The Milnor fiber of the singularity $f(x, y) + zg(x, y) = 0$.

Master's thesis

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1 Introduction

In this note we give a description of the Milnor fiber of a hypersurface singularity of the form $(\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$, $(x, y, z) \mapsto f(x, y) + zg(x, y)$ for germs $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. We will only require that f and g have no common factors. This singularity is not isolated; the singular set will be the z-axis. In fact, we determine the diffeomorphism type of the Milnor fiber in terms of a simultaneous embedded resolution graph of f and g. Singularities of this type play an important role in the investigations of sandwiched singularities, as described in [2].

1.1 Topological description of hypersurface singularities

In this subsection we recall some of the general properties of the Milnor fiber of a holomorphic germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$, the monodromy associated to such a germ, and other invariants related to these two.

Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a hypersurface singularity, denote by B_δ the closed ball with radius δ around the origin in $\mathbb{C}^{n+1}$, and by D_ϵ the closed disc around the origin in $\mathbb{C}$ with radius ϵ. D will denote an arbitrary closed disc in the complex plane. We let $V_f = \{z \in \mathbb{C}^{n+1} : f(z) = 0\}$ and $\mathrm{Sing}(V_f) = \{z \in \mathbb{C}^{n+1} : \partial f = 0\}$. The link of f is defined as $K = V_f \cap \partial B_\delta$ for $0 < \delta \ll 1$.

The Milnor fiber F_f of f is by definition the fiber $f^{-1}(\epsilon) \cap B_\delta$ for $0 < \epsilon \ll \delta \ll 1$. Then F_f is a smooth $2n$ dimensional manifold, and so has the homotopy type of a CW complex. Actually, F_f has the homotopy type of a finite n dimensional CW complex, as proved in [6]. Moreover, if s is the dimension of the singular locus $\mathrm{Sing}(V_f)$, then F_f is $(n - s - 1)$-connected, as proved in [5].

Let $E = f^{-1}(\partial D_\epsilon) \cap B_\delta$. The function $E \to \partial D_\epsilon$, $z \mapsto f(z)$ is a locally trivial fiber bundle with fiber F_f. If $T = \{z \in \partial B_\delta : |f(z)| < \epsilon\}$, we can define another fiber bundle $\partial B_\delta \setminus T \to \partial D_1$, $z \mapsto f(z)/|f(z)$. These two fiber bundles are isomorphic. In fact, there is a bundle-isomorphism $E \to \partial B \setminus T$ which restricts to the identity on ∂T. In particular, we have a diffeomorphism

$$F_f \cong \{z \in \partial B_\delta \setminus T : f(z)/|f(z)| = 1\}. \tag{1}$$

The monodromy of this bundle is a diffeomorphism $m : F \to F$ with the property that this bundle is isomorphic to the bundle given by $F \times I/((p, 0) \sim (m(p), 1)) \to I/(0 \sim 1)$, $(p, t) \mapsto t$. The monodromy is determined by the bundle up to isotopy, and the bundle is determined up to bundle isomorphism by the monodromy. The monodromy induces linear isomorphisms $h_i : H_i(F; \mathbb{C}) \to H_i(F; \mathbb{C})$. In [4], Griffiths gives a discussion of, and references to four different proofs of the theorem stating that the eigenvalues of these monodromy operators are roots of unity.

We call the product

$$\zeta_f(t) = \prod_{i=0}^{n} \det(I - th_i)^{(-1)^i}.$$

the zeta function associated with the singularity f. This product is well defnied because F is a finite CW complex, and so $\dim_{\mathbb{C}} H_*(F;\mathbb{C}) < \infty$. The zeta function behaves multiplicatively in the following sense:

Let C be a subset of F so that $\dim H_*(C;\mathbb{C}) < \infty$ and m_f restricts to a homeomorphism $m_C : C \to C$. Let us call such a subset good with respect to m. Then m_C induces a linear automorphism $h_{C,i}$ on $H_i(C;\mathbb{C})$ and we define

$$\zeta_C(t) = \prod_{i=0}^{\infty} \det(I - th_{C,i})^{(-1)^i}.$$

If A and B are subsets of F so that the interiors of A and B cover X and A, B and $A \cap B$ are good with respect to m, then the identity $\zeta_f(t) = \zeta_A(t)\zeta_B(t)\zeta_{A\cap B}(t)^{-1}$ follows from the Mayer-Vietoris exact sequnce.

The singular fiber of f is defined as

$$F_{f,sing} = \{z : |z| = \delta,\ |f(z)| > 0,\ f(z)/|f(z)| = 1\} \cup K.$$

Usually, $F_{f,sing}$ is not a smooth manifold. By the description (1) of F_f we have an inclusion $\iota : F_f \hookrightarrow F_{f,sing}$.

If f is an isolated singularity, then ι is a homotopy equivalence, as proved in [6]. For non-isolated singularities this does generally not hold.

The monodromy m_f can be extended to a homeomorphism $m_{f,sing} : F_{f,sing} \to F_{f,sing}$, which is called the singular monodromy.

1.2 The case of isolated hypersurface singularities

A singularity given by $f = 0$ for some $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is called isolated if $\mathrm{Sing}(V_f) = \{0\}$. Working with isolated singularities only has proven to give much stronger results than with arbitrary hypersurface singularities. In this case the singular locus $\mathrm{Sing}(V_f)$ has dimension 0, so by an earlier remark, F_f is $n-1$ connected. But a stronger statement holds: If f is isolated, then F_f is homotopically a bouquet of n-spheres. That is,

$$F_f \overset{ht}{\sim} \bigvee_{\mu} S^n.$$

The number μ is clearly an important invariant of the singularity, as it determines completely the homotopy type of the Milnor fiber. This number is called the Milnor number of the singularity, and it has several characterizations:

- Clearly, μ is the n-th Betti number of F_f.

- We have $\mu = \dim_{\mathbb{C}} \mathbb{C}\{z_0, \ldots, z_n\}/(\partial f)$, where (∂f) is the ideal of $\mathbb{C}\{z_0, \ldots, z_n\}$ spanned by the partial derivatives of f. The algebra $\dim_{\mathbb{C}} \mathbb{C}\{z_0, \ldots, z_n\}/(\partial f)$ is called the Milnor algebra of f.

- We have $\mu = \deg G$ where $G : S^{2n+1} \to S^{2n+1}$ is the map

$$G = \frac{(\partial_1 f, \ldots, \partial_{n+1} f)}{\|(\partial_1 f, \ldots, \partial_{n+1} f)\|}.$$

These characterizations are discussed in [6], [7].

1.3 Methods for non-isolated singularities

The results for isolated singularities discussed in the previous subsection do not hold in general. The homotopy type of the fiber is not fully understood; even the Betti numbers can be hard to determine. There are however methods which can be used to determine some of the invariants of the fiber in some cases. Let $\Sigma = \mathrm{Sing}(V_f)$. We can write $\Sigma = \Sigma^s \cup \ldots \cup \Sigma^0$, where $\Sigma^j \setminus \Sigma^{j-1}$ is j dimensional and smooth. Here, $s = \dim \mathrm{Sing}(V_f)$ as before.

1.4 Is F homotopically a bouquet of spheres?

In the case of an isolated hypersurface singularity, the Milnor fiber is homotopically a bouquet of n-spheres. For non-isolated singularities this is no longer true, but one can ask if F is a bouquet of spheres in the expected dimension, i.e. from $n - s$ to n, where s is the dimension of the singular locus.

This following theorem is discussed in [11] which also contains a reference to a proof by the same author.

Theorem 1.1. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ defining such a singularity so that the singular locus is an ICIS, and the transversal singularity outside the origin has type A_1. Then the homotopy type of F_f is either a bouquet of n-spheres, or a bouquet of n-spheres and a single $(n-1)$-sphere.*

The following results can be found in [8].

Theorem 1.2. *Let $n \geq 3$ and $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ whose singular locus is one dimensional. Then F_f is a bouquet of spheres if and only if $H(F_f, \mathbb{Z})$ is free.*

Note that the condition $n \geq 3$ is necessary: If $f(x, y, z) = xyz$, then F_f is homotopically $S^1 \times S^1$. The proof of this statement can be found in [8], but it goes along the following lines:

Let F be a simply connected CW complex so that $H_*(F, \mathbb{Z})$ is free and the Hurewicz homomorphism $\pi_*(F) \to H_*(F, \mathbb{Z})$ is surjective. Then it is possible to construct a map

$$\bigvee_{i \geq 2} \bigvee_{b_i} S^i \to F$$

which induces an isomorphism on homotopy and homology groups, and so by a theorem of Whitehead, it is a homotopy equivalence (here, b_i are the Betti numbers of F). In particular, if $l \geq 3$, $H_i(F, \mathbb{Z}) = 0$ for $i \notin \{l, l-1\}$, $H_*(F, \mathbb{Z})$ is free and F is simply connected then the Hurewicz map is automatically surjective by a theorem of Hurewicz.

Theorem 1.3. *Let us now make the assumption that $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is a germ whose singular locus Σ is a two dimensional ICIS, and that the transversal singularity is A_1 outside a curve in Σ. Then F_f has the same homotopy type as one of the following spaces:*

$$S^n \vee \ldots \vee S^n, \quad S^{n-1} \vee S^n \vee \ldots \vee S^n, \quad S^{n-2} \vee S^n \vee \ldots \vee S^n.$$

For $n = 2$, one must replace the last space on this list with $S^0 \times S^2 \vee \ldots \vee S^2$.

1.5 The present family of germs

In this note we will study the particular germs which have the form $(\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$, $(x, y, z) \mapsto f(x, y) + zg(x, y)$, where f, g are plane curve singularities without common factors. In this case, the singular is the z-axis, in particular, it is one dimensional. However, we can not expect to use the theorems from the previous subsection, since the Milnor fiber is generally not a bouquet of spheres in this case. We find a resolution for this singularity, which can be found by finding a common embedded resolution $V \to \mathbb{C}^2$ for f and g, and then using $V \times \mathbb{C} \to \mathbb{C}^3$. The diffeomorphism type of the Milnor fiber is determined and described by the resolution graph associated to the common embedded resolution of f and g, decorated by the multiplicities of f and g. This means in particular that the boundary of the fiber only depends on the resolution graph obtained by resolving f and g. We also find a simple formula for the zeta function and Euler characteristic of the germ.

These germs were considered by de Jong and van Straten in [2].

The germs considered in this note include the $T_{a,\infty,\infty}$ germs $x^a + xyz = 0$ and the $T_{a,b,\infty}$ germs $x^a + y^b + xyz = 0$, as well as cylinders of plane curve singularities $f(x,y)=0$, or singularities of the form $zg(x,y)=0$. The boundary of the Milnor fiber of these singularities were all considered as examples in [9].

2 Preliminaries

2.1 The topology of singularities on plane curves

We will consider the case when $n = 1$, i.e. when f is a plane curve singularity. Then f is isolated if and only if f has no repeated factors. Write $f = f_1^{q_1} f_2^{q_2} \cdots f_k^{q_k}$ where $f_1, \ldots, f_k$ are the k different factors of f. In this case, K is a smooth 1-dimensional submanifold of ∂B_δ, and T is a tubular neigborhood around K. In fact, there exists a projection $p : \overline{T} \to K$ which is a trivial D-bundle. To be more precise, we have $K = \cup_{i=1}^k K_i$, where $K_i = \{z \in \partial B_\delta : f_i = 0\}$, and $T = \cup_{i=1}^k T_i$, where T_i is the component of T which has K_i as a subset. From the definition it is clear that $\partial F_f \subset \partial T$. The projection p can be chosen in such a way that the restriction $c_i = p|_{F_f \cap \partial T_i} : F_f \cap \partial T_i \to K_i$ is a covering map. This map can be described in terms of the embedded resolution graph of f as follows.

