Fractional order Sobolev spaces

Thesis

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Készítette: Gerencsér Máté

Témavezető: Izsák Ferenc
adjunktus
Alkalmazott Analízis és Számításmatematikai Tanszék

Eötvös Loránd Tudományegyetem
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Contents

1. Introduction

The term "fractional order Sobolev space" might sound like a precise mathematical concept but in fact it is not. There are several methods to fill in the gaps between the traditional Sobolev spaces of integer order and in some cases the function spaces obtained are equivalent, while in other cases they are not. Different approaches focus on generalizing different properties of the Sobolev spaces and each has its own advantage. These generalizations can be interesting and useful both theoretically and in the applications as well. The aim of the thesis is to give an overview of these ideas and apply these techniques to the a non-classical Sobolev space H_{curl}.

The thesis is structured as follows. Chapter 2 summarizes the usual notions used in the following. Chapter 3 describes the different scales of function spaces that are usually referred to as "fractional order Sobolev spaces", based on [1], [2], [5]. Chapter 4 examines the space of $\mathbf{L}^2$ functions whose *curl* is also in $\mathbf{L}^2$, and some fractionalization problems regarding this space. The classical results of the topic follows [4], the rest of the chapter is partially based on [3], partially my own work.

I would like to thank my supervisor, Izsák Ferenc, for his help throughout the making of this thesis.

2. Preliminaries

We will use the following notions throughout the thesis.

A domain is an open subset $\Omega \subset \mathbb{R}^n$. A bounded domain is called Lipschitz, if for every point x of its boundary $\partial\Omega$ there exists a neighborhood U of x such that $U \cap \partial\Omega$ is a graph of a Lipschitz-continous function.

For a normed space X by default $||\cdot||_X$ denotes its norm. The relation $f(u) \leq Cg(u)$ ($cf(u) \leq g(u)$) denotes that there exists a fixed $C > 0$ ($c > 0$) for which the inequality holds for all u from a given space that is always clear from the context. This constant can change from line to line but is always independent from u. The relation $f(u) \sim g(u)$ denotes that $cg(u) \leq f(u) \leq Cg(u)$. Two norms $||\cdot||$ and $||\cdot||'$ on a normed space U are said to be equivalent if $||u|| \sim ||u||'$.

The space $\mathcal{D}(\Omega)$ consists of the compactly supoorted infinitely many times differentiable functions with the topology defined by the convergence: $\phi_n \to \phi$ if and only if there is a compact set $K \subset \Omega$ with $supp(\phi_n), supp(\phi) \subset K$ and for all multiindex α $\partial^\alpha \phi_n \to \partial^\alpha \phi$ uniformly. Its dual, $\mathcal{D}'$ is the space of the continuous linear functionals, or, the distributions, with the weak topology. For every locally integrable f corresponds a distribution with the effect $\phi \to \int \sum_{j=1}^n f_j \phi_j$. For a normed function space V in which $\mathcal{D}$ is imbedded, we define its dual to be the subset of distributions which extends uniquely to V.

The Schwartz space of functions u on $\mathbb{R}^n$ with $\sup |\partial^\alpha u(x) x^\beta| < \infty$ for all α, β multiindices is denoted by $\mathcal{S}$. The elements of $\mathcal{S}'$, the dual of $\mathcal{S}$ are called tempered distributions. The Fourier transformation operator, which can be defined on $\mathcal{S}'$, is denoted by $\mathcal{F}$, and the notation $\hat{u} = \mathcal{F}(u)$ is also used. On $\mathcal{S}$, we define $\mathcal{F}(\phi)(\xi) = (2\pi)^{-n/2} \int \exp(i < x, \xi >) dx$ and we extend it to $\mathcal{S}'$ by $\mathcal{F}u(\phi) = u(\mathcal{F}\phi)$.

Given a domain $\Omega \subset \mathbb{R}^n$ we use the notation $\mathbf{L}^p(\Omega)$ for the space of functions u with $||u||^p_{\mathbf{L}^p(\Omega)} = \int_\Omega |u|^p < \infty$ ($1 \leq p < \infty$). When it does not cause confusion, we use the abbreviation $||\cdot||_{\mathbf{L}^p(\Omega)} = ||\cdot||_p$. The standard Sobolev space of the functions with their α-th partial derivatives in $\mathbf{L}^p(\Omega)$ for all $|\alpha| \leq k$ is denoted with $W^{k,p}(\Omega)$, accompanied with the norm $||u||_{W^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} ||\partial^\alpha u||_p$ ($k \in \mathbb{N}, 1 \leq p < \infty$). Here we use the derivatives in the distributional sense, i.e. for $u \in \mathcal{D}'$, $\phi \in \mathcal{D}$, we define $\partial^{alpha} u(\phi) = (-1)^{|\alpha|} u(\partial^\alpha(\phi))$. We distinguish the special case $H^k(\Omega) = W^{k,2}(\Omega)$. We also use the abbreviation $\mathbf{L}^p = \mathbf{L}^p(\mathbb{R}^n)$, and similarly with the Sobolev spaces. We also use this convention for the spaces introduced later.

Given two Banach spaces X and Y their direct sum is denoted by $X \oplus Y$ and consists of the formal sums $x + y$ where $x \in X, y \in Y$, and $||x + y||_{X \oplus Y} = (||x||_X^2 + ||y||_Y^2)^{1/2}$. Note that changing 2 to any $1 \leq p < \infty$ we obtain equivalent Banach spaces. If X and Y are Hilbert spaces, the norm can also be characterized by the identities $||x + 0||_{X \oplus Y} = ||x||_X$, $||0 + y||_{X \oplus Y} = ||y||_Y$, and the orthogonality of the components. Given a closed subspace U in a Hilbert space P_U denotes the projection operator to U.

3. Classical scales of function spaces

This section aims to cover most of the possible definitions of fractional order Sobolev spaces that can be found in the literature and describe their relations to each other. To avoid confusion, we will omit the term "fractional order Sobolev space" and use other common names for these spaces instead. We will formulate the different but equivalent definitions in forms of theorems. For the detailed proofs we refer to [1], [2], [5].

3.1. Real interpolation

Given two Banach spaces X_0 and X_1, both continously imbedded in a Banach space X with a $X_0 \cap X_1 \neq \{0\}$ - such a pair is called an interpolation couple -, interpolation methods provide ways to construct intermediate spaces between them. In many cases, including the ones we will deal with, X_0 is continuously imbedded in X_1. The two main different methods are the real and complex interpolation but we will now only go into details with the real method.

The intersection $X_0 \cap X_1$ and the algebraic sum $X_0 + X_1$ are themselves Banach spaces with the norms

$$||u||_{X_0 \cap X_1} = \max\{||u||_{X_0}, ||u||_{X_1}\},$$

$$||u||_{X_0+X_1} = \inf\{||x_0||_{X_0} + ||x_1||_{X_1} \, | u = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}.$$

The intersection is continuously imbedded in X_j and X_j is continuously imbedded in the algebraic sum for $j = 0, 1$. In general, we say that a Banach space U is intermediate between X_0 and X_1 if $X_0 \cap X_1$ is continuously imbedded in U and U is continuously imbedded in $X_0 + X_1$. When $X_0 \subset X_1$, this equals the intuitive requirement that and intermediate space has to be "larger" than X_0 and "smaller" than X_1.

