Sigma(s,T),x\rangle}\nu(ds,dx), \end{aligned} \tag{43}$$

for $s \in [0,T]$. Moreover $W^T_t := W_t - \int\sqrt{c_s}\Sigma(s,T^*)ds$ is a Brownian motion.

Proof Define the density process Z by

$$Z_t = \frac{\mathsf{d}\mathbb{P}_T|_{\mathcal{F}_t}}{\mathsf{d}\mathbb{P}|_{\mathcal{F}_t}} = \exp\left(-\int_0^t \theta_s(\Sigma(s,T))\mathsf{d}s + \int_0^t \Sigma(s,T)\mathsf{d}L_s\right).$$

Since from A.1 we know that we can take $u(s) = \Sigma(s,T)$ and $Y(s,x) = \exp(\langle\Sigma(s,T),x\rangle)$, and by the Lemma A.1 and with the drift condition, we have that $\mathcal{E}(H) = Z$, where H is defined as in the lemma. Then again by Lemma A.2 we can use Proposition A.2, with which we are done. □

A.3 Proof of completeness of the Lévy Forward Rate Model

This section is based on the articles [Tomas Björk, 1997], [Eberlein et al., 2005] and we will use the book [Jacod & Shiryaev, 2003]. Our aim will be to prove the following theorem, which is Theorem 1.1 in [Eberlein et al., 2005]:

Theorem A.1 (Theorem 3.1). *Under assumptions of the Froward Rate Model if*

- *the dimension of the driving process L_t is* 1*, or*
- *the dimension of the vector space span*$(\Sigma(t,T) : t \leq T)$ *is at most* 1 *for almost all* T*,*

then the model is complete.

Remark Our main reference in this section is [Eberlein et al., 2005]. We will prove the completeness of the model in the same way as in [Eberlein et al., 2005], but the theorems and proofs are usually simpler. The reason the simplicity is that we will assume that the drift condition is satisfied, which is not assumed in [Eberlein et al., 2005], but derived through the during the proof of completeness.

A.3.1 Preparations

In this section, the drift condition in the Forward Rate Model is still valid. $\mathbb{P}$ is a martingale measure on Ω under which L is a time-inhomogeneous Lévy process in the form

$$L_t = \int_0^t b_s \mathsf{d}s + \int_0^t \sqrt{c_s} \mathsf{d}W_s + \int_0^t \int_{\mathbb{R}^d} x(\mu - \nu)(\mathsf{d}s, \mathsf{d}x).$$

The definition of completeness depends on the filtration. There are two choices for the filtration. We will assume in the following that our original filtration $\mathcal{F}$ is the filtration generated by the process L. From the market, we can only gather the information contained in the bond prices, thus from the economic point of view, we are interested in the equivalent martingale measure with respect to the filtration generated by the bond price processes, which we will denote by $\mathcal{G}$.

We will define six sets of equivalent measures. $\mathcal{Q}_{\mathcal{F}}$, $\mathcal{Q}_{\mathcal{F},loc}$, $\mathcal{Q}'_{\mathcal{F}}$, $\mathcal{Q}'_{\mathcal{F},loc}$, $\mathcal{Q}_{\mathcal{G}}$ and $\mathcal{Q}_{\mathcal{G},loc}$, where the subscript $\mathcal{F}$, $\mathcal{G}$ denotes measures with respect to filtrations $\mathcal{F}$ and $\mathcal{G}$ respectively. The subscript *loc* denotes that the discounted bond price processes are local martingales with respect to measures in the sets. $\mathcal{Q}'_{\mathcal{F}}$ and $\mathcal{Q}'_{\mathcal{F},loc}$ denotes the sets of equivalent measures under which the process L is still a time-inhomogeneous Lévy process.

The first step is to characterize the set of equivalent semi-martingale measures $\mathcal{Q}_{\mathcal{F},loc}$.

Recall the definition of $\mathcal{Y}$: $\mathcal{Y}$ is the set of pairs (u, Y), where

- u is a predictable process with

$$\int_0^{T^*} u(s)c_s u(s)\mathsf{d}s < \infty, \tag{44}$$

- Y is a positive real valued $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$ measurable function with the condition

$$\int_0^{T^*} \int_{\mathbb{R}^d} (Y(s,x)-1)^2 \wedge |Y(s,x)-1| \, \nu(\mathsf{d}s, \mathsf{d}x) < \infty. \tag{45}$$

Define the process

$$N_t := \int_0^t \sqrt{c_s} u(s)\mathsf{d}W_s + \int_0^t \int_{\mathbb{R}^d} (Y(s,x)-1)(\mu-\nu)(\mathsf{d}s, \mathsf{d}x) \tag{46}$$

for $(u, Y) \in \mathcal{Y}$. Note that N is well defined as we saw in Lemma A.1.

We define a equivalence relation on $\mathcal{Y}$, by setting $(u, Y) \sim (u', Y')$ if

$(u-u')c(u-u') = 0$ $\mathsf{d}\mathbb{P}\mathsf{d}s$-almost everywhere, and $Y = Y'$ $\mathsf{d}\mathbb{P}\nu$-almost everywhere.

We will factorize $\mathcal{Y}$ by the equivalence relation, and we will think about $\mathcal{Y}$ as a set of equivalence classes.

Remark The equivalence relation is defined such that for two equivalent pairs, the same process N corresponds to.

Recall that in the Forward Rate Model, we assumed that there is a bond for every maturity date $T \in [0, T^*]$. Further on, it will turn out to be handy if we consider that there might be less bonds on the market. Let $J \subset [0, T^*]$ denote the set of maturity dates for which there is a bond on the market.