Let $\Gamma_f = (\mathcal{V}, \mathcal{E})$ be the embedded resolution graph of some fixed embedded resolution of f. Here $\mathcal{V}$ is the set of vertices and $\mathcal{E}$ the set of edges. Write $\mathcal{V} = \mathcal{W} \amalg \mathcal{A}_f$ where $\mathcal{A}_f$ consists of the arrowhead vertices of Γ, and $\mathcal{W}$ consists of the nonarrowhead vertices. The elements of $\mathcal{A}_f$ correspond to the branches of f, so there is a natural correspondece between the arrowhead vertices of Γ and the components K_i of K, let $a_i \in \mathcal{A}_f$ correpond to K_i. For each $a_i \in A_f$ there exists a uniqe $w_i \in \mathcal{W}$ so that $(w_i, a_i) \in \mathcal{E}$. The map f has multiplicity q_i on a_i, let m_i be its multiplicity on v_i. Then $F_f \cap T_i$ has (q_i, m_i) components, and restricting c_i to any of these components gives a covering of degree $q_i/(q_i, m_i)$. The singular fiber $F_{f,sing}$ of f is homeomorphic to the space $F_f/\sim$ where the equivalence relations $\sim$ are given by $z_1 \sim z_2$ if and only if $z_1, z_2 \in F_f \cap T_i$ for some i, and $c_i(z_1) = c_i(z_2)$ For convenience we define the map $c : \partial F_f \to K$ by $c|_{F_f \cap \overline{T}} = c_i$.

The monodromy $m_f : F_f \to F_f$ can be chosen so that it preserves this equivalence relations, that is, if $x_1 \sim x_2$, we can assume that $m(x_1) \sim m(x_2)$. Therefore, we get a homeomorphism $F_{f,sing} \to F_{f,sing}$ induced by the monodromy. This homeomorphism coincides with the singular monodromy already constructed. Note that $F_{f,sing} = A \cup B$ where A is homotopically eqivalent to F_f and both B and $A \cap B$ are homotopically equivalent to the disjoint union of copies of S^1. The monodromy $m_{f,sing}$ restricts to a homeomorphism $A \to A$ which coincides with the monodromy m_f. Also, $m_{f,sing}$ permutes the connencted components of B and $A \cap B$. (In fact, the map $B \to B$, $b \mapsto m_{f,sing}(b)$ is homotopically equivalent to the identity id_B). If we identify these components with S^1, we can choose an orientation on them so that the restrictions $m_{f,sing}|_B$ and $m_{f,sing}|_{A\cap B}$ preserve this orientation. This implies that $\zeta_B(t) = \zeta_{A\cap B}(t) = 1$, and therefore $\zeta_f(t) = \zeta_{f,sing}(t)$.

Example 2.1. (a) If $f(x, y) = x^d$ for $d \geq 2$ then the Milnor fiber of f is a disjoint union of d copies of D. The resolution graph of f consists of a single non-arrowhead vertex v and a single arrowhead vertex a, and the multiplicity of f on this vertex is $q = d$. The link K has a single component, so we have a covering map $c : \partial F_f \to K$. Restriction c to any of the components of ∂F_f gives a covering map of degree 1, i.e. a homeomorphism. Thus, $F_{f,sing}$ is homeomorphic to the disjoint union of d copies of closed discs with their boundaries identified. This shows that $F_{f,sing}$ has the same homotopy type as the wedge of $d-1$ copies of the two-sphere, i.e. $F_{f,sing} \overset{\sim}{=} \vee_{d-1} S^2$.

The monodromy m_f permutes the connected components of F_f cyclically, and we get $\zeta_f(t) = \zeta_{f,sing}(t) = t^d - 1$.

(b) Let $f(x, y) = x^a + y^b$ with $a, b \geq 2$ and $(a, b) = 1$. Then f is isolated, so $F_{f,sing} = F_f$. In [6] §9 it is shown that F_f has the homotopy type of $\vee_\mu S^1$, where $\mu = (a-1)(b-1)$, and that the characteristic polynomial $\Delta(t)$ of the induced isomorphism $H_1(F; \mathbb{C}) \to H_1(F; \mathbb{C})$ is given by

$$\Delta(t) = \frac{(t^{ab} - 1)(t - 1)}{(t^a - 1)(t^b - 1)}.$$

It follows that

$$\zeta_f(t) = \frac{(1 - t^a)(1 - t^b)}{1 - t^{ab}}.$$

(c) Let $f(x, y) = (xy)^m$. Then F_f is the disjoint union of m copies of $S^1 \times I$ (here $I = [0, 1]$). The resolution graph of f has one non-arrowhead vertice on which f has multiplicity $2m$, and two arrowhead vertices both on which f has multiplicity m. The link K consists of two components K_1 and K_2. For each of the components C of F_f we have $\partial C = S^1 \amalg S^1$, and each of the maps $c_{\partial C}$ is a homeomorphism. Thus, $F_{f,sing}$ is a disjoint set of m copies of $S^1 \times I$ with the boundaries identified. In particular, if $m = 2$, then $F_{f,sing}$ is a torus. If $m \geq 2$, then $F_{f,sing}$ is not homotopically equivalent to a bouquet of spheres, not even if we allow spheres of different dimensions.

Homotopically, F_f is a disjoint union of m copies of S^1, and m_f permutes these components without reversing orientation. Thus, $\zeta_f(t) = \zeta_{f,sing}(t) = 1$.

Remark 2.2. In these examples, the zeta function can also be calculated using A'Campo's formula, which is described in [1].

2.2 Handles and boundary-connected sums

We will use handles to describe the Milnor fiber. More precisely, we will use 2 dimensional 4-handles in our construction. Chapter 4 of [3] gives a presentation of the theory we will need.

Let X be a 4-manifold and $\psi : (\partial D) \times D \to \partial X$ an embedding. We obtain a new manifold $X \cup_\psi (D \times D)$ by taking the disjoint union $X \amalg (D \times D)$ and then identifying any point $x \in (\partial D) \times D$ with $\psi(x) \in \partial X$. The map ψ induces an isomorphism between the normal bundles of $(\partial D) \times \{0\}$ in $(\partial D) \times D$ and $\psi((\partial D) \times \{0\})$ in ∂X. Since $(\partial D) \times \{0\} \subset (\partial D) \times D$ already comes with a canonical framing, this isomorphism can be specified by a framing on $\psi((\partial D) \times \{0\})$. The diffeomorphism type of the resulting manifold is determined by the following data (see for example [3]):

- The embedding $\psi|_{(\partial D)\times\{0\}}$ of $(\partial D) \times \{0\} \cong S^1$ into ∂X.

- The framing of the normal bundle of $\psi|_{(\partial D)\times\{0\}}$.

It will be convenient to be able to specify an alternative framing on $(\partial D) \times \{0\}$ rather then on $\psi((\partial D) \times \{0\})$. For $j \in \mathbb{Z}$, let us call $(\partial D) \times D$ with the framing $S^1 \times D \to (\partial D) \times D$, $(e^{i\theta}, z) \mapsto (e^{i\theta}, e^{ij\theta} z)$ a 4 dimensional 2-handle with the j-th framing. With this notation, the canonical framing on $(\partial D) \times \{0\}$ is the 0-th framing.

2.3 Plumbed manifolds

The description that we will provide will rely on plumbed manifolds associated with plumbing graphs. A plumbing graph Γ is a graph whose vertices v are decorated with an Euler number e_v and a genus g_v.

A 4 dimensional plumbed manifold associated to the graph Γ is constructed as follows: For each vertex $v \in \mathcal{V}(\Gamma)$ we have a compact (real) 2-dimensional surface E_v and a locally trivial disc bundle $b_v : T_v \to E_v$ with Euler number (or first chern-class) e_v. For each edge $(v, w) \in \mathcal{E}(\Gamma)$ we choose a small closed disc-shaped neigborhood U_1 and U_2 in E_v and E_w. Choose trivializations $t_v : D \times D \to b_v^{-1}(U_1)$ and $t_w : D \times D \to b_w^{-1}(U_2)$ and glue together the points $t_v(x, y)$ and $t_w(y, x)$. The plumbed 4-manifold associated to the graph Γ is the disjoint union of the bundles T_v, glued together according to the rules defined by the edges.

A 3 dimensional plumbed manifold associated to the graph Γ is the boundary of the corresponding 4 dimensional plumbed manifold.

The graph Γ can also have arrowhead vertices. An arrowhead vertex a must be the end-vertex of exactly one edge, the other end vertex being a non-arrowhead vertex w. Then a represents a generic fiber over E_w, the fiber being a disc in the case of a 4 dimensional plumbed manifold, but an S^1 in the case of a 3 dimensional manifold.

The graph Γ can have dash-edges. Such an edge should have one non-arrowhead end vertex w, and an arrowhead end-vertex. The dash-edge indicates that the preimage $b_w^{-1}(U)$ of a small closed disc-shaped neigborhood should be removed. In [10] the vertex w is decorated with a new number r_w, the number of dash-edges with w as end-vertex, rather then presenting this date with new edges.

A 3 dimensional plumbed manifold whose graph has dash-edges will have boundary.

For a precise discussion of plumbed manifolds see [10], [9].

3 Description of the fiber

3.1 The fiber as a subset of a resolution

Let $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be any plane curve singularities without common factors and define

$$\Phi(x, y, z) = f(x, y) + zg(x, y).$$

Now consider a fixed common embedded resolution $\phi : V \to \mathbb{C}^2$ of f and g. The resolution graph of this embedded resolution will be denoted by Γ. For the graph Γ we have the set of vertices $\mathcal{V}(\Gamma)$ and the set of edges $\mathcal{E}(\Gamma)$. We have $\mathcal{V}(\Gamma) = \mathcal{W}(\Gamma) \amalg \mathcal{A}(\Gamma)$ where $\mathcal{W}(\Gamma)$ is the set of non-arrowhead vertices and $\mathcal{A}(\Gamma)$ the set of arrowhead vertices. We decompose $\mathcal{A}(\Gamma)$ further as $\mathcal{A}(\Gamma) = \mathcal{A}_f(\Gamma) \amalg \mathcal{A}_g(\Gamma)$, where elements of $\mathcal{A}_f(\Gamma)$ and $\mathcal{A}_g(\Gamma)$ correspond to conponents of the strict transform of f and g respectively. A vertex $v \in \mathcal{V}(\Gamma)$ corresponds to a component E_v of the exceptional divisor $\phi^{-1}(0)$, or the strict transform of f or g. In each case, we denote by m_v the multiplicity of f on E_v, and l_v the multipilcity of g on E_v. In particular, $m_v = 0$ if and only if $v \in \mathcal{A}_g$ and $l_v = 0$ if and only if $v \in A_f$.

