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Stable exchanges

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Contents

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1 Introduction

Given a simple digraph $D = (V, A)$, a set of disjoint directed cycles is called an **exchange**. Every $v \in V$ has a strictly ordered preference list containing the nodes to which there is an arc from v. We say that u gets v in the exchange, if uv is an arc of one of the directed cycles in the exchange. We say that $v \in V$ is **covered** by the exchange E, if v belongs to a cycle of length at least two in E. An exchange is called **stable**, if there is no directed cycle C such that for each arc $e = uv$ of C u is not covered by the exchange or u prefers v over what he got in the exchange. An exchange is called **strongly stable**, if there is no directed cycle C not in the exchange such that for each arc $e = uv$ of C u is not covered by the exchange or e is in the exchange or u prefers v over what he got in the exchange. In both cases the node set of a violating cycle C is called a **blocking coalition**.

An important application of this model is kidney exchange. Currently the best known treatment for kidney failure is transplantation. Since there are a large number of people on the deceased donor waiting list, the more efficient solution is living donation. However, a kidney of a willing living donor is often not suitable for the patient for immunological reasons. Therefore incompatible patient-donor pairs might want to exchange kidneys with other pairs in the same situation. Kidney exchanges have been organized in several countries, for an overview of the different approaches see [11], [3]. In the model described above, the nodes of the digraph correspond to the incompatible patient-donor pairs, $uv \in A$ if and only if the kidney of the donor corresponding to v is suitable for the patient corresponding to u. Each patient has a strict preference order over the kidneys suitable for him. In an exchange the patient-donor pairs exchange kidneys backwards along the cycles.

Shapley and Scarf [14] showed that the stable exchange problem (SE) is always solvable, and a stable exchange can be found by the Top Trading Cycles (TTC) algorithm proposed by Gale. In section 2 we describe this algorithm, we also describe the Top Trading Cycles and Chains algorithm [11], and we introduce a structure by Tan [15] called stable partition, which can be used for finding 2-way stable exchanges.

In case of kidney exchanges the cycles in the exchange should be short, since all operations along a cycle have to be carried out at the same time (to avoid someone backing out). If all the cycles in the exchange have length at most l, we call it an **l-way exchange**. An exchange is called **b-way stable**, if there is no blocking coalition of size at most b. The definition is analogous for strong stability. Biró and McDermid [1] proved that the decision problem of finding a 3-way stable 3-way exchange is NP-complete, and asked whether a polynomial time algorithm exists for the decision problem of finding a 2-way stable 3-way exchange. In section 3 we prove that the problem is NP-complete even in complete digraphs. We also prove that the decision problem of finding a b-way strongly stable l-way exchange is NP-complete for any $b \geq 2,\ l \geq 3$ and the same result holds for b-way stable l-way exchanges and stable l-way exchanges.

An instance might admit more than one stable exchanges, therefore it is a natural goal to maximize the number of covered nodes in the exchange. The complexity of this problem was mentioned as an open problem in [2] as well as the same question for 2-way stable exchanges. In section 4 we show that deciding if an instance admits a complete stable exchange is NP-complete and the same holds for b-way stable exchanges for any $b \geq 2$. Roth and Postlewaite [12] proved that the exchange found by the TTC algorithm is strongly stable and it is the only strongly stable solution. However, there might be more than one b-way strongly stable exchanges. We prove that deciding if an instance admits a complete b-way strongly stable exchange is NP-complete for any $b \geq 2$. We show that if the digraph is symmetric, then TTC is a $\frac{1}{2}$-approximation algorithm, while the stable partition algorithm is a $\frac{2}{3}$-approximation algorithm for maximizing the number of covered nodes in a 2-way stable exchange.

In section 5 we prove that it is NP-hard to maximize the number of 2-cycles in a (2-way) stable exchange.

In section 6 we study the 2-way (strongly) stable pairwise exchange with chains problem, where there are a few special ”altruist” nodes in the digraph, and we only allow 2-cycles and chains ending in altruists in the exchange. We define 3 types of stability concepts, and we show that the problem is NP-complete even if there is only 1 altruist node. The problem is also NP-complete if there is no restriction on the number of altruists, but the length of the chains are at most a given constant. These results hold for all the defined types of stability concepts, except the case where the lengths of the chains are 1, in which case for one type of stability there is a polynomial time algorithm. We show that the problem is solvable in polynomial time if the number of altruists and the length of the chains are both restricted by given constants.