Define $\mathcal{Y}_m(J)$ for a $J \subset [0, T^*]$ as the set of pairs $(u, Y) \in \mathcal{Y}$ for which

- for all $T \in J$

$$\langle \Sigma(t,T), c_t u(t) \rangle + \int_{\mathbb{R}^d} (Y(t,x)-1)(e^{\Sigma(s,T),x} - 1)F_t(\mathsf{d}x) = 0, \tag{47}$$

- and for the process N is defined above we have

$$\mathbb{E}(\mathcal{E}(N)_{T^*}) = 1. \tag{48}$$

A.3.2 Further assumption

Apart from the assumptions of the model, we need a technical assumptions taken from [Tomas Björk, 1997]:

Assumption 4 (Assumption 6.1 in [Tomas Björk, 1997]). *Predictable representation property*

Any local martingale M with respect to $\mathbb{P}$ has the form

$$M_t = M_0 + \int_0^t \psi(s) dW_s + \int_0^t \int_{\mathbb{R}^d} \Psi(s,x)(\mu - \nu)(ds, dx)$$

where ψ is a predictable process whith

$$\int_0^{T^*} |\psi(s)|^2 \, ds < \infty \ \mathbb{P}\text{-a.s},$$

and Ψ is a $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$ measurable function with

$$\int_0^{T^*} \int_{\mathbb{R}^d} |\Psi(s,x)| \wedge \Psi(s,x)^2 \nu(ds, dx).$$

Remark Assumption 6.1 in [Tomas Björk, 1997]

A.3.3 The key theorem

Theorem A.2 (Theorem 3.1 in [Eberlein et al., 2005]). *There is a one-to-one correspondence between the probabilities in $\mathcal{Q}_{\mathcal{F},loc}$ and the set set of equivalence classes $\mathcal{Y}_m(J)$. Moreover, the one-to-one correspondence is given by the following: for each measure $Q \in \mathcal{Q}_{\mathcal{F},loc}$, there is a pair $(u, Y) \in \mathcal{Y}_m(J)$ for which the density process satisfies $\frac{dQ}{d\mathbb{P}}\big|_{\mathcal{F}_t} = \mathcal{E}(N)_t$ where $\mathcal{E}(N)$ is Doléans-Dade exponential of N.*

Proof The proof is based on the sketch of the proof of Theorem 3.1 in [Eberlein et al., 2005].

$Q \in \mathcal{Q}_{\mathcal{F},loc} \Rightarrow \exists (u, Y) \in \mathcal{Y}_m(J)$:

Let Q be an equivalent measure. Theorem 3.24 III.§3d in [Jacod & Shiryaev, 2003] tells us that there is a predictable process u with

$$\int_0^{T^*} \langle u(s), c_s u(s) \rangle \, \mathrm{d}s < \infty \ \mathbb{P}\text{-almost surely}$$

and a $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$ measurable positive real valued function Y, for which

$$\int_0^{T^*} \int_{|x| \le 1} |(Y(s,x) - 1)x| \, \nu(\mathrm{d}s, \mathrm{d}x) < \infty \ \mathbb{P}\text{-almost surely}.$$

As in the proof of Proposition 5.6 in [Tomas Björk, 1997], we have that the process

$$t \mapsto \int_0^t \int_{\mathbb{R}^d} (\sqrt{Y(s,x)} - 1)^2 \nu(\mathrm{d}s, \mathrm{d}x)$$

is dominated by the Hellinger process $h(1/2, \mathbb{P}, Q)$, which is Q-a.s finite (thus it is $\mathbb{P}$-a.s finite) by Theorem 2.1 IV.§2.a in [Jacod & Shiryaev, 2003]. It is easy to check that there are positive constants K_1, K_2 for which $K_1(\sqrt{y} - 1)^2 \le |y - 1| \wedge (y -$

$1)^{\leq} K_2(\sqrt{y}-1)^2$ for $y>0$, thus we can conclude that (u,Y) constructed above is in $\mathcal{Y}$.

Now we have to prove that (u,Y) satisfies (47) and (48). Let us denote the density process by $Z_t := \frac{\mathsf{d}Q}{\mathsf{d}\mathbb{P}}\big|_{\mathcal{F}_t}$. With Assumption 4 and with $Q \sim \mathbb{P}$, thus $Z_t > 0$ $\mathbb{P}-$a.s the conditions of Corollary 5.22 III.§5.a in [Jacod & Shiryaev, 2003] are satisfied, thus $Z = \mathcal{E}(N)$ where N is the local martingale defined above. Since Z is a density process, we have that $\mathbb{E}\mathcal{E}(N)_{T^*} = \mathbb{E}Z_{T^*} = 1$, hence (48) is satisfied.

We only have to prove condition (47).