For any given $x \in X_0 + X_1 \subset X$, $t \geq 0$, $1 \leq p < \infty$ define

$$K_p(t, x) = \inf\{||x_0||^p_{X_0} + ||tx_1||^p_{X_1} \, | x = x_0 + x_1, x_0 \in X_0, x_1 \in X_1\}^{1/p}.$$

The usual approach takes $p = 1$, but it will be clear that all choice of p result in equivalent Banach spaces. Since in the case of Hilbert spaces the choice $p = 2$ will turn out to be the suitable one, we introduce the abbreviation $K = K_2$. Now take $1 \leq q < \infty$ and $0 < \theta < 1$. The interpolation space $(X_0, X_1)_{\theta,q}$ consists of the vectors $x \in X_0 + X_1$ for which

$$||x||^q_{(X_0,X_1)_{\theta,q}} = \int_0^\infty [t^{-\theta} K(t,x)]^q \frac{dt}{t} < \infty.$$

THEOREM 3.1 *Let (X_0, X_1) be an interpolation couple, $0 < \theta < 1$, and $1 \leq q < \infty$. Then $(X_0, X_1)_{\theta,q}$ is a Banach space and its norm satisfies*

$$\frac{1}{\sqrt{2}} ||u||_{X_0+X_1} \leq \frac{||u||_{(X_0,X_1)_{\theta,q}}}{\int_0^\infty [t^{-\theta} \min\{1,t\}]^q \frac{dt}{t}} \leq \sqrt{2}\, ||u||_{X_0 \cap X_1},$$

therefore $(X_0, X_1)_{\theta,q}$ is an intermediate space between X_0 and X_1.

The extremal cases for $q = \infty$ and/or $\theta = 0, 1$ can also be defined, the corresponding theorems are often trivial, but to avoid technical difficulties we will not deal with these cases.

Chopping the defining integral to integrals between 2^j and 2^{j+1} one can show the following discretization theorem.

THEOREM 3.2 *Let (X_0, X_1) be an interpolation couple, $0 < \theta < 1$, and $1 \leq q < \infty$. Then for all $x \in X_0 + X_1$*

$$||x||^q_{(X_0,X_1)_{\theta,q}} \sim \sum_{j=-\infty}^{\infty} 2^{-jq\theta} (K(2^j, x))^q.$$

THEOREM 3.3 *Let (X_0, X_1) be an interpolation couple, $0 < \theta < 1$, and $1 \leq q < \infty$. Then $X_0 \cap X_1$ is dense in $(X_0, X_1)_{\theta,q}$.*

Let us introduce a functional similar to K, this time on $X_0 \cap X_1$:

$$J(t,u) = \max\{||u||_{X_0}, ||tu||_{X_1}\}.$$

This functional can be used to define interpolation in another way resulting in equivalent spaces, but more importantly, it can be used to formulate the Reiteration

Theorem. First, we define an intermediate space X to be in the class $\mathcal{H}(\theta, X_0, X_1)$, if

$$K(t,u) \leq Ct^{\theta} \|u\|_X \text{ for all } u \in X, \text{ and}$$

$$\|u\|_X \leq Ct^{-\theta} J(t,u) \text{ for all } u \in X_0 \cap X_1.$$

LEMMA 3.4 *Let (X_0, X_1) be an interpolation couple, $0 < \theta < 1$, and $1 \leq q < \infty$. Then $(X_0, X_1)_{\theta,q} \in \mathcal{H}(\theta, X_0, X_1)$.*

THEOREM 3.5 (REITERATION THEOREM) *Let (X_0, X_1) be an interpolation couple, $0 < \lambda < 1$, $1 \leq q < \infty$, $0 \leq \theta_0 < \theta_1 \leq 1$, and Y_0, Y_1 intermediate spaces between X_0 and X_1 such that $Y_j \in \mathcal{H}(\theta_j, X_0, X_1)$, $j = 0, 1$. Let $\theta = (1-\lambda)\theta_0 + \lambda\theta_1$. Then*

$$(Y_0, Y_1)_{\lambda,q} = (X_0, X_1)_{\theta,q}.$$

The immediate consequence - and the reason for the name of the theorem - is that with the previous notations

$$((X_0, X_1)_{\theta_0,q_0}, (X_0, X_1)_{\theta_1,q_1})_{\lambda,q} = (X_0, X_1)_{\theta,q}$$

where $1 \leq q_0, q_1 < \infty$ are arbitrary.

It is also an important property of the interpolation that its effect on the dual spaces can be expressed easily.

THEOREM 3.6 (DUALITY THEOREM) *Let (X_0, X_1) be an interpolation couple, $0 < \theta < 1$, and $1 < q < \infty$ and assume that $X_0 \cap X_1$ is dense in both X_0 and X_1. Define q' by $1/q + 1/q' = 1$. Then (X_1^*, X_0^*) is also an interpolation couple and*

$$(X_1^*, X_0^*)_{\theta,q} = (X_0, X_1)^*_{1-\theta,q'}.$$

It is worth to note another simple identity in which this kind of change of parameters appear:

$$(X_0, X_1)_{\theta,q} = (X_1, X_0)_{1-\theta,q}.$$

When interpolating between two Hilbert spaces, it is natural to expect that the result is a Hilbert space as well. By checking the paralelogram identity the following theorem provides a sufficient condition.

THEOREM 3.7 *Let* (X_0, X_1) *be an interpolation couple consisting of two Hilbert spaces and* $0 < \theta < 1$*. Then* $(X_0, X_1)_{\theta,2}$ *is also a Hilbert space.*

3.2. Besov spaces

The scale of Besov spaces is obtained by using the real interpolation method to create intermediate spaces between Sobolev spaces. We have to note that in general the classical Sobolev spaces are not "closed" under interpolation, i.e. $W^{k,p}(\Omega)$ usually does not equal to $(\mathbf{L}^p(\Omega), W^{m,p}(\Omega))_{k/m,q}$.

LEMMA 3.8 *Let* Ω *be a Lipschitz domain. Let* $0 < k < m$ *be integers and* $1 \le p < \infty$*. Then*

$$W^{k,p}(\Omega) \in \mathcal{H}(k/m, \mathbf{L}^p(\Omega), W^{m,p}(\Omega)).$$

DEFINITION 3.9 *Let* $0 < s < \infty$*,* $1 \le p \le \infty$*, and* $1 \le q < \infty$*. Let* m *be the smallest integer larger than* s*. Then the Besov space* $B^{s,p,q}(\Omega)$ *is defined by*

$$B^{s,p,q}(\Omega) = (L^p(\Omega), W^{m,p}(\Omega))_{s/m,q}.$$

The Reiteration Theorem and the previous Lemma show us that in fact

$$B^{s,p,q}(\Omega) = (W^{k,p}(\Omega), W^{m,p}(\Omega))_{\theta,q}$$

for any $k < s < m$ with k, m integers, $0 < \theta < 1$, and $s = (1-\theta)k + \theta m$ and also

$$B^{s,p,q}(\Omega) = (B^{s_1,p,q_1}(\Omega), B^{s_2,p,q_2}(\Omega))_{\theta,q}$$

for any $0 < s_1 < s < s_2$, $0 < \theta < 1$, and $1 \le q_1, q_2 \le \infty$ with $s = (1-\theta)s_1 + \theta s_2$.

The special cases $B^{s,p,p}(\Omega)$ are often referred as the Slobodeckij spaces and have an important role of characterizing the traces of functions in $W^{m,p}(\Omega)$. The trace of a smooth function on $\mathbb{R}^{n+1}$ is defined by restricting it to the subspace $\{(x_1, \ldots, x_n, 0)\}$, and the trace operator is extended in a usual way.