Let $f' = f \circ \phi$, $g' = g \circ \phi$ and $F'_f = (f')^{-1}(\epsilon) \cap \phi^{-1}(B_\delta) = \phi^{-1}(F_f)$. The map $V \setminus \phi^{-1}(0) \to \mathbb{C}^2 \setminus \{(0, 0)\}$, $r \to \phi(r)$ is a diffeomorphism. In particular, it restricts to a diffeomorphism $F'_f \to F_f$.

We have a map $\phi \times \mathrm{id}_{\mathbb{C}} : V \times \mathbb{C} \to \mathbb{C}^3$ which restricts to a diffeomorphism $(V \setminus \phi^{-1}(0)) \times \mathbb{C} \to \mathbb{C}^3 \setminus \{(0, 0, z) : z \in \mathbb{C}\}$. We set $\Phi' = \Phi \circ (\phi \times \mathrm{id}_{\mathbb{C}})$, and $F' = (\phi \times \mathrm{id}_{\mathbb{C}})^{-1}(F)$. Clearly, F' is diffeomorphic to F.

For each $w \in \mathcal{W}$ we may choose a small tubular neigborhood T_w around

E_w in V and a map $b_w : T_w \to E_w$ which is a smooth disc bundle. These neigborhoods can be chosen so that they satisfy the following property:

If $w, w' \in \mathcal{W}$ and $(w, w') \in \mathcal{E}$, then we have $b_w^{-1}(E_w \cap E_{w'}) = E_{w'} \cap T_w$ and $b_w^{-1}(E_w \cap T_{w'}) = T_w \cap T_{w'}$. If $w \in \mathcal{W}$, $a \in \mathcal{A}$ and $(w, a) \in \mathcal{E}$, then $b_w^{-1}(E_w \cap E_a) = E_a \cap T_w$. Then the set $T = \cup_{w \in \mathcal{W}} T_w$ is the plumbed 4-manifold with plumbing graph Γ.

If $w, w' \in \mathcal{W}(\Gamma)$ and $e = (w, w') \in \mathcal{E}(\Gamma)$, then we let $T_e = T_w \cap T_{w'}$. If $w \in \mathcal{W}(\Gamma)$ and $a \in \mathcal{A}(\Gamma)$ so that $e = (w, a) \in \mathcal{E}(\Gamma)$, then we pick a small disc-shaped neigborhood U_a in E_w around $E_w \cap E_a$ and let $T_e = b_w^{-1}(U_a)$.

The Milnor fiber F can be described in terms of the embedded resolution graph Γ, with the additional arrowhead vertices, and all vertices decorated by the multiplicities of f' and g'. This description will rely on which of the two functions f' and g' has higher multiplicities on the exceptional divisors. The following definition makes this more precise.

Definition 3.1. (a) For a subgraph Γ' of Γ, let

$$\mathcal{V}_\Gamma(\Gamma') = \{v \in \mathcal{V}(\Gamma) \setminus \mathcal{V}(\Gamma') : \exists v' \in \mathcal{V}(\Gamma'),\, (v, v') \in \mathcal{E}(\Gamma)\},$$

$$\mathcal{E}_\Gamma(\Gamma') = \{(v', v) \in \mathcal{E}(\Gamma) : v' \in \mathcal{V}(\Gamma'),\, v \in \mathcal{V}_\Gamma(\Gamma')\}.$$

(b) Split Γ into two parts, Γ_1 and Γ_2. The graph Γ_1 is the subgraph of Γ generated by the non-arrowhead vertices $w \in \mathcal{W}(\Gamma)$ for which $m_w \leq l_w$, and Γ_2 is generated by the non-arrowhead vertices $w \in \mathcal{W}(\Gamma)$ for which $m_w > l_w$. This means that $v \in \mathcal{V}(\Gamma)$ is a vertex of Γ_1 (resp, Γ_2) if and only if $v \in \mathcal{W}(\Gamma)$ and $m_v \leq l_v$ (resp, $m_v > l_v$). Two vertices in Γ_i are adjacent in Γ_i if and only they are adjacent in Γ for $i = 1, 2$.

Let T_1 be the closure of the set

$$\left(\bigcup_{v \in \mathcal{V}(\Gamma_1)} T_v \right) \setminus \left(\bigcup_{v \in \mathcal{V}_\Gamma(\Gamma_1) \setminus A_g(\Gamma)} T_v \right).$$

Essentially, T_1 is the plumbed 4-manifold with plumbing graph Γ_1, but we must leave out a neigborhood of the strict transform of f, and the exceptional divisors which belong to Γ_1. Let $F_{\Gamma_1} = F' \cap (T_1 \times \mathbb{C})$.

In the following picture, the pair (m, l) indicates that f (resp, g) has multiplicity m (resp, l) on the divisor.

Similarly, let T_2 be the closure of the set

$$\left(\bigcup_{v \in \mathcal{V}(\Gamma_2)} T_v \right) \setminus \left(\bigcup_{v \in \mathcal{V}_\Gamma(\Gamma_2) \setminus A_f(\Gamma)} T_v \right).$$

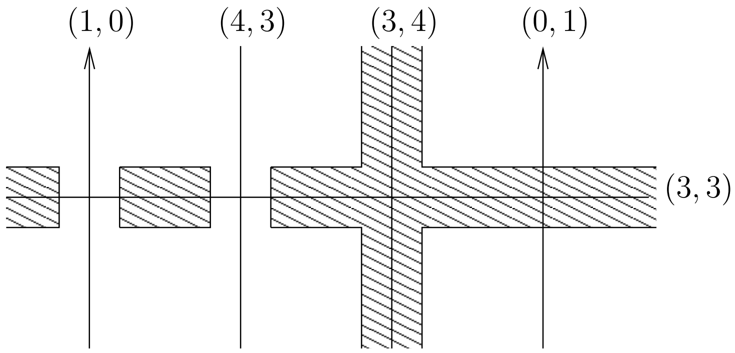

Figure 1: The set T_1.

Again, T_2 is similar to the plumbed 4-manifold with plumbing graph Γ_2, but this time we must leave out a neigborhood of the strict transform of g, and the exceptional divisors which belong to Γ_1.

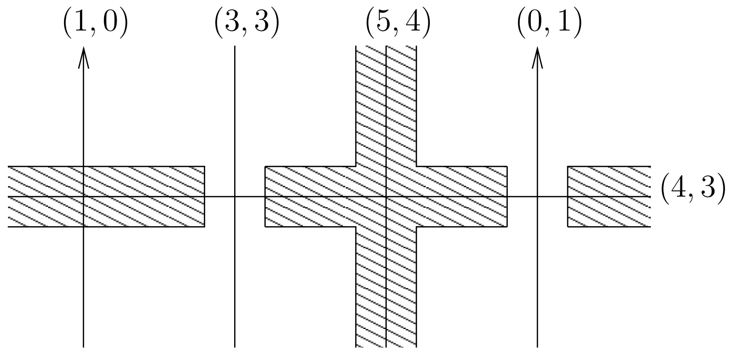

Figure 2: The set T_2.

Let $F_{\Gamma_2} = F' \cap (T_2 \times \mathbb{C})$.

(c) Let Y_2 be the closure of the set

$$(\partial T_2) \setminus \left(\bigcup_{v \in \mathcal{V}_\Gamma(\Gamma_2) \setminus \mathcal{A}_f(\Gamma)} T_v \right).$$

Then Y_2 is a 3-manifold with boundary. Alternatively, Y_2 is the plumbed 3-manifold with boundary, obtained from the graph Γ_2, with the following modification: For any $(v, v') \in \mathcal{E}_\Gamma(\Gamma_2)$ with $v \in \mathcal{V}(\Gamma_2)$ and $v' \notin \mathcal{V}(\Gamma_2) \cup \mathcal{A}_f(\Gamma)$ we remove a tubular neigborhood of a generic fiber over E_v. In [10] this is indicated by decorating the vertex v with the numbers $[g, r]$, where g is the genus of E_v, and r is the number of neigborhoods removed. In [9] this is indicated by adding a dash-edge to the graph, with end vertices v and an arrowhead vertex v'.

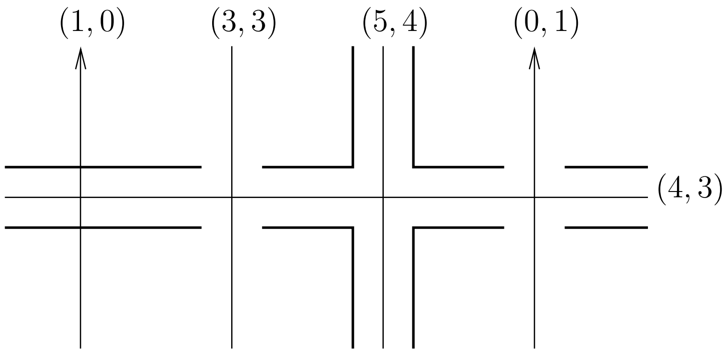

Figure 3: The set Y.

The diffeomorphism type of F can be explicitly described by the graph Γ as follows:

Let

$$T_1' = T_1 \cup \left(\bigcup_e T_e \right).$$

where the union ranges over edges $e \in \mathcal{E}(\Gamma)$ which have one end-vertex in $\mathcal{V}(\Gamma_1)$ and the other end-vertex in $\mathcal{V}(\Gamma_2)$.

Let W' be a closed tubular neigborhood around $F_f \cap T_1'$ in T_1' so that $W' \cap \partial T_1'$ is a tubular neigborhood around $F_f \cap \partial T_1'$ in $\partial T_1'$. Then W' is diffeomorphic to $(F_f \cap T_1') \times D$, because the normal bundle of $F_f \cap T_1'$ is oriented and $F_f \cap T_1'$ is a 2 dimensional surface with boundary, so it has the homotopy type of a 1 dimensional CW complex.

For an edge $e = (w, a) \in \mathcal{E}(\Gamma)$ where $w \in \mathcal{V}(\Gamma_1)$ and $a \in \mathcal{A}_f(\Gamma)$, let u_e, v_e be coordinates on T_e so that v_e is constant on the fibers of b_w, T_e is presented in these coordinates as $\{(u_e, v_e) : |u_e|, |v_e| \leq 1\}$ and $\phi(r) = 0$ if and only if $u_e(r) = 0$ for $r \in T_e$. Choose η so small that $|u_e(r)| > \eta$ for any $r \in W' \cap T_e$. Let W_e' be the set $\{r \in T_e : \eta \leq |u_e(r)|\}$. Then let $A' = W' \cup (\cup_e W_e')$. Here, the union ranges over edges e for which W_e' is defined.

Let W'' be obtained by removing a small tubular neighborhood around $\phi^{-1}(0)$ from T_2. This means that we remove a small open tubular neighborhood around each irreducible component of the exceptional divisor, and we assume that the union of these small neighborhoods form a small plumbed 4-manifold (without the boundary) as is constructed from the graph Γ_2. Let $e' = (w, a) \in \mathcal{E}(\Gamma)$ where $w \in \mathcal{V}(\Gamma_2)$ and $a \in \mathcal{A}_g(\Gamma)$. Let u, v be coordinates on T_e so that the map $r \mapsto (u(r), v(r))$ takes T_e to $D_1 \times D_1$. Assume that v is constant on the fibers of b_w, and that $W'' \cap T_e \subset \partial W''$ is given by $\{r \in \partial T_2 \cap T_e : \eta \leq |u| \leq 1, \ |v| = 1\}$ for some $\eta \ll \epsilon$. Let $C_e \subset W'' \cap T_e$ be described in coordinates as $C_e = \{(u,v) : u^{m_w} = \epsilon\}$. For any $c \in C_e$ we have a direct sum decompostion $T_c\partial W = T_c C_e \oplus \langle \partial_1, \partial_2 \rangle$ where ∂_1 and ∂_2 are the tangent vectors which differentiate with respect to the real and imaginary part of u. This gives a framing on C_e, which allows us to attach handles along the components of C_e.