Notice that the discounted bond price processes with the drift condition can also be written as a stochastic exponential by Lemma A.1:

$$\frac{B(t,T)}{B_t} = B(0,T)\mathcal{E}(H^T),$$

where

$$H_t^T = \int_0^t \Sigma(s,T)\sqrt{c_s}\mathsf{d}W_s + \int_0^t \int_{\mathbb{R}^d} (e^{\langle \Sigma(s,T),x\rangle}-1)(\mu-\nu)(\mathsf{d}s,\mathsf{d}x).$$

Thus if the discounted bond price process is a Q local martingale, than we also have that H^T is a Q local martingale. Under Q, the Brownian motion W will be a semi martingale, and its bounded variation term is $\int_0^\bullet \sqrt{c_s}u(s)\mathsf{d}s$. The Q-compensator of μ becomes

$$\tilde{\nu}(\mathsf{d}s,\mathsf{d}x) = Y(s,x)\nu(\mathsf{d}s,\mathsf{d}x).$$

Hence

$$H_t^T = \int_0^t \Sigma(s,T)\sqrt{c_s}\mathsf{d}\tilde{W}_s + \int_0^t \int_{\mathbb{R}^d} (e^{\langle \Sigma(s,T),x\rangle}-1)(\mu-\tilde{\nu}) +$$
$$\int_0^t \langle \Sigma(s,T), c_s u(s)\rangle\,\mathsf{d}s + \int_0^t \int_{\mathbb{R}^d} (Y(s,x)-1)(e^{\langle \Sigma(s,T),x\rangle}-1)\nu(\mathsf{d}s,\mathsf{d}x),$$

where $\tilde{W}_t = W_t - \int_0^t \sqrt{c_s}u_s\mathsf{d}s$.

Since H is a local martingale, we know that

$$\langle \Sigma(t,T), c_t u(t)\rangle + \int_{\mathbb{R}^d} (Y(t,x)-1)(e^{\langle \Sigma(t,T),x\rangle}-1)F_t(\mathsf{d}x) = 0$$

for $T \in J$, since the Ito integral and the integral with respect to $\mu-\nu$ is a local martingale.

So we can conclude that for every local martingale measure Q, there is a pair $(u,Y) \in \mathcal{Y}_m(J)$.

Proof of the other direction $(u,Y) \in \mathcal{Y}_m(J) \Rightarrow \exists Q \in \mathcal{Q}_{\mathcal{F},loc}$:

Take a pair $(u,Y) \in \mathcal{Y}_m(J)$. We can construct the process N, since $(u,Y) \in \mathcal{Y}$. Define the process $Z := \mathcal{E}(N)$. The process Z is non-negative and it is a local martingale since N is a local martingale. Moreover $\mathbb{E}Z_{T^*} = 1$ by (48), thus Z is a martingale. Since $Y(s,x) > 0$, thus $\Delta N_s > 0$ $\mathbb{P}$-a.s, thus by Theorem 4.61 I.§4d in [Jacod & Shiryaev, 2003] get that $Z_{T^*} > 0$ $\mathbb{P}$-a.s.

Up to now, we know that Z is a positive martingale with expected value 1.

Hence we can define a measure Q by $\frac{\mathsf{d}Q}{\mathsf{d}\mathbb{P}} := Z_{T^*}$. $Q \sim \mathbb{P}$ since Z $\mathbb{P}$ a.s positive.

As we saw in the previous part of the proof, (48) assures that H is a local martingale under Q, hence we are done, since Q is an equivalent measure under which the discounted bond price processes are local martingales, i.e $Q \in \mathcal{Q}_{\mathcal{F},loc}$. □

Remark In the last proof, we shown a bit more, namely if $Q \in \mathcal{Q}_{\mathcal{F},loc}$, then its density process is $\mathcal{E}(N)$. Moreover we characterized all of the equivalent measures: dropping the condition (48) in the definition of $\mathcal{Y}_m$, then we get the set of pairs (u,Y) with the property that for every equivalent measure Q there is a pair (u,Y) for which $\frac{\mathsf{d}Q}{\mathsf{d}\mathbb{P}} = \mathcal{E}(N)_{T^*}$.

Remark The measure $\mathbb{P}$ corresponds to the pair $(0,1)$.

A.3.4 Connections between $\mathcal{Q}_{\mathcal{F},loc}$, $\mathcal{Q}'_{\mathcal{F},loc}$ and $\mathcal{Q}'_{\mathcal{F}}$.

In this section our aim is to prove the following theorem, which is one part of Theorem 4.1 in [Eberlein et al., 2005]:

Theorem A.3. *Assume that $d = 1$. The set $\mathcal{Q}_{\mathcal{F},loc}$ contains more than one point if and only if $\mathcal{Q}'_{\mathcal{F},loc}$ contains more than one point.*

We need two technical lemmas. The first one is a version of Lemma 4.3 in [Eberlein et al., 2005]

Lemma A.3. *Let $d = 1$. For any at most countable dense subset J' of J, $\mathcal{Y}_m(J) = \mathcal{Y}_m(J')$.*

Remark For any $J \subset [0,T^*]$ there exists such J', since the Eucledian topology of $[0,T^*]$ has a countable basis.

Proof The proof is inspired by the proof of Lemma 4.3 in [Eberlein et al., 2005]. Clearly it is enough to prove that for all $(u,Y) \in \mathcal{Y}_m(J')$ we also have $(u,Y) \in \mathcal{Y}_m(J)$. Since J' is at most countable, there is a $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that for $\forall \omega_0 \in \Omega_0$

$$\int_0^{T^*} \int_{\mathbb{R}^d} |Y(s,x,\omega_0) - 1| \wedge (Y(s,x,\omega_0) - 1)^2 \nu(\mathsf{d}s,\mathsf{d}x) < \infty \quad (49)$$

$$\Sigma(t,T)c_t u(t,\omega_0) + \int_{\mathbb{R}^d} (e^{\Sigma(t,T)x} - 1)(Y(t,x,\omega_0) - 1)F_t(\mathsf{d}x) = 0 \text{ for } T \in J'. \quad (50)$$

We want to prove that $(u, Y) \in \mathcal{Y}_m(J)$. Since $(u, Y) \in \mathcal{Y}_m(J')$, we know that $(u, Y) \in \mathcal{Y}$, and $\mathbb{E}(\mathcal{E}(N)_{T^*}) = 1$. The only statement left to prove: (50) is true for all $T \in J$ for $\omega_0 \in \Omega_0$. Fix ω_0.