THEOREM 3.10 *Let* $1 < p < \infty$ *and* $m > 0$ *be an integer. Then* $u \in B^{m-1/p,p,p}(\mathbb{R}^n)$ *if and only if* u *is the trace of a function in* $W^{m,p}(\mathbb{R}^{n+1})$*.*

The Theorem does not apply in the case of $p = 1$. However, it is known that the trace of a function from $W^{m,1}$ is in $W^{m-1,1}$. There is also a trace imbedding theorem for Besov spaces.

THEOREM 3.11 *Let $1 \leq p < \infty$, $1 \leq q < \infty$, and $s > 0$ such that $s - 1/p > 0$. Then the trace operator is continuous from $B^{s,p,q}(\mathbb{R}^n)$ to $B^{s-1/p,p,q}(\mathbb{R}^{n-1})$.*

Repeating taking traces gives imbedding theorems to spaces on $\mathbb{R}^k$ for sufficiently large k. These theorems extend to traces on sufficiently smooth surfaces of sufficiently high dimension as well. In case there exists a suitable extension operator from Ω, these theorems also extend to functions in $B^{s,p,q}(\Omega)$.

It is sometimes useful to examine what more well-known spaces includes the Besov space in question. This motivates the imbedding theorems similar to the following one.

THEOREM 3.12 *Let $1 < p < \infty$, $1 \leq q < \infty$, $s > 0$, such that $sp > n$. Then $B^{s,p,q}$ is imbedded in the space of continuous and bounded functions.*

The norms of the Besov spaces on $\mathbb{R}^n$ have a more intristic equivalent expressed with the $\mathbf{L}^p$-modulus of continuity. First, for a point $h \in \mathbb{R}^n$ and a function $u \in \mathbf{L}(\mathbb{R}^3)$ define u_h to be the mapping x to $u(x-h)$. Let $\Delta_h u = u - u_h$, $\omega_p(u,h) = ||\Delta_h u||_p$, and for positive integers m let $\omega_p^{(m)}(u,h) = ||(\Delta_h)^m u||_p$.

THEOREM 3.13 *Let $1 < p < \infty$, $1 \leq q < \infty$, $m > s > 0$ with m being an integer and $u \in \mathbf{L}^p(\mathbb{R}^n)$. Then $u \in B^{s,p,q}(\mathbb{R}^n)$ if and only if*

$$\int_{\mathbb{R}^n} \left[|h|^{-s}\omega_p^{(m)}(u,h)\right]^q \frac{dh}{|h|^n} < \infty.$$

Moreover, the q-th root of the expression above is equivalent to $||\cdot||_{B^{s,p,q}}$.

For non-integers $s = \lfloor s \rfloor + \{s\}$, where $\lfloor s \rfloor \in \mathbb{N}$ and $\{s\} \in (0,1)$, a closely related intristic norm can be defined using the $\lfloor s \rfloor$-th order derivatives:

THEOREM 3.14 *Let $1 < p < \infty$, $1 \leq q < \infty$, $s > 0$, and $u \in W^{\lfloor s \rfloor,p}(\mathbb{R}^n)$. Then $u \in B^{s,p,q}(\mathbb{R}^n)$ if and only if*

$$\sum_{|\alpha|=\lfloor s \rfloor} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\partial^\alpha u(x) - \partial^\alpha u(x-h)|^p \right)^{q/p} |h|^{-\{s\}q} \frac{dh}{|h|^n} < \infty.$$

There is another equivalent intristic norm that will show some kind of relation between the Besov spaces and the Triebel-Lizorkin spaces defined in the next subsection. To this end, first let Φ be an even function on the real line with the properties $\Phi(t) = 1$ for $|t| \leq 1$, $\Phi(t) = 0$ for $|t| \geq 2$, and $|\Phi(t)| \leq 1$ for all t. For each integer i let $\phi_i(t) = \Phi(t/2^{i+1}) - \Phi(t/2^i)$ and for any $\xi \in \mathbb{R}^n$ let $\psi_i(\xi) = \phi_i(|\xi|)$. Finally, define the operator $T_i u = \mathcal{F}^{-1}(\xi \to \psi_i(\xi)\hat{u}(\xi))$. One can regard the functions $T_i u$ as dyadic parts of u with nearly disjoint frequencies.

THEOREM 3.15 *The norm $||u||_{B^{s,p,q}(\mathbb{R}^n)}$ is equivalent to*

$$\left[\sum_{j=-\infty}^{\infty}\left(\int_{\mathbb{R}^n}(1+2^{sj})^p|T_j u(x)|^p dx\right)^{q/p}\right]^{1/q}.$$

3.3. Triebel-Lizorkin spaces

DEFINITION 3.16 *Let $0 < s < \infty$, $1 \leq p < \infty$, and $1 \leq q < \infty$. Then the Triebel-Lizorkin space $F^{s,p,q}(\Omega)$ is defined by*

$$F^{s,p,q}(\mathbb{R}^n) = \{u : ||u||_{F^{s,p,q}(\mathbb{R}^n)} = \left[\int_{\mathbb{R}^n}\left(\sum_{j=-\infty}^{\infty}(1+2^{sj})^q|T_j u(x)|^q\right)^{p/q} dx\right]^{1/q} < \infty\}.$$

Clearly $F^{s,p,p} = B^{s,p,p}$, and therefore the Slobodeckij spaces are included in this scale as well. Another important special case is obtained by generalizing the well-known fact that for any k integer $u \in W^{k,p}$ if and only if the function $\xi \to (1 + |\xi|^2)^{k/2}\hat{u}(\xi)$ is the Fourier transform of a function from $\mathbf{L}^p$. The spaces we get by changing k to any $s \geq 0$ in this property are often referred to as the Bessel potential spaces.

THEOREM 3.17 *Let $0 < s < \infty$ and $1 \leq p < \infty$. Then $u \in F^{s,p,2}$ if and only if the function $\xi \to (1 + |\xi|^2)^{s/2}\hat{u}(\xi)$ is the Fourier transform of a function from $\mathbf{L}^p$.*

Since $(1+|\xi|^2)^{s/2}$ can be bounded from above and below by constant times $1+|\xi|^s$, the previous condition can be rephrased that require u and $\xi \to \mathcal{F}^{-1}|\xi|^s\hat{u}(\xi)$ to be in $\mathbf{L}^p$. The latter expression is also known as the fractional Laplacian of u, and so the Bessel potential spaces can be obtained by generalizing the definition of Sobolev spaces by the derivatives of the function. The fractional Laplacian is one of the possible generalizations of the differentiation operator (see [7], Chapter 2), it simply fractionalizes the positive operator $-\Delta$. It also follows from the previous Theorem that $F^{m,p,2} = W^{m,p}$ for integers m.

It also turns out that the Bessel potential spaces can be obtained from the classical Sobolev spaces the same way as the Besov spaces if we use the complex interpolation method instead of the real interpolation.