Let A'' be the the manifold obtained by attaching a 4 dimensional 2-handle to each of the components of C_e (with the framing described in the previous paragraph) for all edges $e = (w,a)$ with $w \in \mathcal{V}(\Gamma_2)$ and $a \in \mathcal{A}_g(\Gamma)$. The handles should have the $(-k)$-th framing, as defined in (2.2), where k is the multiplicity of g on a.

The set A is defined as $A' \cup W''$ with the handles glued to each of the sets C_e as already described. More abstractly, we can define the space A as follows:

The set $A' \cap W''$ is a closed tubular neigborhood around a submanifold of $\partial A'$ or $\partial W''$. The union $A' \cup W''$ is a manifold with boundary (assuming that we smooth out some corners on the boundary) which can clearly be constructed as $A' \amalg W''/\sim$ where the equivalence relation identifies points in $A' \cap W''$ in the obvious way. This gives $A' \cup W''$ as the boundary connected sum of A' and W'' as abstract manifolds. The attaching spheres (attaching circles) of the handles which are attached to W'' to obtain A'' do not intersect the set of points of W'' which are glued to points of A' in this construction. Therefore, we can construct a boundary connected sum of A' and A'' using essentially the same gluing as when constructing $A' \cup W''$ as a boundary connected sum. This construction yields the same space A.

The main result in this note is that this construction gives the Milnor fiber of Φ.

Theorem 3.2. *(i) The Milnor fiber F of the singularity $\Phi = f + zg$ and A have the same diffeomorphism type.*

(ii) The monodromy m_Φ can be chosen so that the subsets W', W'' and W'_e are invariant (for edges e where W'_e is defined). Its restriction to $W' = (F_f \cap T_1) \times D$ sends (r,z) to $(m_f(r), z)$ where m_f is the monodromy of f, chosen so that $F_f \cap T_1$ is an invariant subset of F_f.

The restrictions $m_\Phi|_{W'_e}$ and $m_\Phi|_{W''}$ are isotopic to the identity.

For an edge $e = (w,a)$ with $w \in \mathcal{V}(\Gamma_2)$ and $a \in \mathcal{A}_g(\Gamma)$, the monodromy permutes the m_w handles corresponding to e cyclically.

3.2 Corollaries

Corollary 3.3. *(a) If $m_w \leq l_w$ for all $w \in \mathcal{W}(\Gamma)$, then F and $F_{f,sing}$ have the same homotopy type.*

(b) If $m_w > l_w$ for all $w \in \mathcal{W}(\Gamma)$, then F has the same homotopy type as the plumbed 3-manifold with plumbing graph Γ_2 with $\sum_{a\in\mathcal{A}_g(\Gamma_2)} m_{w_a}$ open balls removed from it.

Proof. (a) In this case, $W' = F_f \times D$ has the same homotopy type as F_f. To construct A' we glue $(\partial F_f) \times D$ to $\amalg_{\mathcal{A}_f(\Gamma)} D_1 \times (D_1 \setminus D_\eta)$. Up to homotopy, this is equivalent to gluing ∂F_f to $\amalg_{\mathcal{A}_f(\Gamma)} S^1$. It is easy to see that this gluing is precisely the construction of $F_{f,sing}$.

(b) In this case, $W'' = Y_2 \times I$ where Y_2 is the plumbed 3-manifold with plumbing graph Γ_2 with a dash-edge added in place of any edge in $A_g(\Gamma)$. In particular, W'' has the same homotopy type as this plumbed manifold. Up to homotopy, A'' is obtained by adding 2-cells to this plumbed 3-manifold along m_{w_a} disjoint meridians. This is equivalent to starting with the plumbed 3-manifold whose plumbing graph is Γ_2 without any dash-arrows, and then removing m_w small balls for each edge $(w, a) \in \mathcal{E}(\Gamma)$ with $w \in \mathcal{V}(\Gamma_2)$ and $a \in \mathcal{A}_g(\Gamma)$. □

Corollary 3.4. *(a) The Euler characteristic of F is given by the formula*

$$\chi(F) = \sum_{w\in\mathcal{V}(\Gamma_1)} m_w(2-\delta_w) + \sum_{a\in\mathcal{A}_g(\Gamma_2)} m_{w_a}.$$

(b) The zeta function associated to Φ is given by the formula

$$\zeta_\Phi(t) = \zeta(t) = \left(\prod_{w\in\mathcal{V}(\Gamma_1)} (t^{m_w}-1)^{2-\delta_w}\right)\left(\prod_{a\in\mathcal{A}_g(\Gamma_2)} (t^{m_{w_a}}-1)\right)$$

where δ_w is the number of edges in Γ having w as one of its end-vertices.

Proof. It is enough to prove (b), since $\chi(F) = \deg \zeta(t)$.

For $e = (w, a)$ with $w \in \mathcal{V}(\Gamma_1)$ and $a \in \mathcal{A}_f(\Gamma)$ We have $\zeta_{W'_e}(t) = 1$ because the monodromy acts trivially on $H_*(W'_e)$ and W'_e has the same homotopy type as S^1. The set $W'_e \cap W'$ has the homotopy type of a disjoint union of copies of S^1, and the monodromy permutes these components, preserving their orientation. Thus $\zeta_{W'_e\cap W'}(t) = 1$. A similar argument shows that $\zeta_{A'\cap A''}(t) = 1$. Using the result of [1] we get

$$\zeta_{A'}(t) = \prod_{w\in\mathcal{V}(\Gamma_1)} (t^{m_w}-1)^{2-\delta_w}.$$

The monodromy acts trivially on $H_*(W'')$ so $\zeta_{W''}(t) = (1-t)^{\chi(W'')}$ where $\chi(W'')$ is the Euler characteristic of W''. But W'' has the homotopy type of a closed 3-manifold from which subsets of the type $S^1 \times D$ have been removed. Hence, we have $\chi(W'') = 0$. For each $e = (w, a) \in \mathcal{E}(\Gamma)$ where $w \in \mathcal{V}(\Gamma_2)$ and $a \in A_g(\Gamma)$ we attach m_w 2-handles, the intersection of these handles with W'' is up to homotopy a disjoint union of copies of S^1, which the monodromy permutes, preserving their orientation. The monodromy permutes the handles cyclically, so we get

$$\zeta_{A''}(t) = \zeta_{W''}(t) \cdot \prod_{a \in \mathcal{A}_g(\Gamma_2)} (t^{m_{w_a}} - 1).$$

Now we get $\zeta_\Phi(t) = \zeta_{A'}(t)\zeta_{A''}(t)\zeta_{A' \cap A''}(t) = \zeta_{A'}(t)\zeta_{A''}(t)$ which is the desired result. □

Example 3.5. Let $f(x, y) = x^d$ and $g = y^d$ where $d \geq 2$. Then we can choose the resolution V so that $\mathcal{V}$ has a single element, say $\mathcal{V} = \{v\}$. Then $m_v = l_v = d$, so we can apply (3.3)(a). The Milnor fiber F associated to Φ has the same homotopy type as $F_{f,sing}$, which is up to homotopy a bouquet of $d - 1$ two-spheres. Note that in spite of this, Φ is not isolated. The zeta function of this singularity is $\zeta(t) = t^d - 1$.

4 Proof of theorem (3.2)

4.1 Theorem (3.2) restated

The following theorem is a more technical reformulation of theorem (3.2). The set W' is replaced with the set F_{Γ_1}, the set W'' is replaced with F_{Γ_2} and A' and A'' are obtained by enlargeing W' and W'' appropriately.

Theorem 4.1. *(a) The set F_{Γ_1} is diffeomorphic to $(F_f \cap T_1) \times D$. In fact, the map $F_{\Gamma_1} \to D_\delta$, induced by the projection $V \times \mathbb{C} \supset V \times D_\delta \to D_\delta$, $(r, z) \mapsto z$ is a trivial fiber bundle with fiber $F_f \cap T_1$.*

(b) Let $w \in \mathcal{V}(\Gamma_1)$ and $a \in \mathcal{A}_f(\Gamma)$ so that $e = (w, a) \in \mathcal{E}(\Gamma)$. Let (u, v) be some coordinates on T_e so that v is constant on the fibers of b_w, we have $\phi(r) = 0$ if and only if $u(r) = 0$ for $r \in T_e$, and that the image of the map (u, v) is $D_1 \times D_1$. Then, for some small $\eta > 0$, there exists a diffeomorphism

$$\sigma_e : F' \cap (T_e \times \mathbb{C}) \to (D_1 \setminus D_\eta^\circ) \times D_1$$

so that $\sigma_e(r, z) = (u(r), v(r))$ whenever $|v(r)| = 1$ (here, D_η° is the open disc with center 0 and radius η).

(c) The set F_{Γ_2} is diffeomorphic to $Y_2 \times I$.

(d) If $v \in \mathcal{V}(\Gamma_2)$ and $a \in \mathcal{A}_g(\Gamma)$ so that $e = (w,a) \in \mathcal{E}(\Gamma)$, then F is diffeomorphic to $\overline{F' \setminus (T_e \times \mathbb{C})}$ with m_v copies of 4-dimensional 2-handles attached along $F'_f \cap \partial T_e \subset \overline{F' \setminus (T_e \times \mathbb{C})}$. If u, v are coordinates on T_e so that v is constant on the fibers of b_w, then $F'_f \cap \partial T_e$ has the framing induced by these coordinates (similarly as in the discussion before (3.2)). The handle should have the $(-k)$-th framing where k is the vanishing order of g on E_a.

(e) Let $e = (v_1, v_2) \in \mathcal{E}(\Gamma)$, where $v_1 \in \mathcal{V}(\Gamma_1)$ and $v_2 \in \mathcal{V}(\Gamma_2)$. Then F' is diffeomorphic to $\overline{F' \setminus (T_e \times \mathbb{C})}$, after identifying a tubular neigborhood around $(F'_f \cap (\partial T_e \cap \partial T_{v_1})) \times \{0\}$ in $\partial(\overline{F' \setminus (T_e \times \mathbb{C})})$ with a tubular neigborhood of $(F'_f \cap (\partial T_e \cap \partial T_{v_2})) \times \{0\}$ in $\partial(\overline{F' \setminus (T_e \times \mathbb{C})})$. The identification is obtained from a trivialisation of the normal bundle (or a tubular neigborhood) around $(F_f \times \{0\}) \cap (T_e \times \mathbb{C}) \subset (F' \cap (T_e \times \mathbb{C}))$ which is a disjoint union of copies of $S^1 \times I$.

(f) The monodromy m_Φ restricts to a diffeomorphism $F_{\Gamma_1} \to F_{\Gamma_1}$. Identifying F_{Γ_1} with $(F_f \cap T_1) \times D$, this diffeomorphism is given by $(r, z) \mapsto (m_f(r), z)$, where the monodromy m_f of f is chosen so that it maps the set $(F'_f \cap T_1)$ to itself.