We will split the LHS of (50) into the following parts:

$$\begin{aligned}
U_1(t,T) &:= \Sigma(t,T)c_t u(t,\omega_0) \\
U_2(t,T) &:= \int_{|x|\le 1} (e^{\Sigma(t,T)x} - 1)(Y(t,x,\omega_0) - 1)F_t(\mathsf{d}x) \\
U_3(t,T) &:= \int_{|x|>1} (1 - Y(t,x,\omega_0))F_t(\mathsf{d}x) \\
U_4(t,T) &:= -\int_{|x|>1} e^{\Sigma(t,T)x} F_t(\mathsf{d}x) \\
U_5(t,T) &:= \int_{|x|>1} e^{\Sigma(t,T)x} Y(t,x,\omega_0)F_t(\mathsf{d}x)
\end{aligned}$$

It is enough to prove that all the functions above are well defined if we take $T \in J$, and they are also continuous in T.

(a) U_1, U_3

We have that the function $T \mapsto \Sigma(t,T)$ is continuous by its definition. It is trivial that the functions $U_1(t,T), U_3(t,T)$ are well-defined for $T \in J$ and continuous in T.

(b) U_2 Since $|\Sigma(t,T)| \le M$, we have

$$\int_{|x|\le 1} \left|e^{\Sigma(t,T)x} - 1\right| |Y(t,x,\omega_0) - 1| \, F_t(\mathsf{d}x) \le \int_{|x|\le 1} M\,|x|\,|Y(t,x,\omega_0) - 1| \, F_t(\mathsf{d}x) < \infty.$$

The last equality follows from (49) and from the fact that $\int_{|x|\le 1} x^2 F_t(\mathsf{d}x) < \infty$. Again the dominated convergence theorem yields the desired properties of U_2.

(c) U_4

By the condition $|\Sigma(t,T)| \le M$, we have that that $e^{\Sigma(t,T)x} \le \exp(M \sum_{j=1}^d |x_j|)$.

By Assumption $\mathbb{EM}'$ we have that $\int_{|x|>1} \exp(M \sum_{j=1}^d |x_j|)F_t(\mathsf{d}x) < \infty$, thus we can use the dominated convergence theorem, and get that $U_4(t,T)$ is well defined for $T \in J$ and it is continuous in T.

(d) U_5

$$U_5(t,T) = \int_{x>1} e^{\Sigma(t,T)x} Y(s,x,\omega_0)F_t(\mathsf{d}x) + \int_{x<-1} e^{\Sigma(t,T)x} Y(t,x,\omega_0)F_t(\mathsf{d}x)$$

Take $T \in J$, and a sequence $(T_n) \subset J'$ such that $\Sigma(t,T_n) \nearrow T$ or $\Sigma(t,T_n) \searrow T$, then applying the monotone convergence theorem for the two integrals separately gives U_5 is well defined for $T \in J$. The monotone convergence theorem does not tell us that $U_5(t,T)$ is finite or not.

We can see from (a), (b) and (c) that $U_i(t,T)$ is finite for $T \in J$ and $i = 1,2,3,4$. (50) is true for $T \in J'$, and we know, that for each $T \in J$, there is a sequence $(T_n) \subset J'$ for which $T_n \to T$ and $U_i(t,T_n) \to U_i(t,T)$ for $i = 1,\dots,5$. Hence we can conclude that (50) is true for all $T \in J$ and $U_5(t,T)$ is finite, which completes the proof. $\square$

Lemma A.4. *Let M be a time-inhomogeneous Lévy process which is a local martingale, then the $\mathcal{E}(M)$ is a martingale.*

Remark The lemma is extracted from Proposition 4.4 in [Tomas Björk, 1997] and references were added in the proof.

Proof We know by Theorem 4.61 in I.§4f [Jacod & Shiryaev, 2003] that $\mathcal{E}(M)$ is a local martingale, since M is a local martingale. As we saw in the construction of time-inhomogeneous Lévy processes, M has the form

$$M_t = M_t^1 + M_t^2 + M_t^3,$$

where M_t^1 is the continuous martingale part of M with a deterministic term, M_t^3 is a square integrable martingale, M_t^2 is the process associated with the jumps of M greater than 1. Thus we can write

$$M = M' + M^3,$$

where $M' = M^1 + M^2$. From the construction of M, we also have that M' and M^3 are independent, thus $\mathcal{E}(M) = \mathcal{E}(M')\mathcal{E}(M^3)$.

M' is a square integrable martingale plus a deterministic term, so we can use Lemma 2 in V.3 [Potter, 2004] and deduce that $\mathbb{E}(\sup_{t\in[0,T^*]} \mathcal{E}(M')) < \infty$.