DEFINITION 3.18 *Let (X_0, X_1) be an interpolation couple. Let $\mathcal{A}$ denote the collection of bounded analytic functions f from the strip $\{\theta + i\tau | 0 < \theta < 1\}$ to $X_0 + X_1$ that extend continously to the boundary with the property $f(j + \tau) \in X_j$ and $||f(j + i\tau)||_{X_j} \to 0$ as $|\tau| \to \infty$, for $j = 0, 1$. For $0 < \theta < 1$ the complex interpolation space between X_0 and X_1 is*

$$[X_0, X_1]_\theta = \{u \in X_0 + X_1 | \exists f \in \mathcal{A} : f(\theta) = u\}$$

with the norm

$$||u||_{[X_0,X_1]_\theta} = \inf\{\max\{\sup_\tau ||f(i\tau)||_{X_0}, \sup_\tau ||f(1 + i\tau)||_{X_1}\} | f(\theta) = u\}.$$

Although we omit the detailed description of this method, it is worth to note that it has similar properties to the real method. For example the analogues of the Reiteration Theorem and the Duality Theorem hold. We also have the following identities that show the connection between the two methods

$$(X_0, X_1)_{\theta,1} \subset [X_0, X_1]_\theta \subset (X_0, X_1)_{\theta,\infty},$$

$$([X_0, X_1]_{\theta_0}, [X_0, X_1]_{\theta_1})_{\lambda,q} = (X_0, X_1)_{(1-\lambda)\theta_0+\lambda\theta_1,q},$$

$$[(X_0, X_1)_{\theta_0,q_0}, (X_0, X_1)_{\theta_1,q_1}]_\lambda = (X_0, X_1)_{(1-\lambda)\theta_0+\lambda\theta_1,q},$$

where q is defined by $1/q = (1 - \lambda)/q_0 + \lambda/q_1$.

THEOREM 3.19 *Let $0 < s < \infty$, $0 < \theta < 1$, $m \le k$ integers with $(1 - \theta)m + \theta k = s$, and $1 \le p < \infty$. Then*

$$F^{s,p,2} = [W^{m,p}, W^{k,p}]_\theta.$$

Similar imbedding theorems holds for the Triebel-Lizorkin spaces as for the Besov spaces. For example, analogously to the one mentioned in the previous subsection, the following theorem holds.

THEOREM 3.20 *Let $1 < p < \infty$, $1 \leq q < \infty$, $s > 0$, such that $sp > n$. Then $F^{s,p,q}$ is imbedded in the space of continuous and bounded functions.*

Finally, we summarize the relations between the spaces introduced:

$$F^{s,p,q} \subset B^{s,p,q} \text{ if } q \leq p,$$

$$B^{s,p,q} \subset F^{s,p,q} \text{ if } p \leq q,$$

$$B^{s,p,q_0} \subset B^{s,p,q_1} \text{ if } q_0 \leq q_1,$$

$$F^{s,p,q_0} \subset F^{s,p,q_1} \text{ if } q_0 \leq q_1.$$

In the special case $p = q = 2$, $B^{s,2,2} = F^{s,2,2}$ with the special cases $F^{m,2,2} = H^m$ in case m is integer, and furthermore we have

$$B^{s,2,2} = (B^{s_0,2,2}, B^{s_1,2,2})_{\theta,2} = [B^{s_0,2,2}, B^{s_1,2,2}]_\theta$$

for $0 < s_0 < s < s_1 < \infty$ with $(1-\theta)s_0 + \theta s_1 = s$.

REMARK. Some of the definitions and the theorems were stated only in the case $\Omega = \mathbb{R}^n$. These properties can be extended to more general domains via the use of an extension operator. In [6] it is shown that there exists an extension operator $\mathcal{E}$ such that it simultaneously and boundedly extends functions in $B^{s,p,q}(\Omega)$ to $B^{s,p,q}(\mathbb{R}^n)$ and $F^{s,p,q}(\Omega)$ to $F^{s,p,q}(R^n)$ if the domain Ω is "nice" enough, for example, if it is a Lipschitz domain. Here $F^{s,p,q}(\Omega)$ is defined as restrictions of functions on $F^{s,p,q}(\mathbb{R}^n)$. The spaces $B^{s,p,q}(\Omega)$ were already defined, but from the existence of this extension operator it follows that they can also be defined through restriction.

4. The fractionalization of H_{curl}

In this section we deal with the possibilities of the fractionalization of the non-standard Sobolev space $H_{curl} = \{u \in (\mathbf{L}^2)^3 | curl(u) \in (\mathbf{L}^2)^3\}$. This space plays an important role in the theory of Maxwell equations. Even the extension to integer orders is not evident, we present two approaches.

First recall some properties of the *curl* operator. It is defined by

$$curl(u) = \left(\frac{\partial u_3}{\partial_2} - \frac{\partial u_2}{\partial_3}, \frac{\partial u_1}{\partial_3} - \frac{\partial u_3}{\partial_1}, \frac{\partial u_2}{\partial_1} - \frac{\partial u_1}{\partial_2} \right)$$

where derivatives are understood in the distributional sense. Its Fourier counterpart is the vectorial product with the longitudinal direction:

$$\mathcal{F}(curl(u))(\xi) = i(\xi \times \hat{u}(\xi)).$$

The same formula holds if we replace *curl* by $(curl)^j$, $\xi\times$ by $\xi\times^j$, and i by i^j for any $j \in \mathbb{N}$. Here the operator $\xi\times^j$ is defined by

$$\xi \times^j f(\xi) = \underbrace{\xi \times \cdots \xi \times}_{j} f(\xi)$$

and similarly for $(curl)^j$. The space H_{curl} can be decomposed

$$H_{curl} = \mathrm{Ker}(curl) \oplus \mathrm{Ker}(curl)^{\perp}$$

where the second component is compactly imbedded in $(\mathbf{L}^2)^3$. This is known as the Helmhlotz decomposition. The Fourier transform of a function in the kernel of the *curl* points in the longitudinal direction almost everywhere. Conversely, if a function from $(\mathbf{L}^2)^3$ has a Fourier transform that points in the longitudinal direction, its *curl* is zero. Therefore it is easy to see that if $u \in \mathrm{Ker}(curl)^{\perp}$, then its Fourier transform has to point perpendicular to the longitudinal direction.

It is known that the sequence called the De Rahm-diagram

$$(H^1)^3 \xrightarrow{grad} H_{curl} \xrightarrow{curl} H_{div} \xrightarrow{div} (\mathbf{L}^2)^3 \qquad (1)$$

has the property that the image of each operator is included in the kernel of the next one, where

$$H_{div} = \{u \in (\mathbf{L}^2)^3 | div(u) = \frac{\partial u_1}{\partial_1} + \frac{\partial u_2}{\partial_2} + \frac{\partial u_3}{\partial_3} \in \mathbf{L}^2\}.$$

Similarly, if our initial space is $(\mathbf{L}^2(\Omega))^3$, we can define $H_{curl}(\Omega)$, $H_{div}(\Omega)$ for any domain. Assume that Ω is bounded and has the property that there exist L open connected surfaces $\Sigma_1, \Sigma_2, \dots \Sigma_L$ such that

- Σ_l is an open part of a smooth surface,
- $\partial\Sigma_l \subset \partial\Omega$,
- $\Sigma_l \cap \Sigma_m = \emptyset$ if $l \neq m$,
- For any point $x \in \partial\Omega$ there is an integer $r_x \in \{1, 2\}$ and a $\rho_x > 0$ such that for all $0 < \rho < \rho_x$ the intersection of Ω with the ball with center x and radius ρ has r_x connected components, each one being a Lipschitz domain.

Under these conditions the diagram (1) also has the property that the image of each operator is a closed subspace of finite codimension in the kernel of the next operator [4].

4.1. The scale H^s_{curl}

If we consider the *curl* operator as a kind of differentiation, it is natural to introduce the following spaces

$$H^n_{curl} = \{u \in (\mathbf{L}^2)^3 | curl^j(u) \in (\mathbf{L}^2)^3 \text{ for } j = 1, 2, \dots n\}.$$

H^n_{curl} is a Hilbert space with the norm $||u||^2_{H^n_{curl}} = \sum_{j=0}^n ||curl^j(u)||_2^2$. We can extend this definition for any non-integer $s > 0$ by defining

$$H^s_{curl} = ((\mathbf{L}^2)^3, H^{\lceil s \rceil}_{curl})_{s/\lceil s \rceil, 2} \tag{2}$$

where $\lceil s \rceil$ is the smallest integer larger than s.