(g) The monodromy m_Φ restricts to a diffeomorphism $F_{\Gamma_2} \to F_{\Gamma_2}$ which is isotopic to the identity map.

(h) If $(w, a) \in \mathcal{E}(\Gamma)$ with $a \in \mathcal{A}_g$ and $w \in \mathcal{V}(\Gamma_2)$, then the monodromy permutes the m_w copies of 2-handles constructed in (d) cyclically.

The following paragraphs describe general guidelines for notation used throughout the proof.

Let $w \in \mathcal{V}(\Gamma)$. Pick a small disc-shaped neigborhood $U \subset E_w$ and let $m = m_w$, $l = l_w$. If $U \cap E_v \neq \varnothing$ for some $v \in \mathcal{V}_w = \{v \in \mathcal{V} : (w, v) \in \mathcal{E}\}$, then let $n = m_v$ $k = l_v$, otherwise let $n = k = 0$.

By assuming that U and T_w are chosen small, there exist coordinates (u, v) and a function α_p on $b_w^{-1}(U)$ so that $f' = u^m v^n$ and $g' = \alpha_p u^l v^k$. Since α does not vanish, there exist positive real constants α_1, α_2 so that $0 < \alpha_1 < |\alpha| < \alpha_2$ on $b_w^{-1}(U_p)$. These constants are independent of our choice of δ and ϵ.

Since $\cup_{w \in \mathcal{W}(\Gamma)} E_w$ is compact, it is covered by finitely many such discs, say $\phi^{-1}(0) = \cup_{p \in P} U_p$ for some finite index set P. Let the coordinates on U_p be u_p, v_p so that $f = u_p^m v_p^n$ and $g = \alpha_p u_p^l v_p^k$ for the appropriate values of m, n, l, k. Since P is finite, we may assume that the functions α_p, $p \in P$ are all bounded by the same constants α_1, α_2.

We choose $0 < \epsilon \ll \delta \ll 1$. For the proof of (4.1)(a), let $S_1 = F' \cap (T_1 \times \mathbb{C})$. We start with a lemma.

4.2 The structure of F_{Γ_1}

Lemma 4.2. *Let $r \in T_1$ and $z \in \mathbb{C}$ with $|z| \leq \delta$ and $\Phi'(r,z) = \epsilon$. Then $|f'(r)| < 2\epsilon$ and $|x'(r)|, |y'(r)| < \delta$.*

Proof. We may assume that $r \in b_w^{-1}(U_p)$ for some $p \in P$ and $w \in \mathcal{V}(\Gamma_1)$. We also assume that if U_p intersects some E_v with $v \neq W$ then $v \notin \mathcal{A}_f, \mathcal{V}(\Gamma_2)$. Write $(u,v) = (u_p(r), v_p(r))$ and $\alpha = \alpha_p(r)$. In coordinates we have $\Phi'(r,z) = u^m v^n + z\alpha u^l v^k$ where $m \leq l$ and $n, k = 0$ or $n \leq k$. Then the equation $\Phi'(r,z) = \epsilon$ gives

$$\left| \frac{u^m v^n - \epsilon}{\alpha u^l v^k} \right| = |z| \leq \delta.$$

If $|f'(r)| = |u^m v^n| \geq 2\epsilon$ then the triangle inequality gives

$$\left| \frac{u^m v^n - \epsilon}{\alpha u^l v^k} \right| \geq \frac{|u^m v^n| - \frac{1}{2}|u^m v^n|}{|\alpha u^l v^k|} \geq \frac{1}{2\alpha_2} |u^{m-l} v^{n-k}| > \delta$$

because $|u|, |v| < 1$, $m \leq l$, $n \leq k$ and δ is chosen small independent of α_2. This shows that $|f'(r)| = |u^m v^n| < 2\epsilon$. The lemma is proven as soon as we prove the following claim:

Claim: We can choose ϵ small enogh so that if $|f'(r)| < 2\epsilon$ then $|x'(r)| < \delta$ and $|y'(r)| < \delta$.

To see this, note that the sets $\{t \in T_1 : |f'(t)| > 2\epsilon_0\}$, for all $\epsilon_0 > 0$, form an open cover of the compact set $\{t \in T_1 : |x'(t)| \geq \delta\}$. This implies that for ϵ small enough, we have $\{t \in T_1 : |x'(t)| \geq \delta\} \subset \{T \in T_1 : |f'(t)| > 2\epsilon\}$. The same holds if we replace x' with y'. □

What the lemma says is that $F_{\Gamma_1} = \{(r,z) \in T_1 \times \mathbb{C} : \Phi'(r,z) = \epsilon,\ |z| \leq \delta\}$. Since the function $Z : T_1 \times \mathbb{C} \to \mathbb{C}$, $Z(r,z) = z$ behaves well when restricted to $\Phi'^{-1}(\epsilon)$, it will give information on the set F_{Γ_1}.

Lemma 4.3. *The derivative of the map $Z : F_{\Gamma_1} \to \mathbb{C}$, $(r,z) \mapsto z$ is surjective at every point, so is the derivative of $Z|_{F' \cap \partial T_1}$.*

Proof. Let $q \in F_{\Gamma_1}$. Then $q \in b_w^{-1}(U_p)$ for some $p \in P$ and $w \in \mathcal{V}(\Gamma_1)$. On this neigborhood we have the coordinates $(u,v) = (u_p, v_p)$. We will show that the map $(r,z) \mapsto (v(r), z)$ will give coordinates on a neigborhood around q in F_{Γ_1}. This is clearly enough to prove the first statement of the lemma. For this, we only need to show that $\partial_u \Phi'(q,z) \neq 0$, since the result will then follow from the implicit function theorem. We get

$$\begin{aligned} \partial_u \Phi'(q,z) &= \partial_u (u^m v^n + z\alpha u^l v^k) = m u^{m-1} v^n + z((\partial_u \alpha) u^l v^k + \alpha l u^{l-1} v^k) \\ &= u^{m-1} v^n (m + z u^{l-m} v^{k-n} (\partial_u \alpha u + \alpha l)). \end{aligned}$$

We have $u \neq 0$, and we can have $v = 0$ only if q is on the strict transform of g. Since f and g have no common factors, this would mean that $n = 0$, and we get $\partial_u \Phi'(q, z) = m u^{m-1} \neq 0$. Since $m \leq l$ and $n \leq k$ we have

$$|z u^{l-m} v^{k-n} (\partial_u \alpha u + \alpha l)| \leq \delta(|\partial_u \alpha| + |\alpha|) < m.$$

The last inequality is valid because δ is chosen small with respect to $|\partial_u \alpha|$ and $|\alpha|$. This shows that $m + z u^{l-m} v^{k-n} (\partial_u \alpha u + \alpha l) \neq 0$, and thus $\partial_u \Phi'(q, z) \neq 0$.

This proves the first statement of the lemma. For the second statement, if $q \in F' \cap \partial T_1$, then $q \in b_w^{-1}(\partial U_p)$ for some $p \in P_1$, and we may assume that for some coordinate v in an neigborhood of $b_w(q)$ in U_p, we have $v(b_w(q)) = 0$ and that the image of v is $\{v : \mathrm{Re} v \geq 0, |v| < 1\}$. Then the previous argument shows that the map $(r, z) \mapsto (\mathrm{Re} v(r), z)$ provides coordinates in a neigborhood around q in $F' \cap \partial T_1$. This proves the second statement in the theorem. □

Proposition 4.4. *The map $Z : F_{\Gamma_1} \to \overline{D}$ is a trivial fibration, whose fiber is diffeomorphic to $F_f \cap T_1$.*

Proof. The map Z is clearly a proper map. Therefore, lemma (4.3) and Ehresmann's fibration theorem imply together that the map is a locally trivial fibration. In fact this fibration is trivial, since $\overline{D}$ is contractible.

The fiber $Z^{-1}(0)$ is the subset of $T_1 \times \{0\}$ given by $f' = \epsilon$, so it is diffeomorphic to $F_f' \cap T_1$. □

The proposition proves (4.1)(a).

4.3 Local picture close to the strict transform of f

Let $(w, a) \in \mathcal{E}(\Gamma)$ for some $w \in \mathcal{V}(\Gamma_1)$ and $a \in \mathcal{A}_f$. We have $T_e = b_w^{-1}(\overline{U_e})$ for some disc shaped neigborhood U_e around $E_w \cap E_a$ in E_w. Choose a function u on T_e so that $u^l = g'$. This is possible since g' vanishes with multiplicity $l = l_w$ on E_w, but does otherwise not vanish on T_e. Then it is possible to coose a function v so that $f' = u^m v^n$ on T_e. We assume that $U_e = \{v : |v| < \rho\}$ for some $\rho > 0$, and that in these coordinates, T_e is the polydisc $\{(u, v) : |u|, |v| \leq \rho\}$. The number ρ should be thaught of as small, since the tubular neighborhood T_w is choosen small. However, ρ does not depend on the choice of δ, ϵ. Thus, we have $0 \ll \epsilon \ll \delta \ll \rho \ll 1$.

If r is a point in T_e so that $g'(r) = 0$, then $u(r) = 0$, so we have $f'(r) + z g'(r) = 0$. Otherwise, if $g'(r) \neq 0$, then the equation $f'(r) + z g'(r) = \epsilon$ can be solved for z. This means that the map $F' \cap (T_e \times \mathbb{C}) \to T_e$, $(r, z) \mapsto r$ is injective.

Since this map is continous and its domain is compact, it is a homeomorphism onto its image $X_e \subset T_e$. In fact, if we solve for z, we see that

$$X_e = \{r : |u|, |v| \leq \rho, \frac{|u^m v^n - \epsilon|}{|u|^l} \leq \delta\}. \tag{2}$$

We will show that X_e is diffeomorphic to $I \times S^1 \times I$. First, consider the set $X_e^1 = X_e \cap \{|v|^n \leq \epsilon/\rho^m\}$. Let r be a point in X_e^1 with coordinates (u, v), and denote by $z(r)$ the unique point in $\mathbb{C}$ for which $(r, z) \in F' \cap (T_e \times \mathbb{C})$. We get

$$\begin{aligned} \partial_u z(r) &= \frac{\partial}{\partial u}\left(\frac{u^m v^n - \epsilon}{u^l}\right) = (m-l)u^{m-l}v^n + l\epsilon u^{-l-1} \\ &= ((m-l)u^{m+1}v^n + l\epsilon)u^{-l-1} \neq 0 \end{aligned}$$

since (assuming $\rho \leq l/|m-l|$)

$$|(m-l)u^{m+1}v^n| \leq |(m-l)\rho^{m+1}v^n| \leq |l\rho^m v^n| < l\epsilon$$

This means that the function $z : X_e^1 \to \mathbb{C}$ has surjective derivative everywhere, and so by the implicit function theorem, this means that the level set $\{|z| = \delta\}$ is a submanifold of X_e^1, and that the projection of this submanifold onto the v-axis is a submersion. Now the relative Ehresmann theorem implies that this projection $X_e^1 \to D_{\epsilon/\rho^m}$ is a locally trivial fiber bundle. But this bundle is trivial because D_{ϵ/ρ^m} is contractible. The fiber of this fiber bundle is

$$X_e^1 \cap \{v = 0\} = \{(u, v) : |u| \leq \rho, \frac{\epsilon}{|u^l|} \leq \delta\},$$

so there is a diffeomorphism $X_e^1 \cong \{u : (\epsilon/\delta)^{1/l} \leq |u| \leq \rho\} \times D_{\epsilon/\rho^m}$.