M^3 is a non-homogeneous compound Poisson process with Lévy measure $\mathbf{1}_{|x|>1} F_t(\mathsf{d}x)\mathsf{d}t$ and intensity $\lambda_t = F_t(|x| > 1)$. Define $\Lambda := \int_0^{T^*} \lambda_t \mathsf{d}t$. Theorem 4.61 in I.§4f [Jacod & Shiryaev, 2003] gives that $\mathcal{E}(M^3)_t = \prod_{s\le t}(1 + \Delta M_s^3)$. Hence:

$$\begin{aligned}
\mathbb{E}(\sup_{t\le T^*} \mathcal{E}(M^3)_t) &\le \mathbb{E}(\prod_{s\le t} \left|1 + \Delta M_s^3\right|) \\
&= e^{-\Lambda} \sum_{n=0}^{\infty} \frac{\Lambda^n}{n!} \left(\frac{1}{\Lambda} \int_0^{T^*} \lambda_t \left(1 + \frac{1}{\lambda_t} \int_{|x|>1} |x| \, F_t(\mathsf{d}x) \right) \mathsf{d}t \right) \\
&= e^{-\Lambda} \sum_{n=0}^{\infty} \frac{\Lambda^n}{n!} \left(1 + \frac{\delta}{\Lambda} \right)^n = e^{\delta} < \infty
\end{aligned}$$

with $\delta = \int_0^{T^*} \left(\lambda_t + \int_{|x|>1} |x| \, F_t(\mathsf{d}x) \right) \mathsf{d}t$. So we have that

$$\mathbb{E}(\sup_{t\le T^*} \mathcal{E}(M^3)_t) = \mathbb{E}(\sup_{t\le T^*} \mathcal{E}(M')_t) \mathbb{E}(\sup_{t\le T^*} \mathcal{E}(M^3)_t) < \infty.$$

Hence $\mathcal{E}(M)$ is a martingale. $\square$

Corollary A.3 (Proposition 4.4 in [Eberlein et al., 2005]). $\mathcal{Q}'_{\mathcal{F},loc} = \mathcal{Q}'_{\mathcal{F}}$.

Proof Proof of Proposition 4.4 [Eberlein et al., 2005]. Notice that here we do not need the condition $d = 1$. Clearly $\mathcal{Q}'_{\mathcal{F},loc} \supset \mathcal{Q}'_{\mathcal{F}}$, so we only have to prove that if $Q \in \mathcal{Q}'_{\mathcal{F},loc}$, then $Q \in \mathcal{Q}'_{\mathcal{F}}$. As we saw in the proof of Theorem A.2, we have that the processes H^T are local martingales for $T \in J$. Under Q L is still a time inhomogeneous Lévy process, which means that the pair (u, Y) associated to Q is deterministic, thus H^T is a time-inhomogeneous Lévy process. Lemma A.4 gives that $\mathcal{E}(H^T)$ is a martingale, which gives that Q is an equivalent martingale measure.
□

Proof of Theorem A.3 The proof follows the proof of Theorem 4.1 in [Eberlein et al., 2005]. It is enough to prove that if $\mathcal{Q}_{\mathcal{F},loc}$ contains more than one point, then so does $\mathcal{Q}'_{\mathcal{F},loc}$. By Lemma A.3, we can suppose that J is at most countable. As in the proof of Lemma A.3, there is a set $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ for which (49) and (50) hold. Now take an arbitrary $\omega_0 \in \Omega_0$, and define the following functions:

$$u'(t,\omega) := u(t,\omega_0) \;\; Y'(t,x,\omega) := Y'(t,x,\omega_0).$$

Since u' and Y' are deterministic, we can see that the the process N is a time-inhomogeneous Lévy process. N is also a semi-martingale so we can deduce that $\mathcal{E}(N)$ is a martingale by Lemma A.4, which means that $\mathbb{E}(\mathcal{E}(N)_{T^*}) = 1$. So by the definition of $\mathcal{Y}_m(J)$, we can see that $(u', Y') \in \mathcal{Y}_m(J)$.

Let $Q' \in \mathcal{Q}_{\mathcal{F},loc}$ associated with (u', Y'). Than under Q' L is a time-inhomogeneous Lévy process, since (u', Y') are deterministic. We are done, since if there are two different elements of $\mathcal{Q}_{\mathcal{F},loc}$, than we can construct two different measures in $\mathcal{Q}'_{\mathcal{F},loc}$.
□

Theorem A.3 combined with Corollary A.3 gives:

Corollary A.4 (Theorem 4.5 in [Eberlein et al., 2005]). *Assume $d = 1$. The set $\mathcal{Q}_{\mathcal{F},loc}$ contains more than one point if and only if $\mathcal{Q}'_{\mathcal{F}}$ contains more than one point.*

A.3.5 Changing the filtration

For each $t \in [0, T^*]$ define the vector space $E_t = E_t(J)$ by the subspace of $\mathbb{R}^d$ generated by the vectors $\Sigma(t,T)$ for $T \in J$. Let Π_t be the orthogonal projection to E_t.

Notice that since the function $T \mapsto \Sigma(t,T)$ is continuous, and E_t is a closed in $\mathbb{R}^d$, thus $E_t(J) = E_t(J')$ where $J' \subset J$ is a countable dense subset of J. This remark gives that the the function $t \mapsto E_t$ is measurable, and $t \mapsto \Pi_t$ is measurable as well.

Define

$$\bar{L}_t := \int_0^t \Pi_s \mathsf{d} L_s.$$

Note that $\bar{L}$ is also a time-inhomogeneous Lévy process, and by $\Sigma(t, T) = \Pi_t \Sigma(t, T)$, we have that

$$\int_0^t \Sigma(s, T) \mathsf{d} L_s = \int_0^t \Sigma(s, T) \Pi_s \mathsf{d} L_s = \int_0^t \Sigma(s, T) \mathsf{d} \bar{L}_s.$$

With this notation, we can rewrite the discounted bond price process as

$$\frac{B(t, T)}{B_t} = \exp\left(- \int_0^t \theta_s(\Sigma(s, T)) \mathsf{d}s + \int_0^t \Sigma(s, T) \mathsf{d} \bar{L}_s \right).$$

Proposition A.3 (Proposition 5.1 in [Eberlein et al., 2005]). *$\mathcal{G}_t = \bar{\mathcal{F}}_t$, where $\bar{\mathcal{F}}$ is the filtration generated by the process $\bar{L}$.*

Proof The same as proof of Proposition 5.1 in [Eberlein et al., 2005]. Clearly $\mathcal{G}_t \subset \bar{\mathcal{F}}_t$.