Before moving on to examine these spaces further, we establish some simple properties of the real interpolation.

PROPOSITION 4.1 *Let X, Y, V, W be Banach spaces, $1 \le q < \infty$, and $0 < \theta < 1$. Then*

$$(X \oplus V, Y \oplus W)_{\theta,q} = (X, Y)_{\theta,q} \oplus (V, W)_{\theta,q}$$

with equivalent norms. In case $q = 2$, the same holds for Hilbert spaces.

PROOF. The equality of the sums simply follows from considering

$$\frac{1}{\sqrt{2}}\left(t\,||x||_X + ||y||_Y + t\,||v||_V + ||w||_W\right) \le t\,||x+v||_{X\oplus V} + ||y+w||_{Y\oplus W} \le$$

$$\le t\,||x||_X + ||y||_Y + t\,||v||_V + ||w||_W\,.$$

After taking infimums, multiplying by t^θ, taking q-th powers, and integrating with respect to the measure $\frac{dt}{t}$ on $[0, \infty)$, we get the inequalities required.

In case $q = 2$ it remains to show that for any $z + u \in (X \oplus V, Y \oplus W)_{\theta,q}$, z is perpendicular to u.

$$4 < z, u >= ||z+u||^2_{(X\oplus V, Y\oplus W)_{\theta,q}} - ||z-u||^2_{(X\oplus V, Y\oplus W)_{\theta,q}} =$$

$$= \int_0^\infty t^{-2\theta}\left((K(t, z+u))^2 - (K(t, z-u))^2\right)\frac{dt}{t}.$$

Note that the infimum defining K can be decomposed:

$$K(t, z+u) = \inf\{||t(x+v)||^2_{X\oplus V} + ||y+w||^2_{Y\oplus W}\,|z+u = (x+v)+(y+w)\}^{1/2} =$$

$$= \inf\{||tx||^2_X + ||y||^2_Y\,|z = x+y\}^{1/2} + \inf\{||tv||^2_V + ||w||^2_W\,|u = v+w\}.$$

$K(t, z-u)$ has exactly same form, except that in the second infimum v and w are multiplied by -1. This does not affect the value of the infimum, therefore $K(t, z+u)) - (K(t, z-u)) = 0$ and $< z, u >= 0$. ■

PROPOSITION 4.2 *Let U, V, W be subspaces of a Hilbert space X, and suppose that W is closed, $1 \le q < \infty$, and $0 < \theta < 1$. Then*

$$(U, V)_{\theta,q} \cap W = (U \cap W, V \cap W)_{\theta,q}$$

with equivalent norms.

PROOF. Set $w \in W$. For every decomposition $w = u + v$ we can take $w = (P_{U\cap W}u + P_{V\cap W}v) + (P_{U\cap W^\perp}u + P_{V\cap W^\perp}v)$ instead. The second term is in $W^\perp$ by defnition, and it is also a difference of two elements of W, therefore it is in $W \cap W^\perp$ and thus it is zero. The norm of the components can only decrease with this modification, therefore $K_{U\cap W,V\cap W}(t,w) \leq K_{U,V}(t,w)$, while the other inequality is trivial. The equivalence of the two K functionals then implies the equivalence of the norms. ■

PROPOSITION 4.3 *Let U, V be Banach spaces with U compactly imbedded in V, $1 \leq q < \infty$, and $0 < \theta < 1$. Then $(U,V)_{\theta,q}$ is also compactly imbedded in V.*

PROOF. Let $\{z_n\}$ be a sequence in the unit ball of $(U,V)_{\theta,q}$. Using the discrete version of interpolation and K_1 there are $u_j^n \in U$ and $v_j^n \in V$ such that $u_j^n + v_j^n = z_n$ and

$$\sum_{j=-\infty}^{\infty} \left[2^{-j\theta}(2^j \left|\left|u_j^n\right|\right|_U + \left|\left|v_j^n\right|\right|_V)\right]^q \leq 2 \left|\left|z_n\right|\right|_{(U,V)_{\theta,q}} \leq 2.$$

We have $\left|\left|u_j^n\right|\right|_U \leq 2^{j(\theta-1)+1}$, therefore $\{u_j^n\} \subset V$ has a Cauchy subsequence for all j and with the diagonal argument we get a subsequence $\{z_{m_k}\}$ such that $\{u_j^{m_k}\} \subset V$ is Cauchy for all j. Fix $\epsilon > 0$ and choose j so that $2^{j\theta} < \epsilon$. Then we have $\left|\left|v_j^n\right|\right|_V < 2\epsilon$ and therefore if N is chosen so that for $m_k, m_l > N$ $\left|\left|u_j^{m_k} - u_j^{m_l}\right|\right|_V < \epsilon$, then $\left|\left|z_{m_k} - z_{m_l}\right|\right|_V < 5\epsilon$. ■

This statement is true in a more general way, but this weaker version easily implies a stronger one. Note that from the Schauder theorem it follows that if U is compactly imbedded in V then V^* is compactly imbedded in U^*. In case U and V are reflexive spaces we can apply the previous proposition to the dual spaces. By the Duality Theorem we obtain that U is compactly imbedded in $(U,V)_{\theta,q}$ and by the Reiteration Theorem we can conclude that $(U,V)_{\theta_1,q}$ is compactly imbedded in $(U,V)_{\theta_2,q}$ for $0 < \theta_1 < \theta_2 < 1$.

Let $H^n_{0,curl}$ denote the orthogonal complement of Ker($curl$) in H^n_{curl}. Note that the inner product, and therefore the orthogonality, is the same in every H^n_{curl} if one of the functions is in Ker($curl$). Proposition 4.1 shows that it suffices to interpolate between these spaces. It turns out that they are subspaces of the traditional Sobolev spaces.

PROPOSITION 4.4
$$(H^n)^3 \cap Ker(curl)^\perp = H^n_{0,curl}$$

with equivalent norms.

PROOF. Let $u \in \mathrm{Ker}(curl)^\perp$. For the Fourier transform of such a function $\xi \perp \hat{u}(\xi)$ holds almost everywhere, therefore

$$||u||_{H^n_{0,curl}} \sim \sum_{j=0}^{n} \left|\left|curl^j(u)\right|\right|_2 = \sum_{j=0}^{n} \left|\left|\xi \to \xi \times^j \hat{u}(\xi)\right|\right|_2 =$$

$$= \sum_{j=0}^{n} \left|\left|\xi \to |\xi \times^j \hat{u}(\xi)|\right|\right|_2 = \sum_{j=0}^{n} \left|\left|\xi \to |\xi|^j |\hat{u}(\xi)|\right|\right|_2 \sim \left|\left|\xi \to (1+|\xi|^2)^{n/2} |\hat{u}(\xi)|\right|\right|_2 =$$

$$= \left|\left|\xi \to (1+|\xi|^2)^{n/2} \hat{u}(\xi)\right|\right|_2 \sim ||u||_{(H^n)^3}.$$

■

Using the results from the interpolation of Sobolev spaces we can write the following

COROLLARY 4.5

$$H^s_{curl} = Ker(curl) \oplus ((B^{s,2,2})^3 \cap Ker(curl)^\perp) \tag{3}$$

as Hilbert spaces, where by Proposition 4.3 the second component is compactly imbedded in $(\mathbf{L}^2)^3$.

This is a slightly generalized form of the Helmholtz decomposition. The consequence of this decomposition is that these spaces "behave well" under interpolation, i.e.

$$(H^{s_1}_{curl}, H^{s_2}_{curl})_{\theta,2} = H^s_{curl}$$

for any $0 \le s_1 < s < s_2$ with $(1-\theta)s_1 + \theta s_2 = s$.