Now consider the set $X_e^2 = X_e \cap \{|v|^n \geq \epsilon/\rho^m, |u| \geq \rho/2\}$. If (u, v) are in this set, then we get: If $t \in [0, 1]$ is such that $|tv|^n \geq \epsilon/\rho$, then

$$\frac{|u^m(tv)^n - \epsilon|}{|u|^l} \leq \frac{|u^m v^n - \epsilon|}{|u|^l}.$$

This follows from the a simple geometrical fact: If a and b are vectors (in $\mathbb{C}$ or $\mathbb{R}^n$), $t \in [0, 1]$ and $|tb| \geq |a|$, then $|ta - b| \leq |a - b|$. To apply this we need to check: $|u^m(tv)^m| \leq \rho^m \epsilon/\rho^m = \epsilon$.

This means that the map

$$X_e^2 \to X_e^1 \cap X_2^e, \quad (u, v) \to (u, \frac{\epsilon^{1/l}}{\rho^{m/l}|v|}v)$$

is a strong homotopy retraction. We can extend this to a strong homotopy retraction $X_e^1 \cap X_e^2 \to X^1$.

It is easy to check that the homotopy $X_e \times I \to X_e$ given by

$$(u, v) \mapsto \begin{cases} (t^{-n}u, t^{-m}v) & \text{if } |t^{-n}u| \leq \rho,\ |t^m v|^n \geq \epsilon/\rho^m \\ (u, v) & \text{otherwise} \end{cases}$$

yields a strong homotopy retraction $X_e \to X_e^1 \cup X_e^2$.

We have now constructed a strong retraction

$$X_e \to X_e^1 \stackrel{\simeq}{=} (D_\rho \setminus D^\circ_{(\epsilon/\delta)^l}) \times D_{(\epsilon/\rho^m)^{1/l}}.$$

It is easy to see from this retraction that X_e can be constructed by gluing $I \times I \times S^1$ to X_e^1 by an embedding $I \times S^1 \times \{0\} \to \partial X_e^1$, and then gluing another copy of $I \times I \times S^1$ to this space $(X_e^1 \cup X_e^2)$, again by an embedding $I \times S^1 \times \{0\} \to \partial(X_e^1 \cup X_e^2)$. We can therefore replace X_e by the set $\{(u,v) : \eta \leq |u| \leq \rho,\ |v| \leq \rho\}$. This proves (3.2)(b).

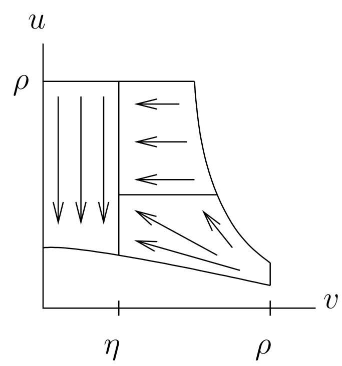

Figure 4: A homotopy of X_e.

4.4 The structure of F_{Γ_2}

We have a map $\pi_1 : V \times \mathbb{C} \to V$ given by $\pi_1(r,z) = r$. The set F' can be given by explicitly the equations $\Phi'(r,z) = \epsilon$, $N(r) = \|(x,y)\| \leq \delta$, $|z| \leq \delta$, where $\|\cdot\|$ is a norm on $\mathbb{C}^2$. We may essentially choose any (smooth) norm for this purpouse. In particular, we may multiply the norm with a positive real constant which is fixed before choosing δ and ϵ.

The multiplicity of g is always less then that of f on any divisor which intersects T_{Γ_2}, and strictly less on $\phi^{-1}(0)$. Therefore, the function f/g is holomorphic on T_{Γ_2}, and it vanishes on $\phi^{-1}(0) \cap T_{\Gamma_2}$. That is, the order of vanishing of the function f/g is 1 or higher on the components of $\phi^{-1}(0) \cap T_{\Gamma_2}$. For technical purposes, we scale the norm function $N(r) = \|(x,y)\|$ with a large enough constant so that the following holds: There exists a $\delta^* > 0$ so that for any $r \in T_{\Gamma_2}$ with $N(r) < \delta^*$ we have $|f'(r)/g'(r)| < N(r)/2$. Afterwards, we assume that $\delta < \delta^*$.

Since g does not vanish on $T_{\Gamma_2} \setminus \phi^{-1}(0)$, the equation $\Phi(r,z) = \epsilon$ can be solved for z with $r \in T_{\Gamma_2} \setminus \phi^{-1}(0)$. That is, the restriction $\pi_1|_{F'_{\Gamma'}}$ is injective.

Since its domain is compact, it is a homeomorphism onto its image. This shows that $F'_{\Gamma'}$ is homeomorphic to the set

$$\{r \in T_{\Gamma'} : \phi(r) \neq 0,\, N(r) \leq \delta,\, |z(r)| \leq \delta\}.$$

Here $z(r)$ is the unique point $z(r) \in \mathbb{C}$ for which $\Phi(r, z(r)) = \epsilon$.

We know that

$$\{r \in T_{\Gamma'} : N(r) \leq \delta\} \tag{3}$$

is a tubular neighborhood around $\phi^{-1}(0) \cap T_{\Gamma_2}$ in T_{Γ_2}. What we need to show is that

$$\{r \in T_{\Gamma'} : \phi(r) = 0 \text{ or } |z(r)| > \delta\} \tag{4}$$

is a tubular neighborhood around $\phi^{-1}(0)$ inside (3).

Around any point r in $\phi^{-1}(0) \cap T_{\Gamma_2}$, the function $|z|^{-1}$ can be described in real coordinates (x_1, x_2, x_3, x_4) as $(x_1^2 + x_2^2)^n (x_1^2 + x_2^2)^k$ for some integers $n > 0$, $k \geq 0$. If $r \in \phi^{-1}(0) \cap \partial T_{\Gamma_2}$, then we may assume that r has coordinates $(0, 0, 0, 0)$ and that ∂T_{Γ_2} is given as $x_4 = 0$. Using Morse theory, the following claim will prove that (4) is in fact a tubular neighborhood around $\phi^{-1}(0)$ inside (3), proving (3.2)(c).

Claim. For any point $r \in T_{\Gamma_2}$, we have either $|z(r)|^{-1} \geq \delta^{-1}$, or that the derivative of $|z|^{-1}$ (and $|z|^{-1}|_{\partial T_{\Gamma_2}}$ if $r \in \partial T_{\Gamma_2}$) at r is surjective.

Proof. We choose the constants δ, ϵ as before, i.e. $0 \ll \epsilon \ll \delta \ll 1$. Let $r \in T_{\Gamma_2}$. Choose a neighborhood around r with coordinates u, v so that $f = u^m v^n$, $g = \alpha u^l v^k$, where α is a non-vanishing holomorphic function (as before, we can start by choosing finitely many such charts covering T_{Γ_2}, allowing us to have a universal lower bound for $|\alpha|$, and an upper bound for $|\partial_u \alpha|$). The integers m, n, l, k satisfy $m > l > 0$ and $n > k \geq 0$ or $n = k = 0$. If $r \in \partial T_{\Gamma_2}$, then we assume that the boundary is given in coordinates as $\mathrm{Re} v = 0$.

First consider the case when $|g'(r)| < 2\epsilon/\delta$. This gives $|u^l v^k \alpha| < 2\epsilon/\delta$. In case that $k = 0$, we get $|u|^l |\alpha| < 2\epsilon/\delta$. If however $k \neq 0$, then we have $n \neq 0$, and we can reverse the roles of u and v if neccesary and get $|u^{l+m}\alpha| \leq 2\epsilon/\delta|\alpha|$. In any case, we assume that $|u^{l+m}\alpha| \leq 2\epsilon/\delta$ (the main point is to bound $|u|$ by something which depends on ϵ, making the bound small with respect to any other variable). This yields in particular

$$|mu^m| \leq \delta|(\partial_u \alpha)u - \alpha l| \tag{5}$$

if ϵ is small enough with respect to δ and $|\alpha|$. Notice also that since we insist on $|u|$ being small, the magnitude of $|(\partial_u \alpha)u - \alpha l|$ is essentially $|\alpha l|$, which is bounded below by a constant which does not effect the choice of ϵ.

Now, assuming that $|z(r)|^{-1} < \delta^{-1}$ (and $|z(r)| \neq \infty$), we want to show that

$$\frac{\partial z}{\partial u}(r) = \frac{\partial}{\partial u}\left(\frac{u^m v^n - \epsilon}{\alpha u^l v^k}\right) \neq 0. \tag{6}$$

From this it follows that the derivative of $|z|^{-1}$ at r is surjective. We have

$$\frac{\partial}{\partial u}\left(\frac{u^m v^n - \epsilon}{\alpha u^l v^k}\right) = \frac{m u^{m-1} v^n \alpha u^l v^k - (u^m v^n - \epsilon)((\partial_u \alpha) u^l - \alpha l u^{l-1}) v^k}{(\alpha u^l v^k)^2}.$$

In order to prove that the right hand side of this equation does not vanish, we will show that

$$m u^{m-1} v^n \alpha u^l v^k > (u^m v^n - \epsilon)((\partial_u \alpha) u^l - \alpha l u^{l-1}) v^k. \tag{7}$$

By assumption we have $|z|^{-1} < \delta^{-1}$, that is, $|u^m v^n - \epsilon| > \delta |\alpha u^l v^k|$. In order to prove (7), it is therefore enough to prove

$$|m u^{m-1} v^n \alpha u^l v^k| > |\delta \alpha u^l v^k ((\partial_u \alpha) u^l - \alpha l u^{l-1}) v^k|.$$

By cancellation, this is eqivalent to $|m u^{m-l} v^{n-k}| \leq \delta |(\partial_u \alpha) u - \alpha l|$ which follows from (5) (since $|v|^{n-k} \leq 1$). This finishes the case when $|g'(r)| < 2\epsilon/\delta$.

In the case when $|g'(r)| \geq 2\epsilon/\delta$ we use the inequality

$$|z(r)| = \left|\frac{f'(r) - \epsilon}{g'(r)}\right| \leq \left|\frac{\epsilon}{g'(r)}\right| - \left|\frac{f'(r)}{g'(r)}\right|.$$

In the beginning of this section we concluded that unless $N(r) > \delta$, we had $|f'(r)/g'(r)| < \delta/2$, and by assumption we have $|\epsilon/g'(r)| < \delta/2$. This gives $|z(r)| < \delta$, and the claim has been proved. □

4.5 Local picture close to the strict transform of g

Let $w \in \mathcal{V}(\Gamma_2)$ and $a \in \mathcal{A}_g(\Gamma)$ so that $(w, a) \in \mathcal{E}(\Gamma)$. We pick a disc shaped coordinate neigborhood U_e around $E_w \cap E_a$ in E_w, and look closely at the set $T_e = b_w^{-1}(\overline{U_p})$. We may assume that we have coordinates (u, v) on T_e with the properties $f = u^m$ and $g = u^l v^k$, where $m = m_w$, $l = l_w$ and $k = l_a$. We have $m_a = 0$ since f does not vanish on E_a. For convenience, we will identify points in T_e with their coordinates (u, v). Via this correpondance, we may assume that T_e is precisely the set $\{(u, v) : |u|, |v| \leq \rho\}$ for some positive real number ρ.