Proof of the converse $\mathcal{G}_t \supset \bar{\mathcal{F}}_t$:

We can replace J by J', and get that $\mathcal{G}_t$ will decrease, since there will be less processes which generate the filtration $\mathcal{G}_t$.

So we can suppose that J is countable, thus we can write $J = \{T_1, T_2, \ldots\}$. The filtration $\mathcal{G}_t$ is generated by the processes

$$X_t^{T_i} := \int_0^t \Sigma(s, T_i) \mathsf{d} L_s.$$

Our aim is to express $\bar{L}$ in terms of X^{T_i} for $i = 1, 2 \ldots$.

Define $\kappa(t, i)$ as the smallest $j \geq 1$ for which $\dim(\operatorname{span}(\Sigma(t, T_k), k = 1, \ldots, j)) = i$. Then the dimension of E_t is $d_t = \inf\{i : \kappa(t, i+1) = \infty\}$. Define a $d \times d_t$ matrix G_t, where the ith column of G_t is $\Sigma(t, T_{\kappa(t,i)})^*$, where the subscript * denotes the transpose. Define

$$B_t := G_t^* G_t.$$

Then we have

$$\Pi_t = G_t B_t^{-1} G_t^*.$$

Hence we have

$$\begin{aligned}
\bar{L}_t &= \int_0^t \Pi_t \mathsf{d} L_s \\
&= \int_0^t G_t B_t^{-1} G_t^* \mathsf{d} L_s \\
&= \int_0^t G_t B_t^{-1} \sum_{i=1}^{d_t} \Sigma(t, T_{\kappa(t,i)}) \mathsf{d} L_s \\
&= \sum_{i=1}^{d_t} \int_0^t G_t B_t^{-1} \mathsf{d} X_s^{T_{\kappa(t,i)}},
\end{aligned}$$

from which the claim follows. □

Combining Proposition A.3 with Corollary A.4 gives:

Theorem A.4 (Theorem 5.2 in [Eberlein et al., 2005]). *Assume that* $\dim E_t \leq 1$ *for almost every* $t \in [0, T^*]$, *then* $\mathcal{Q}_{\mathcal{G},loc}$ *contains more then one point if and only if* $\mathcal{Q}'_{\mathcal{G}}$ *contains more than one point.*

A.3.6 Further preparations for Theorem 3.1

We will introduce some more sets, which are the 'local' versions of the sets defined in section A.3.1.

For $u \in \mathbb{R}^d$ and $t \in [0, T^*]$, we denote the set of pairs (u, f) by $\mathcal{Y}_{t,u}$ where $v \in \mathbb{R}^d$ and f is a Borel function on $\mathbb{R}^d$ which satisfies the following conditions:

$$\int_{\mathbb{R}^d} |f(x)| \wedge |f(x)|^2 F_t(\mathsf{d}x) < \infty \tag{51}$$

$$\int_{|x|>1} |f(x)| e^{\langle u,x\rangle} F_t(\mathsf{d}x) < \infty \tag{52}$$

$$\langle u, c_t v\rangle + \int_{\mathbb{R}^d} (e^{\langle u,x\rangle} - 1) f(x) F_t(\mathsf{d}x) = 0. \tag{53}$$

Notice that the integral $\int_{\mathbb{R}^d} (e^{\langle u,x\rangle} - 1) f(x) F_t(\mathsf{d}x)$ is finite, since $\int_{\mathbb{R}^d} |x|^2 \wedge 1 F_t(\mathsf{d}x) < \infty$ and (52) holds. Define the set $\mathcal{Y}'_{t,u}$ as the non-degenerate elements of $\mathcal{Y}_{t,u}$: $\mathcal{Y}'_{t,u}$ consists of pairs $(v, f) \in \mathcal{Y}_{t,u}$ for which

$$\langle v, c_t v\rangle + \int_{\mathbb{R}^d} |f(x)| F_t(\mathsf{d}x) > 0.$$

For any subset $J \subset [0, T^*]$ set

$$\mathcal{U}_J := \left\{ t : \bigcap_{T \in J} \mathcal{Y}'_{t,\Sigma(t,T)} \neq \emptyset \right\}.$$

We will denote the Lebesgue measure on $\mathbb{R}$ by λ.

Theorem A.5 (Theorem 6.1 in [Eberlein et al., 2005]). *If* $\lambda(\mathcal{U}_{J'}) = 0$, *for some countable* $J' \subset J$, *then the set* $\mathcal{Q}_{\mathcal{F},loc}$ *has one element.*

Proof Follows the proof of Theorem 6.1 in [Eberlein et al., 2005]. Recall that by the construction of the model, we already have one measure in $\mathcal{Q}_{\mathcal{F},loc}$, namely $\mathbb{P}$, which is associated to the pair (u, Y), where u is the constant 0, and Y is constant 1.