4.2. The scale $H^{\cdot s}_{curl}$

There is another common way to extend the definition of H_{curl} to integer indices, found in e.g [4]. In this case we are not interested in the higher order *curl*s of the functions, but instead the smoothness of the function and the *curl* of the function is simultaneously prescribed. Define H^{n}_{curl} by

$$H^{n}_{curl} = \{u \in (H^n)^3 | curl(u) \in (H^n)^3\}.$$

These are also Hilbert spaces with the norm $||u||^2_{H^{n}_{curl}} = ||u||^2_{(H^n)^3} + ||curl(u)||^2_{(H^n)^3}$. It is possible to derive a similar decomposition to (3) for these spaces. Obviously, if $u \in \mathrm{Ker}(curl)$ then $u \in H^{n}_{curl}$ if and only if $u \in (H^n)^3$.

PROPOSITION 4.6 *Let Ker(curl)$^\perp$ denote the orthogonal complement of Ker(curl) in* $(\mathbf{L}^2)^3$. *Then*

$$H^{n}_{curl} \cap Ker(curl)^\perp = (H^{n+1})^3 \cap Ker(curl)^\perp$$

with equivalent norms.

PROOF. Similarly to the proof of Proposition 4.4, consider a function $u \in \mathrm{Ker}(curl)^\perp$ for which therefore $\xi \perp \hat{u}(\xi)$ holds. Using again the equivalent Sobolev norm and the unitarity of the Fourier transform we can write

$$||u||_{H^{n}_{curl}} \sim \big|\big|\xi \to (1+|\xi|^2)^{n/2}|\hat{u}(\xi)|\big|\big|_2 + \big|\big|\xi \to (1+|\xi|^2)^{n/2}|\xi \times \hat{u}(\xi)|\big|\big|_2 =$$

$$= \big|\big|\xi \to (1+|\xi|^2)^{n/2}|\hat{u}(\xi)|\big|\big|_2 + \big|\big|\xi \to |\xi|(1+|\xi|^2)^{n/2}|\hat{u}(\xi)|\big|\big|_2 \sim$$

$$\sim \big|\big|\xi \to (1+|\xi|^2)^{(n+1)/2}|\hat{u}(\xi)|\big|\big|_2 \sim ||u||_{(H^{n+1})^3}.$$

■

Observe that if $u \in \mathrm{Ker}(curl) \cap (H^n)^3$ and $v \in \mathrm{Ker}(curl)^\perp \cap (H^n)^3$, then they are orthogonal with respect to the $(H^n)^3$ inner product as well. Indeed, taking partial derivatives from the functions does not change the direction of the Fourier transform of the function, and thus $\mathcal{F}(\partial^\alpha u)$ points in the longitudinal direction, and $\mathcal{F}(\partial^\alpha v)$ points perpendicular to the longitudinal direction. Therefore $< \mathcal{F}(\partial^\alpha u), \mathcal{F}(\partial^\alpha v) >= 0$ pointwise. We thus get the decomposition

$$H^{n}_{curl} = ((H^n)^3 \cap \mathrm{Ker}(curl)) \oplus ((H^{n+1})^3 \cap \mathrm{Ker}(curl)^\perp)$$

as Hilbert spaces and we can again conclude that

$$(H^{\cdot m}_{curl}, H^{\cdot n}_{curl})_{\theta,2} = H^{\cdot s}_{curl}$$

for any $0 \le m < s < n$ integers with $(1-\theta)m+\theta n = s$. We can also extend this scale to any non-integer s by using the above interpolation equation as a definition. By the same interpolation properties as we used in (3), we then obtain the decomposition, now for all $s \ge 0$,

$$H^{\cdot s}_{curl} = ((B^{s,2,2})^3 \cap \mathrm{Ker}(curl)) \oplus ((B^{s+1,2,2})^3 \cap \mathrm{Ker}(curl)^{\perp}).$$

4.3. Fractionalization of the *curl* operator

Similarly to the fractional versions of the Laplacian operator, it is possible to fractionalize the *curl* as well through fractionalizing the appropriate operator acting on the Fourier transform. A natural expectation for these operators to form a semi-group, and also we wish that for integer indices we get back the $curl^n$ operator defined earlier. Let $u \in \mathrm{Ker}(curl)^{\perp} \cap \mathcal{S}$ and examine how the *curl* operator changes $\hat{u}$ at a point ξ. Both $\hat{u}(\xi)$ and $\xi \times \hat{u}(\xi)/i$ falls in the subspace $\xi^{\perp}$. In fact $\xi \times \hat{u}(\xi)/i = A\hat{u}(\xi)$, where

$$A = A_{\xi} = |\xi| R_{\pi/2} = |\xi| \begin{pmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{pmatrix}$$

is a linear operator acting on $\xi^{\perp}$. Similarly, $\xi \times^n \hat{u}(\xi)/i^n = A^n \hat{u}(\xi)$. The s-th power of A can be easily defined by $A^s = |\xi|^s R_{s\pi/2}$ for any $s \ge 0$ and this leads to the definition

$$curl^s(u) = \mathcal{F}^{-1}(\xi \to i^s A^s_{\xi} \hat{u}(\xi)).$$

This definition can be extended to any $u \in \mathrm{Ker}(curl)^{\perp} \cap (\mathbf{L}^2)^3$: since $\hat{u}$ is locally integrable, so is $\xi \to i^s A^s_{\xi}\hat{u}(\xi)$ and therefore we can take its inverse Fourier transform. In fact this definition extends to any tempered distribution having locally integrable Fourier-transform. It is easy to see that $curl^s(curl^t(u)) = curl^{s+t}(u)$ and we defined $curl^n$ so that coincides with the n-th power of the *curl* operator if n is an integer.

If $u \in \mathrm{Ker}(curl) \cap (\mathbf{L}^2)^3$ then we can simply take $curl^s(u) = 0$ for $s > 0$. Then we can extend $curl^s$ to $(\mathbf{L}^2)^3$ linearly. It is clear that the properties $curl^s(curl^t(u)) =$

$curl^{s+t}(u)$ and $curl^n = (curl)^n$ still hold. This extension therefore satisfies our expectations and also, it is closely related to the scale of spaces H^s_{curl}.

PROPOSITION 4.7 *Let $u \in (\mathbf{L}^2)^3$. Then $u \in H^s_{curl}$ if and only if $curl^s(u) \in (\mathbf{L}^2)^3$.*

This shows that this fractionalization of the *curl* could also be used to give another equivalent definition of H^s_{curl} quite analogously to the traditional Sobolev spaces.