As in the previous subsections, we consider the coordinate map $F' \cap (T_e \times \mathbb{C}) \to T_e$, $(r, z) \mapsto r$. Call this map π_1. Unfortunately, this map will not be injective. However, if we restrict π_1 to some subset where $v \neq 0$, we will get an injective map. Let $0 < \eta < \rho$. The map $F' \cap ((T_e \setminus \{|v| \geq \eta\}) \times \mathbb{C}) \to T_e$,

$(r, z) \mapsto r$ is injective, in fact it is a diffeomorphism onto its image. The set $F' \cap (T_e \times \mathbb{C})$ is the union of two subsets:

$$\begin{aligned} F' \cap (T_e \times \mathbb{C}) &\cong (\{(u,v,z) \in F' : \eta \le |v| \le \rho\}) \cup (\{(u,v,z) \in F' : |v| \le \eta\}) \\ &= A_\eta \cup H_\eta. \end{aligned}$$

Here, A_η is mapped diffeomorphically onto its image in T_e by π_1. In coordinates, this image is determined by the equations $u^m + zu^l v^k = \epsilon$ and $\|(x', y')\| < \delta$. The first equation translates to $|u^m - \epsilon|/|u^l v^k| \le \delta$, and the second equation can be replaced by $|u| \le \delta$. We get

$$\pi_1(A_\eta) = \left\{ (u,v) : |u| \le \delta, \left| \frac{u^m - \epsilon}{u^l v^k} \right| \le \delta, \, \eta \le |v| \le \rho \right\}. \tag{8}$$

Let $(u, v) \in \pi_1(A_\eta) \cap T_e$. For any v' with $|v| \le |v'| \le \rho$ it is clear from (8) that $(u, v') \in \pi_1(A_\eta)$. This shows that $\pi_1(\{(u,v,z) \in F' : |v| = \rho\}$ is a strong deformation retract of $\pi_1(A_\eta)$. In fact, this shows that there is a diffeomorphism $\overline{F' \setminus \pi_1^{-1}(H_\eta)} \to \overline{(F' \setminus T_e) \times \mathbb{C}}$ which takes $(u, v) \in F'$ with $|v| = \eta$ to $(u, \rho v/|v|)$. Thus, for any $\eta \le \rho$, we have $F' = (H_\eta \amalg \overline{F' \setminus (T_e \times \mathbb{C})})/\lambda$, where λ is the map

$$\lambda : \{(u,v) \in H_\eta : |v| = \eta\} \to \overline{F' \setminus (T_e \times \mathbb{C})}, \, (u,v) \mapsto (u, \rho v/\eta, z(u,v)).$$

Here, $z(u, v)$ is the unique $z \in \mathbb{C}$ so that $(u, v, z) \in F'$. If η is small, then the set H_η has a particularly nice description. We will show that the map $H_\eta \to D_\eta \times D_\delta$, $(u, v, z) \mapsto (v, z)$ is an m-covering. The map is obviously proper, and it maps ∂H_η to $\partial(D_\eta \times D_\delta)$. Therefore, we only need to show that it has surjective derivatives at all points, and that the same holds for points on the boundary. For this, it is enough to show that $\partial_u \Phi' \ne 0$ on H_η. From this, the implicit function theorem implies that (v, z) will locally give coordinates on H_η, $(v, \arg z)$ will give local coordinates on $\{|z| = \delta\}$ and $(\arg v, z)$ will give local coordinates on $\{|v| = \eta\}$. We get

$$\partial_u \Phi' = \partial_u(u^m + zu^l v^k) = mu^{m-1} + zlu^{l-1}v^k.$$

Since T_e is compact, and u does not vanish on $F' \cap (T_e \times \mathbb{C})$, the function mu^{m-1} is bounded below by a positive constant. Similarly, the function zlu^{l-1} is bouded above. We can therefore choose η so small that $|mu^{m-1}| > |zlu^{l-1}v^k|$ whenever $|v| < \eta$. From this we get $\partial_u \Phi' \ne 0$ whenever $|v| \le \eta$. This proves that the map $(v, z)|_{H\eta}$ is a covering map.

Now we only need to observe that

$$\begin{aligned} ((v,z)|_{H_\eta})^{-1}(0) &= \{(u,v,z) : v = 0, \, z = 0, \, u^m + zu^l v^k = \epsilon, \, |z| \le \delta\} \\ &= \{(u,0,0) : u^m = \epsilon, \, |z| \le \delta\}. \end{aligned}$$

This shows that the fibers of (v, z) consist of m points. The cover is trivial because the disc $D_\eta \times D_\delta$ is contractible. Therefore, $H_\eta = \amalg_m(D \times D)$.

The set H_η is glued to $\overline{F' \setminus (T_e \times \mathbb{C})}$ by a diffeomorphism $\amalg_m(D \times \partial D) \to \partial(\overline{F' \setminus (T_e \times \mathbb{C})})$. In other words, we attach m copies of 2-handles to $\overline{F' \setminus (T_e \times \mathbb{C})}$. Let us fix one handle h, which has coordinates given by v, z, i.e. we have $h \cong \{(v, z) : |v| \leq \eta, |z| \leq \delta\}$. Let $\lambda_h : \{(v, z) : |v| = \eta, |z| \leq \delta\} \to \{(u, v) : |u| \leq \delta, |v| = \rho\}$ be the attaching map. Then

$$\lambda(\{(v, 0) : |v| = \eta\}) = \{(u_0, v) : |v| = \delta\}$$

for some fixed u_0 for which $u_0^m = \epsilon$. The function z gives a parametrization of the fibers of the tubular neigborhood $(\eta S^1) \times D_\delta$ of the submanifold $(\eta S^1) \times \{0\}$ (thought of as a submanifold of ∂h). This function induces the canonical framing on the handle h. The function $u - u_0$ parametrizes the fibers of a tubular neigborhood of the image $(\rho S^1) \times \{u_0\}$ which is thought of as a submanifold of $\partial(F' \setminus T_e)$. This function induces the framing of the attaching circle which was discussed in section (3).

For some small $\eta' > 0$, the section $(v, 0) \mapsto (v, \eta' v^{-k})$ of the tubular neigborhood of $(\eta S^1) \times \{0\} \subset (\eta S^1) \times D_\delta$ corresponds to a non-vanishing section of the corresponding normal bundle. It can be thought of as the first element of a basis of each fiber, i.e. a framing. This framing will be the $(-k)$-th framing of the handle as discussed in (2.2). Write $\lambda(v, \eta' v^{-k}) = (u_0 + u_1(v), \delta v/\eta)$. Then we have the formula

$$(u_0 + u_1(v))^m - \epsilon = -v^{-k}((u_0 + u_1)^l v^k) = -(u_0 + u_1)^l \tag{9}$$

because the coordinates $(u_0 + u_1, v, v^{-k})$ correspond to an element of F'. Since η' is chosen small, we may assume that $|u_1|$ is small with respect to to $|u_0| = \epsilon^{1/m}$. In particular, we may assume that for any fixed choice of $\kappa > 0$ we have

$$\operatorname{Arg} \frac{(u_0 + u_1(v))^m - \epsilon}{u_0^l} = \operatorname{Arg} \frac{-(u_0 + u_1(v))^l}{u_0^l} \in (-\kappa, \kappa) \qquad \mod 2\pi$$

for all $v \in \eta S^1$ (here, $\operatorname{Arg} : \mathbb{C}^* \to \mathbb{R}/(2\pi)\mathbb{Z}$ is the standard argument function). But we have for $u_1 = u_1(v)$

$$(u_0 + u_1)^m - \epsilon = \left(\sum_{j=0}^{m} \binom{m}{j} u_0^j u_1^{m-j} \right) - \epsilon = m u_0^{m-1} u_1 + u_1^2 \sum_{j=0}^{m-2} \binom{m}{j} u_0^j u_1^{m-j-2} \tag{10}$$

because $u_0^m = \epsilon$. Assuming again that $|u_1(v)|$ is small with respect to $|u_0|$ we can make $|u_1^2 \sum_{j=0}^{m-2} \binom{m}{j} u_0^j u_1^{m-j-2}|$ small with respect to $|m u_0^{m-1} u_1|$. Therefore, we may assume that

$$\operatorname{Arg} \frac{(u_0 + u_1(v))^m - \epsilon}{m u_0^{m-1} u_1(v)} \in (-\kappa, \kappa) \qquad \mod 2\pi \tag{11}$$

Combining (10) and (11) we get

$$\operatorname{Arg}\frac{mu_0^{m-1}u_1(v)}{u_0^l} \in (-2\kappa, 2\kappa) \qquad \mod 2\pi$$

which means that the variable $u_1(v)$ does not wind around the origin. Phrased differently, the loop $I \to \mathbb{C}^*$, $t \mapsto u_1(e^{2\pi it})$ is null-homotopic. In the language of framings, this means that the if we complete the section corresponding to $u_0 + u_1(v)$ in the normal bundle of $\rho S^1 \times \{u_0\} \subset \partial(F' \setminus T_e)$ to a (positive) basis on all fibers, then we get a framing which is equivalent to the framing already constructed. This means that the handle h is in fact attached with the $(-k)$-th framing.

4.6 Edges connecting Γ_1 and Γ_2

Let $v_1 \in \mathcal{V}(\Gamma_1)$ and $v_2 \in \mathcal{V}(\Gamma_2)$ so that $e = (v_1, v_2) \in \mathcal{E}(\Gamma)$. We want to describe $F' \cap (T_e \times \mathbb{C})$. The function g' does not vanish on this set. As in the previous subsections, we can project it down to a subset of T_e. Thus, $F' \cap (T_e \times \mathbb{C})$ is diffeomorphic to $\{r \in T_e : |x'(r)|^2 + |y'(r)|^2 \leq \delta, |z(r)| \leq \delta\}$. Here, z is the function which takes r in the projection of $F' \cap (T_e \times \mathbb{C})$ to the unique point $z \in \mathbb{C}$ so that $\Phi'(r, z(r)) = \epsilon$.

We use the notation $m = m_{v_1}$, $l = l_{v_1}$, $n = m_{v_2}$, $k = l_{v_2}$. By our choice of v_1 and v_2 we have $m \leq l$ and $n > k$.