Suppose that $\mathcal{Y}_m(J')$ contains more than one element. The proof of Theorem A.3 tells us that there is also a deterministic pair $(u', Y') \in \mathcal{Y}_m(J')$ which differs from (u, Y). The definition of $\mathcal{Y}_m(J')$ gives that

$$(u'(t), Y'(t, .) - 1) \in \mathcal{Y}_{t,\Sigma(t,T)} \text{ for } T \in J' \text{ and } t \in [0, T^*].$$

This translates to

$$(u'(t), Y'(t,.) - 1) \in \bigcap_{T \in J'} \mathcal{Y}_{t,\Sigma(t,T)} \text{ for } t \in [0, T^*]. \tag{54}$$

Since $(u, Y) \nsim (u', Y')$, which means that the set of $t \in [0, T^*]$ for which

$$\langle u'(t), c_t u(t) \rangle > 0 \text{ or } \int_{\mathbb{R}^d} |Y(t,x) - 1| \, F_t(\mathsf{d}x) > 0 \tag{55}$$

has positive Lebesgue measure. (54) and (55) gives that the set

$$\left\{ t \in [0, T^*] : (u', Y'(t,.)) \in \bigcap_{T \in J'} \mathcal{Y}'_{t,\Sigma(t,T)} \right\}$$

has positive Lebesgue measure. Hence $\mathcal{U}_{J'}$ has positive measure, which completes the proof. □

Lemma A.5 (Lemma 6.2 in [Eberlein et al., 2005])**.** *Let U be a subset of $\mathbb{R}^d$ whose closure has a nonempty interior, and let $t \in [0, T^*]$. Then the set $\bigcap_{u \in U} \mathcal{Y}'_{t,u}$ is empty.*

Proof Follows the proof of Lemma 6.2 in [Eberlein et al., 2005]. Suppose that $(v, f) \in \bigcap_{u \in U} \mathcal{Y}'_{t,u}$. Take a vector u_0 from the interior of the closure of U. First we will prove that there is a positive δ for which $(v, f) \in \mathcal{Y}_{t,u'}$ if $|u' - u_0| < \delta$.

Let us denote the interior of the closure of U by $\mathrm{int}(\bar{U})$. Since $u_0 \in \mathrm{int}(\bar{U})$, we can conclude that there exists $u_1, \dots, u_{2^d}$ vectors in $\mathrm{int}(\bar{U})$ such that the vectors $u_0 - u_i$ are in the 2^d different quadrants of $\mathbb{R}^d$. Define

$$\delta = \min_{\substack{i = 1, \dots, d \\ j = 1, \dots, 2^d}} \left| u_0^i - u_j^i \right| > 0.$$

Let $D := \left\{ u \in \mathbb{R}^d : |u - u_0| < \delta \right\}$. Take $u \in D$. From the condition (52) we can conclude that

$$\begin{aligned} \int_{|x|>1} |f(x)| \, e^{\langle u', x \rangle} F_t(\mathsf{d}x) &= \int_{|x|>1} |f(x)| \, e^{\langle u' - u_0, x \rangle} F_t(\mathsf{d}x) \\ &\leq \int_{|x|>1} |f(x)| \sum_{j=1}^{2^d} e^{\langle u_i - u_0, x \rangle} F_t(\mathsf{d}x) < \infty. \end{aligned}$$

From this inequality, we can deduce that (52) holds for u' also, and by the Dominated Convergence Theorem we can see that the function $u \mapsto \int_{|x|>1} f(x)(e^{\langle u, x \rangle} - 1) F_t(\mathsf{d}x)$ is continuous on D.

(51) and $\int_{|x| \leq 1} |x|^2 \, F_t(\mathsf{d}x) < \infty$ gives that there is a constant K for which

$$\int_{|x| \leq 1} \left| e^{\langle u, x \rangle} - 1 \right| |f(x)| \, F_t(\mathsf{d}x) \leq K \int_{|x| \leq 1} |x| \, |f(x)| \, F_t(\mathsf{d}x) < \infty \text{ for } u \in D.$$

Now we can use the Dominated Convergence Theorem, and deduce that the function

$$u \mapsto \int_{|x|\leq 1} f(x)(e^{\langle u,x\rangle} - 1)F_t(\mathsf{d}x)$$

is continuous on D. So we can see that

$$u \mapsto \int_{\mathbb{R}^d} f(x)(e^{\langle u,x\rangle} - 1)F_t(\mathsf{d}x)$$

is a continuous function on D, thus (53) is true for $u \in D$. Let f_+ and f_- be the positive and negative parts of the function f. Define the measures $F_\pm(\mathsf{d}x) := f_\pm(x)e^{\langle u_0,x\rangle}F_t(\mathsf{d}x)$.

Since $\int_{|x|\leq 1} f(x)(e^{\langle u_0,x\rangle} - 1)F_t(\mathsf{d}x)$ and $\int_{|x|\leq 1} f(x)(e^{\langle u,x\rangle} - 1)F_t(\mathsf{d}x)$ is finite for $u \in D$, the following is well defined

$$\int_{|x|\leq 1} f(x)(e^{\langle u-u_0,x\rangle} - 1)e^{\langle u_0,x\rangle}F_t(\mathsf{d}x).$$

Thus we can deduce that

$$\int_{|x|\leq 1} |x|\,|f(x)|\,e^{\langle u_0,x\rangle}F_t(\mathsf{d}x)$$

is finite. So we can define

$$\lambda_\pm := \int_{|x|\leq 1} xF_\pm(\mathsf{d}x).$$

With the definitions above, (53) can be rewritten as

$$\langle z, c_t v + \lambda_+ - \lambda_-\rangle + \int_{\mathbb{R}^d}(e^{\langle z,x\rangle} - 1)F_+(\mathsf{d}x) = \int_{\mathbb{R}^d}(e^{\langle z,x\rangle} - 1)F_-(\mathsf{d}x) \text{ for } |z| < \delta. \quad (56)$$

The equation (56) holds also for $z \in \mathbb{C}^d$ where $|\Re z|^2 < \delta$. Take $z = iw$ where $w \in \mathbb{R}^d$. With this substitution, the LHS and RHS of (56) can be considered as the logarithm of the characteristic function of some infinitely divisible distributions. The Lévy-Khinchine triplet of the two distributions are $(c_t v + \lambda_+ - \lambda_-, 0, F_+)$ and $(0, 0, F_-)$. The triplet is unique, which gives that $F_+ = F_-$ and $c_t v + \lambda_+ - \lambda_- = 0$. From $F_+ = F_-$ we can deduce that $\lambda_+ = \lambda_-$, thus $c_t v = 0$.