PROOF. Clearly it is enough to show the claim for $u_0 = P_{\text{Ker}(curl)^\perp} u$. By the decomposition (3), $u_0 \in (B^{s,2,2})^3 = (F^{s,2,2})^3$, and we can use the equivalent norm of Proposition 3.17. Therefore,

$$||u_0||_{(\mathbf{L}^2)^3} + ||u_0||_{H^s_{curl}} \sim ||\xi \to \hat{u_0}(\xi)||_2 + \left|\left|\xi \to (1+|\xi|^2)^{s/2}\hat{u_0}(\xi)\right|\right|_2 \sim$$

$$\sim ||\xi \to \hat{u_0}(\xi)||_2 + ||\xi \to ||\xi|^s \hat{u_0}(\xi)|||_2 =$$

$$= ||\xi \to \hat{u_0}(\xi)||_2 + \left|\left|\xi \to |i^s|\xi|^s R_{s\pi/2}\hat{u_0}(\xi)|\right|\right|_2 = ||u_0||_2 + ||curl^s(u_0)||_2 .$$

Note that this calculation shows that, in fact, if we introduce the norm $(||u||_2^2 + ||curl^s(u)||_2^2)^{1/2}$ on H^s_{curl}, we obtain an equivalent Hilbert space. ■

The analogue of the diagram (1) can be written as

$$(H^1)^3 \xrightarrow{grad} H^s_{curl} \xrightarrow{curl^s} H_{div} \xrightarrow{div} (\mathbf{L}^2)^3$$

for any $s > 0$, and it preserves the property that the image of each operator is included in the kernel of the next one. Indeed, the kernel of $curl^s$ is the same for any $s > 0$, namely it consists of the functions $u \in (\mathbf{L}^2)^3$ whose Fourier transform points in the longitudinal direction almost everywhere. On the other hand, the kernel of *div* consists of the functions $u \in (\mathbf{L}^2)^3$ whose Fourier transform points in perpendicular to longitudinal direction. By definition, all functions of form $curl^s u$ has this property. However, using the fractional *curl* it is easy to show that the inclusion $\text{Im}(curl^s) \subset \text{Ker}(div)$ is not of finite codimension.

PROPOSITION 4.8 *Set $0 < T \leq \infty$ and let the family of linear operators $\{C^s\}_{s\in(0,T)}$ acting on the spaces $\{V_s\}_{s\in(0,T)}$ such that C^sC^t is defined and equals to C^{s+t} for all choice of $s, t > 0$ with $s + t < T$. Suppose furthermore that $Ker(C^s) = Ker(C^t)$ for* *all* $s, t > 0$. *Then* $Im(C^t) \subset Im(C^s)$ *for all* $0 < s < t < T$ *and the inclusions are either strict for every choice of* s *and* t*, or are trivial for every choice.*

Proof. The first part is trivial from $C^t u = C^s(C^{t-s}u)$. For the second part, it is enough to check the actions of C^s on $V_s/\mathrm{Ker}(C^s)$, thus we can assume that the operators are injectives. Suppose that the inclusion $\mathrm{Im}(C^t) \subset \mathrm{Im}(C^s)$ is srtict for a fixed $0 < s < t$. For any $r > q > t$ fix m such that $(t-s)/m \leq r-q$. At least for one $j \in \{1, 2, \ldots, m\}$ one of the inclusions $\mathrm{Im}(C^{s+j(t-s)/m}) \subset \mathrm{Im}(C^{s+(j-1)(t-s)/m})$ is strict, that is, there is a v such that $v = C^{s+(j-1)(t-s)/m}u$, but there is no such w that $v = C^{s+j(t-s)/m}w$. Consequently there is no such w that $v = C^{s+(j-1)(t-s)/m+(r-q)}w$ either. Thus $v' = C^{q-(s+(j-1)(t-s)/m)}v$ equals to $C^q u$, but doesnt equal to $C^r w$ for any w, and thus $\mathrm{Im}(C^r) \subset \mathrm{Im}(C^q)$. We can conclude that there exists an s_0 such that the inclusion of images is trivial if both indices are smaller than s_0, and strict if both indices are larger than s_0. On the other hand, if $\mathrm{Im}(C^s) = \mathrm{Im}(C^{2s}) = \mathrm{Im}(C^s C^s)$ for a fixed s, then C^s maps $\mathrm{Im}(C^s)$ onto itself, therefore $\mathrm{Im}(C^{ms}) = \mathrm{Im}(C^{ns})$ for any n, m integers. Hence s_0 can be only 0 or T, which proves the statement. ■

It now suffices to find an $u \in \mathrm{Im}(curl) \backslash \mathrm{Im}(curl^2)$, and then the Proposition yields that all the intermediate inclusions in

$$\mathrm{Im}(curl^s) \subset \mathrm{Im}(curl^{s/2}) \subset \cdots \subset \mathrm{Im}(curl^{s/2^n}) \subset \cdots \subset \mathrm{Ker}(div)$$

are strict, therefore the codimension of $\mathrm{Im}(curl^s)$ in $\mathrm{Ker}(div)$ is infinite. Consider the function $v = \mathcal{F}^{-1}(\xi \to \chi_{|\xi|<1} f(\xi)/|\xi|)$ where $|f(\xi)| = 1$ and its direction is chosen in a measurable way such that it points perpendicular to the longitudinal direction. The function we are looking for is $u = curl(v)$. Clearly u and v are in $(\mathbf{L}^2)^3$, so $u \in \mathrm{Im}(curl)$. However, $u \notin \mathrm{Im}(curl^2)$, or equivalently, $v \notin \mathrm{Im}(curl)$. Suppose $v = curl(w)$, and consider $w' = P_{\mathrm{Ker}(curl)^\perp} w$. We know that $|\xi \times \hat{w'}(\xi)| = |\xi||\hat{w'}(\xi)| = \chi_{|\xi|<1}/|\xi|$, and therefore $|\hat{w'}(\xi)| = \chi_{|\xi|<1}/|\xi|^2$, so $\hat{w'}$ is not in $(\mathbf{L}^2)^3$. Consequently, neither is w', or, by $||w||_2 \geq ||P_{\mathrm{Ker}(curl)^\perp} w||_2$, neither is w.

Our definition of the fractional *curl*, just like usually the fractional derivatives, is not local. Therefore there is no guarantee that the fractional *curl* of a compactly supported function will also have the same support. If we wish to fractionalize the operator $curl : H_{curl}(\Omega) \to \mathbf{L}^2(\Omega)$ in the case of for example a simply connected bounded Lipschitz-domain, then by the properties mentioned at the beginning of the chapter and the arguments above, the image of $curl^s$ would have to be the same for all $s > 0$. Therefore if a counterexample like in the preceeding is found with compact support, then it proves that such fractionalization can not exist.

As an interesting application, it is described in [3] that with the fractionalization of the *curl* it is possible to generalize the principle of duality in electromagnetics. The source-free Maxwell equations in vacuum for the time-harmonic case with the time dependence $e^{-i\omega t}$ can be written in the form

$$\frac{1}{ik_0} curl(\eta_0 \mathbf{H}) = -\mathbf{E}$$

$$\frac{1}{ik_0} curl(\mathbf{E}) = \eta_0 \mathbf{H}$$

$$div(\eta_0 \mathbf{H}) = 0$$

$$div(\mathbf{E}) = 0.$$

Using the semigroup-property and $div(curl^s) \equiv 0$ we get that if $\mathbf{E}$ and $\mathbf{H}$ are solutions, then so are their fractional duals

$$\mathbf{E}_{fd} = c \cdot curl^s(\mathbf{E}),\ \eta_0 \mathbf{H}_{fd} = c \cdot curl^s(\eta_0 \mathbf{H})$$

for any $s > 0$ and c constant. If we choose $c = 1/(ik_0)^s$, then for the special case $s = 1$ we obtain $\mathbf{E}_{fd} = \eta_0 \mathbf{H}$ and $\eta_0 \mathbf{H}_{fd} = -\mathbf{E}$, the dual fields of the original solution.

4.4. Non-positive indices

The method in the preceeding for the fractionalization of the vectorial product, and therefore the *curl* in fact works for negative, or even complex s. Since in this case ensuring the semigroup property requires $curl^0 = curl^s curl^{-s}$, we have to define $curl^0$ to be the projection to $\mathrm{Ker}(curl)^\perp$ instead of the more natural choice of the identity operator.