Lemma 4.5. *There exist coordinates (u, v) on T_e so that in these coordinates $f' = u^m v^n$ and $g' = u^l v^k$.*

Proof. We can find coordinates (u_0, v_0) on T_e so that $f = u_0^m v_0^n$. In these coordinates, we have $g = \alpha u_0^l v_0^k$ for some function α, which does not vanish on T_e. Since $mk - nl < 0$, there exits a function β_1 on T_e so that $\beta_1^{mk-nl} = \alpha^n$. Let β_2 be a function so that $\beta_2^n = \beta_1^{-m}$. Let $u = \beta_1 u_0$ and $v = \beta_2 v_0$. Then $f = u_0^m v_0^n = \beta_1^m \beta_2^n u^m v^n = u^m v^n$, and $g = \alpha u_0^l v_0^k = \alpha \beta_1^l \beta_2^k u^l v^k$. We have $(\beta_1^l \beta_2^k)^n = \beta_1^{nl} \beta_2^{nk} = \beta_1^{nl} \beta_1^{-mk} = \alpha^n$. Thus, $\alpha \beta_1^l \beta_2^k$ is a constant function, whose value is an n-th root of unity. By choosing our functions β_1, β_2 carefully, we may assume that this constant is 1. This proves the lemma. □

From lemma (4.5), we see that the projection of $F' \cap (T_e \times \mathbb{C})$ to T_e is precisely the set

$$F_e = \left\{ (u, v) : |u|, |v| \leq \rho,\ |x'(u, v)|^2 + |y'(u, v)|^2 \leq \delta,\ \left| \frac{u^m v^n - \epsilon}{u^l v^k} \right| \leq \delta \right\}.$$

We need to describe this set, and how it glues to to $F' \setminus (T_e \times \mathbb{C})$. From the theory of plumbed 4-manifolds, and how the Milnor fiber of plane curves can be embedded into these manifolds, we have $F'_f \cap T_e \cong \amalg_{(m,n)} S' \times I$. Also, ∂T_e is the union of two solid tori. For any of the connected components $L \subset F'_f \cap T_e$ we have $\partial L = S^1 \amalg S^1$. One of these circles lies inside each of the tori. Choose a tubular neigborhood N around $F'_f \cap T_e$ in T_e so that $N \cap \partial T_e$ is a tubular neigborhood around $F'_f \cap \partial T_e$ in ∂T_e. Clearly, N is a trivial bundle, i.e. we have a diffeomorphism $N \to \amalg_{(m,n)} S' \times I \times D$. Let $\kappa : N \to \amalg_{(m,n)} S^1 \times D$ be the projection which simply disregards the second coordinate. Then any fiber of κ contains exactly two points in ∂T_e, one is in ∂T_{v_1} and the other in ∂T_{v_2}, corresponding to the partition of ∂T_e to two tori.

The following proposition reformulates the statement (4.1)(e):

Proposition 4.6. *The Milnor fiber F is diffeomorphic to $\overline{F' \setminus (T_e \times \mathbb{C})}/\kappa$. In other words, we may diregard the interior of $F' \cap (T_e \times \mathbb{C})$, and simply glue together the two points $(r, z), (r', z') \in F' \cap (\partial T_e \times \mathbb{C})$ if $\kappa(r) = \kappa(r')$. This procedure will not alter the diffeomorphism type of F'*

Proof. Let us consider the set

$$F_{e,\theta,\phi} = \{(u, v) \in F_e : \mathrm{Arg} u = \theta,\ \mathrm{Arg} v = \phi\},$$

Then $F_{e,\theta,\phi}$ can be identified with a subset of $[0, \rho] \times [0, \rho]$, via $(t, s) \leftrightarrow (|u|, |v|)$ for $(u, v) \in F_{e,\theta,\phi}$. Using simple calculus it is easy to see that $F_{e,\theta,\phi}$ has exactly one component which is simply connected, and that $\{(u, v) \in F_{e,\theta,\phi} : |v| = \rho\}$ is a closed interval i.e. a one dimensional compact connected manifold with nonempty boundary. Thus, the pair $(F_{e,\theta,\phi}, \{(u, v) \in F_{e,\theta,\phi} : |v| = \rho\})$ is topologycally the same as $([0, 1], \{1\} \times [0, 1])$.

We can extend this description to all of C, namely

$$(C, \{(u, v) \in C : |v| = \rho\}) = (S^1 \times S^1 \times [0, 1] \times [0, 1], S^1 \times S^1 \times [0, 1] \times \{1\}).$$

The inclusion $F' \cap (\overline{(T_{v_2} \setminus T_e)} \times \mathbb{C}) \to F' \cap (T_{v_2} \times \mathbb{C})$ is therefore isotopic to a diffeomorphism $h : F' \cap (\overline{(T_{v_2} \setminus T_e)} \times \mathbb{C}) \to F' \cap (T_{v_2} \times \mathbb{C})$. We may even assume that if $h(r, z) = (r', z')$, and $r \in N$, then $\kappa(r) = r'$, and that h is the identity outside some small neigborhood around $T_e \times \mathbb{C}$. Thus, h extends to a diffeomorphism which satisfies the required properties. $\square$

4.7 The monodromy

Let

$$E = \{(x, y, z, \lambda) \in \mathbb{C}^3 \times S^1 : \|(x, y, z)\| \leq \delta,\ \Phi(x, y, z) = \lambda\epsilon\}.$$

We want to describe the monodromy of the bundle $\mu : E \to S^1$, $(x, y, z, \lambda) \mapsto \lambda$. We have a map $p : E \to V$, $(x, y, z, \lambda) \mapsto \phi^{-1}(x, y)$. Here, $\phi : V \to \mathbb{C}^2$ is the common embedded resolution of f and g as before, and $\phi^{-1}(x, y)$ is well defined because $(x, y) \neq (0, 0)$. We split E into three parts:

Let E_1 be the preimage of the union of T_1 and T_e for any $e = (v, a) \in \mathcal{E}(\Gamma)$ with $v \in \mathcal{V}(\Gamma_1)$ and $a \in \mathcal{A}_f$.

Let E_2 be the preimage of the union of T_2 and T_e for any $e = (v_1, v_2) \in \mathcal{E}(\Gamma)$ with $v_1 \in \mathcal{V}(\Gamma_1)$ and $v_2 \in \mathcal{V}(\Gamma_2)$.

Let E_2 be the preimage of the union of all T_e where $e = (v, a) \in \mathcal{E}(\Gamma)$ with $v \in \mathcal{V}(\Gamma_2)$ and $a \in \mathcal{A}_g$.

Explicitly, we have

$$\begin{aligned} E_1 &= p^{-1}(T_1 \cup (\cup_{e \in \mathcal{E}_\Gamma(\Gamma_1) \setminus \mathcal{E}_\Gamma(\Gamma_2)} T_e)) \\ E_2 &= p^{-1}(T_2 \cup (\cup_{e \in \mathcal{E}_\Gamma(\Gamma_1) \cap \mathcal{E}_\Gamma(\Gamma_2)} T_e)) \\ E_3 &= p^{-1}(\cup_{e \in \mathcal{E}_\Gamma(\Gamma_1) \cap \mathcal{E}_\Gamma(\Gamma_2)} T_e). \end{aligned}$$

Restricting μ to any of these will yield a subbundle of E, whose fibers are described in (4.2)-(4.6). We have $F = \mu^{-1}(1) = \cup_{j=1}^{3}(\mu^{-1}(1) \cap E_j)$. The sets E_j are actually subbundles of E, so we can assume that the monodromy m_Φ of the bundle E acts on each of the smaller fibers $\mu^{-1}(1) \cap E_j$ individually, that is, these sets are invariant under m_Φ.

By (4.2) we have $\mu^{-1}(e^{it}) \cap p^{-1}(T_1) = f^{-1}(\epsilon e^{it} \times D$. From this description it is clear that up to homotopy, we have $m_\Phi(r, z) = (m_f(r), z)$ for any $r \in F_{\Gamma_1}$, where m_f is chosen to map $F_f \cap T_1$ to it self.

For any edge $e = (w, a) \in \mathcal{E}(\Gamma)$ with $w \in \mathcal{V}(\Gamma_1)$ and $a \in \mathcal{A}_f(\Gamma)$, consider $p^{-1}(T_e)$. Using the description of $F' \cap (T_e \times \mathbb{C})$ from (4.3), we have $p^{-1}(T_e) \cong (D_\rho \setminus D_\eta^\circ) \times D_\rho) \times S^1$, so the monodromy acts (up to isotopy) trivially on $F' \cap (T_e \times \mathbb{C})$.

To describe the monodromy action of the bundle K_2, it is enough to look at $p^{-1}(T_2)$, since the inclusion

$$F' \cap (T_2 \times \mathbb{C}) \hookrightarrow F' \cap ((T_2 \cup (\cup_{e \in \mathcal{E}_\Gamma(\Gamma_1) \cap \mathcal{E}_\Gamma(\Gamma_2)} T_e)) \times \mathbb{C})$$

is a homotopy equivalence. From the description of $F' \cap T_2$ in (4.4) we see that for any t the set there is a diffeomorphism $F' \cap (T_2 \times \mathbb{C}) \to \{r \in T_2 : \delta/2 \leq \|(x'(r), y'(r))\| \leq \delta\}$. Using this diffeomorphism on each fiber we obtain a trivialization $K_2 = (F' \cap (T_2 \times \mathbb{C})) \times S^1$. This means that restricting m_Φ to $F' \cap (T_2 \times \mathbb{C})$ we get a map which is isotopic to the identity.

We have seen that the complement $F' \setminus (\cup_e T_e \times \mathbb{C})$, where the union ranges over edges $e = (w, a) \in \mathcal{E}(\Gamma)$ with $w \in \mathcal{V}(\Gamma_2)$ and $a \in \mathcal{A}_g(\Gamma)$, is invariant under m_Φ. Therefore, $F' \cap (\cup_e T_e \times \mathbb{C})$ is also invariant under m_Φ. In fact, it is easy to see that the set H_η, defined in (4.5) is invariant under m_Φ. The set

$\{(r,z) \in H_\eta : z = 0\} = \amalg_{m_w} D$ is naturally identified with a subset of F_f, and it is a strong homotopy retract of H_η. The monodromy m_f permutes these discs cyclically, and restricting m_Φ to the set $\{V \times \{0\} \subset V \times \mathbb{C}$ we get the monodromy of f. This shows that m_Φ must permutes the handles $H_\eta = \amalg_{m_w} D \times D$.

References

[1] N. A'Campo. La fonction zeta d'une monodromie. *Comment. Math. Helvetici*, 50:233–248, 1975.

[2] T. de Jong and D. van Straten. Deformation theory of sandwiched singularities. *Duke Math. Journal*, 95(3), 1998.

[3] R. E. Gompf and A. I. Stipsicz. *4-Manifolds and Kirby calculus*. AMS Graduate Studies in Math. AMS, 1999.

[4] P. A. Griffiths. Periods of integrals of algebraic manifolds: summary of main results and discussion of open problems. *Bull. Amer. Math. Soc.*, 76:228–296, 1970.

[5] M. Kato and Y. Matsumoto. On the connectivity of the Milnor fibre of a holomorphic function at a critical point. *Manifolds Tokyo 1973*, pages 131–136, 1975.

[6] J. Milnor. *Singular Points of Complex Hypersurfaces*. Annals of Math. Studies. Princeton University Press, 1968.

[7] A. Némethi. Some topological invariants of isolated hypersurface singularities. *Bolyai Soc. Math. Studies*, 8:353–413, 1996.

[8] A. Némethi. Hypersurface singularities with 2-dimensional critical locus. *J. London Math. Soc.*, 59(2):922–938, 1999.

[9] A. Némethi and Á. Szilárd. The boundary of the Milnor fibre of non-isolated hypersurface surface singularities, 2009. arxiv:0909.0354.

[10] W. D. Neumann. A calculus for plumbing applied to the topology of complex surface singularities and degenerating complex curves. *Trans. of Amer. Math. Soc.*, 268(2):299–344, 1981.

[11] D. Siersma. The vanishing topology of non isolated singularities. *New Developments in Singularity Theory (Cambridge 2000)*, pages 447–472, 2001.