From the definition of $F_\pm$ we get that $f_+(x)F_t(\mathsf{d}x) = f_-(x)F_(\mathsf{d}x)$, thus

$$\int_{\mathbb{R}^d} |f(x)|\,F_t(\mathsf{d}x) = 0.$$

Hence

$$\langle v, c_t v\rangle + \int_{\mathbb{R}^d} |f(x)|\,F_t(\mathsf{d}x) = 0,$$

thus $(v, f) \notin \mathcal{Y}'_{t,u}$ for any $u \in \mathbb{R}^d$. $\square$

Theorem A.6. *Assume that $d = 1$ and J is dense in $[0, T^*]$, and set*

$$H := \left\{ t \in [0, T^*] : \int_t^{T^*} |\sigma(t,s)| \, \mathsf{d}s = 0, \; c_t + F_t(\mathbb{R}) > 0 \right\}.$$

Then the set $\mathcal{Q}_{\mathcal{F},loc}$ exactly one point if and only if $\lambda(H) = 0$.

Proof Assume that $\lambda(H) = 0$. By Theorem A.5, it is enough to prove that

$$\bigcap_{T \in J'} \mathcal{Y}'_{t,\Sigma(t,T)} = \emptyset. \tag{57}$$

for $t \notin H$.

Since $t \notin H$, then either (a) $c_t + F_t(\mathbb{R}) = 0$ or (b) $c_t + F_t(\mathbb{R}) > 0$ and $\int_t^{T^*} |\sigma(t,s)| \, \mathsf{d}s > 0$.

Case (a) $c_t + F_t(\mathbb{R}) = 0$:

then $\mathcal{Y}'_{t,u} = \emptyset$ for every $u \in \mathbb{R}$ from which (57) follows.

Case (b) $c_t + F_t(\mathbb{R}) > 0$ and $\int_t^{T^*} |\sigma(t,s)| \, \mathsf{d}s > 0$:

Let $U := \{\Sigma(t,T) : T \in J\}$. The function $T \mapsto \Sigma(t,T)$ is continuous thus the closure of U, $\bar{U}$, is an a closed interval. Moreover $\Sigma(t,t) = 0$, hence $0 \in \bar{U}$. Since $\int_t^{T^*} |\sigma(t,s)| \, \mathsf{d}s > 0$, we have $\bar{U} \neq \{0\}$, hence $\bar{U}$ contains an interior point, then Lemma A.5 gives (57).

For the converse, assume that $\lambda(H) > 0$. We will construct a $Q \in \mathcal{Q}_{\mathcal{F},loc}$ which is different from $\mathbb{P}$.

Define the pair (u', Y') as follows:

$$\begin{aligned} u'(t) &:= 0 \\ Y'(t,x) &:= \left\{ \begin{array}{ll} 1 + (1 \wedge |x|) & \text{if } t \in H \\ 1 & \text{if } t \notin H \end{array} \right. . \end{aligned}$$

Since (u', Y') is a pair of deterministic functions, it is easy to check that $(u', Y') \in \mathcal{Y}_m(J)$: $(u', Y') \in \mathcal{Y}$, since $\int_{\mathbb{R}^d} (|x|^2 \wedge 1) F_t(\mathsf{d}x) < \infty$ conditions (47) is trivially satisfied, and Lemma A.4 gives that the process N associated to (u', Y') is a martingale, hence (48) is true. Let Q be the measure associated to the pair (u', Y').

Since $\lambda(H) > 0$, so we have that $Q \neq \mathbb{P}$. □

A.3.7 Proof of Theorem 3.1

Clearly, it is enough to prove that if $\dim(E_t) \leq 1$ for $t \in [0, T^*]$, then there is a unique martingale measure. Let $\bar{L}_t = \Pi_t L_t$ as defined in Section A.3.5. One can find a measurable function $t \mapsto d_t$ such that $d_t \in E_t$, such that if $\dim(E_t) = 1$, then

$d_t \neq 0$. The characteristics of $\bar{L}$ are

$$\begin{aligned}
\bar{b}_t &= \langle d_t, b_t \rangle \\
\bar{c}_t &= \langle d_t c_t, d_t \rangle \\
\bar{F}_t(\mathsf{d}y) &= \begin{cases} \int_{\langle d_t, x\rangle = y} F_t(\mathsf{d}x)\mathsf{d}y & \text{if } d_t \neq 0 \\ 0 & \text{if } d_t = 0. \end{cases}
\end{aligned}$$

Proposition A.3 gives that the filtration $(\mathcal{G}_t)$ is the filtration generated by $\bar{L}$. From the form of the characteristics, we can see that either $\bar{c}_t + \bar{F}_t(\mathbb{R}) = 0$ or $\int_t^{T^*} |\sigma(t,s)\mathsf{d}s > 0|$. Applying Theorem A.6 for $\bar{L}$ gives that $\mathcal{Q}_{\mathcal{G},loc}$ has exactly one element. $\square$

Remark From the proof we can see that we have proved a bit more, namely that there is exactly one local martingale measure.

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