By the virtue of Proposition 4.7, we could define H^s_{curl} to be the space of functions $u \in \mathbf{L}^2$ such that $curl^s \in \mathbf{L}^2$. Observe that for any s, the eigenvalues of $R_{s\pi/2}$ are nonzeros and are same for all ξ, that is, $|R_{s\pi/2} x| \sim |x|$. Thus, if $\Re(s_1) = \Re(s_2) = \sigma$, then

$$|i^{s_1}A_\xi^{s_1}\hat{u}(\xi)| = |\xi^{s_1}||R_{s_1\pi/2}\hat{u}(\xi)| = |\xi|^\sigma|R_{s_1\pi/2}\hat{u}(\xi)| \sim |\xi|^\sigma|R_{s_2\pi/2}\hat{u}(\xi)| = |i^{s_2}A_\xi^{s_2}\hat{u}(\xi)|.$$

It follows that $H^{s_1}_{curl} = H^{s_2}_{curl} = H^\sigma_{curl}$, so the introduction of non-real complex indices does not yield new spaces. On the other hand, even for negative s, since $H^s_{curl} \subset (\mathbf{L}^2)^3$, this definition would provide a scale that clearly lacks the property $(H^s_{curl})' = H^{-s}_{curl}$. Since the analogue property is held by the classical Sobolev spaces, and even the versions of Besov-, and Triebel-Lizorkin spaces for negative indices, we will provide a new definition. First let us determine these dual space similarly to the classical case ([1], Theorem 3.9).

Proposition 4.9 *Set $s > 0$. The dual space of H^s_{curl} is given by*

$$\left\{u \in \mathcal{S}' | u = u_1 + curl^s u_2 | u_1, u_2 \in (\mathbf{L}^2)^3\right\}.$$

Proof. Let us define the operator $curl^s_*$ in exactly the same way as $curl^s$, with the only exception being using the transpose of the matrix $R_{s\pi/2}$. For now the only interesting property is that

$$||curl^s_*(u)||_2 = ||curl^s(u)||_2.$$

This follows from that since $R_{s\pi/2}$ is a rotation, $|R_{s\pi/2}\xi|^2 = |R^T_{s\pi/2}\xi|^2 = |\xi|^2$.

Now consider the space $V = Q_1 \cup Q_2$ consisting of two distinct copies of $\mathbb{R}^n$ and let P be the mapping from H^s_{curl} to $\mathbf{L}^2(V)$ such that $Pu|_{Q_1} = u$ and $Pu|_{Q_2} = curl^s_*(u)$. By the equivalent norm in Proposition 4.7, P is an isometric isomorphism between its domain and its range W. For any L a linear functional on H^s_{curl} we can define L^* on W by $L^*(Pu) = Lu$. By the Hahn-Banach theorem L^* can be extended in a norm preserving way to a linear functional on $\mathbf{L}^2(V)$. Denoting this extension by L', we can use the Riesz Representation theorem for L'. That is, there exists a $v \in \mathbf{L}^2(V)$ such that $L'z =< z, v >$ for all $z \in \mathbf{L}^2(V)$, in particular

$$Lu = L'(Pu) =< u, \overline{v_1} > + < curl^s_*(u), \overline{v_2} > \tag{4}$$

for all $u \in H^s_{curl}$ where the latter inner products are understood in the $(\mathbf{L}^2(\mathbb{R}^n))^3$ sense, and $v_j = \overline{v}|_{Q_j}$ for $j = 1, 2$. Let us use the notation T_f for the regular distribution for any given locally integrable function f. That is, $T_f(\phi) = \int \sum_{j=1}^{3} f_j \phi_j$. Consider the distribution

$$T = T_{v_1} + curl^s(T_{v_2}). \tag{5}$$

We now show that T is extended to H^s_{curl} by L. Taking $\phi \in \mathcal{S}$, we clearly have $T_{v_1}(\phi) =< \phi, \overline{v_1} >$, and

$$curl^s(T_{v_2})(\phi) = \mathcal{F}(curl^s(T_{v_2})(\mathcal{F}^{-1}(\phi)) = \int i^s|\xi|^s \sum_{j=1}^{3} (R_{s\pi/2}\hat{v}_2)_j(\xi)\hat{\phi}_j(-\xi) =$$

$$= \int i^s|\xi|^s \hat{v}_2(\xi)^T R^T_{s\pi/2}\hat{\phi}(-\xi) = \int i^s|\xi|^s \sum_{j=1}^{3} (\hat{v}_2)_j(\xi)(R^T_{s\pi/2}\hat{\phi})_j(-\xi) =$$

$$= \mathcal{F}(v_2)(\mathcal{F}^{-1}(curl^s_*(\phi)) =< curl^s_*(\phi), \overline{v_2} > .$$

Comparing this with (4), we get $T(\phi) = L(\phi)$, so L indeed extends T.

On the other hand, if T is of the form (5), we need to show that it has a unique extension to H^s_{curl}. Since $\mathcal{S}$ is dense in $(\mathbf{L}^2)^3$, which is dense in H^s_{curl}, for any fixed $u \in H^s_{curl}$ we can find a sequence $\{\phi_n\}_{n=1}^{\infty} \subset \mathcal{S}$ converging to it. We can write

$$|T(\phi_k) - T(\phi_l)| \leq | < \phi_k - \phi_l, \overline{v_1} > | + | < curl^s_*(\phi_k - \phi_l), \overline{v_2} > | \leq$$

$$\leq ||\phi_k - \phi_l||_2 \, ||v_1||_2 + ||curl^s(\phi_k - \phi_l)||_2 \, ||v_2||_2 \leq ||\phi_k - \phi_l||_{H^s_{curl}} \, ||v||_{\mathbf{L}^2(V)} .$$

Thus $\{T(\phi_n)\}$ is a Cauchy sequence in $\mathbb{C}$ and so converges to a limit that we can denote by $L(u)$ since it is clear that we obtain the same limit to any other sequence $\{\psi_n\} \subset \mathcal{S}$ converging to u. The functional L is linear and also bounded, since

$$|L(u)| = \lim_{n\to\infty} |T(\phi_n)| \leq \lim_{n\to\infty} ||\phi_n||_{H^s_{curl}} \, ||v||_{\mathbf{L}^2(V)} = ||u||_{H^s_{curl}} \, ||v||_{\mathbf{L}^2(V)} .$$

■

We can now extend the scale H^s_{curl} to negative indices in a natural way, while also ensuring the expectations for duality. Let $\mathcal{A}$ denote the subspace of $\mathcal{S}'$ with distributions whose Fourier transform is locally integrable and its longitudinal component is in $(\mathbf{L}^2)^3$. For any $s \in \mathbb{R}$ set $A_s = \mathcal{A} \cap \{u \in \mathcal{S}' | curl^s(u) \in (\mathbf{L}^2)^3\}$.

PROPOSITION 4.10 *The scale*

$$H^s_{curl} = \begin{cases} (\mathbf{L}^2)^3 \cap A_s & \textit{if } s \geq 0 \\ (\mathbf{L}^2)^3 + A_s & \textit{if } s \leq 0 \end{cases}$$

coincides with the initial definition (2) for $s \geq 0$ *and has the property* $(H^s_{curl})' = H^{-s}_{curl}$.

PROOF. The first claim is simply a rephrasing of Proposition 4.7. The only difference is the intersection with $\mathcal{A}$ which does not change our space since $H^s_{curl} \subset (\mathbf{L}^2)^3 \subset \mathcal{A}$ for $s \geq 0$. The second part is Proposition 4.9 for $s \geq 0$, while the case $s \leq 0$ follows from the reflexivity of Hilbert spaces. ■

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