In section 7 we introduce the basic definitions regarding parameterized complexity, and show that the 2-way stable 3-way exchange and the 2-way stable pairwise exchange with chains problems are W[1]-hard.

All the NP-hardness reductions are from the k-clique in k-partite graph problem, which is specified as follows:
Instance: An integer k, and a k-partite graph $G = (V_1 \cup V_2 \cup ... \cup V_k, E)$.
Question: Is there a clique of size k in G?

The NP-completeness and W[1]-completeness of this problem was proved in [6].

1.1 Related work

An instance of the stable marriage problem (SM) consists of n men and n women. Each person has a strictly ordered preference list containing all members of the opposite sex. The problem is to find a matching which is stable in a sense that there is no blocking pair, i.e. a man and a woman who prefer each other over their partners in the matching. The Gale-Shapely algorithm [7] always finds a stable matching in an instance of SM.

In the stable roommates problem (SR) there are $2n$ persons, each of whom ranks all the others in strict order of preference. The goal is to find a complete stable matching. Gale and Shapely [7] gave an instance of SR for which no stable matching is possible. Irving [8] proposed an $O(n^2)$ time algorithm which finds a complete stable matching if there is one, or reports that none exists.

The stable roommates with incomplete lists problem (SRI) is a generalization of SR, where each person's preference list only contains his acceptable partners. The problem can be represented by a graph, where there is an edge between two persons if and only if they are acceptable to each other. Here the number of people is not necessarily even, and the stable matching does not need to be complete. However, the same persons are matched in every stable matching and Irving's algorithm can be extended to SRI [9].

The stable exchange problem and the definition of b-way stable l-way exchanges have already been described above. We may assume that if $uv \in A$ in an instance of the 2-way stable 2-way exchange problem, then $vu \in A$, since otherwise uv does not belong to any 2-cycle or blocking coalition. A digraph satisfying this property is called a **symmetric digraph**. We call two arcs in opposite directions between the same two nodes a **bidirected edge**. The 2-way stable 2-way exchange problem is equivalent to SRI hence solvable in polynomial time.(We can replace the bidirected edges with edges and vice versa.) Irving [10] proved that it is NP-complete to decide if an instance admits a stable 2-way exchange, and the same holds for 3-way stable 2-way exchanges.

Another generalization of SM is the so-called 3-dimensional stable marriage problem (3DSM). Here there are three sets: men, women and dogs. The sets have cardinality n. Each man has a strict preference order over all the woman-dog pairs. The preference lists of the women and dogs are defined analogously. A matching is a set of n disjoint families, that is triples of the form (man, woman, dog). A matching is stable if there is no blocking family, i.e. a family such that all of its members prefer this family over their current family in the matching. Ng and Hirschberg [13] proved that the problem of deciding whether a stable matching exists is NP-complete. They mentioned the cyclic 3DSM as an open problem, where men only care about women, woman only care about dogs and dogs only care about men. In case of strong stability the cyclic 3DSM problem is NP-complete [1]. If the preference lists may be incomplete we refer to the problem as cyclic 3DSMI. Here the cardinality of the sets are not necessarily equal, and the matching does not need to cover everyone. Biró and McDermid [1] showed that it is NP-complete to decide if an instance of cyclic 3DSMI admits a stable matching. Cyclic 3DSMI is equivalent to the 3-way stable 3-way exchange problem in tripartite graphs, therefore the NP-completeness result applies to this problem as well.

2 Classic algorithms

2.1 Top Trading Cycles

We are given a digraph $D = (V, A)$. Every $v \in V$ has a strictly ordered preference list containing the nodes to which there is an arc from v. The Top Trading Cycles algorithm [14] described below always finds a strongly stable exchange.

First let $X = V$. We create a new digraph by taking the nodes of X, and from each node drawing an arc to its most preferred node in X (if he has one). Case 1: the resulting digraph does not contain a directed cycle. Then there is at least one node whose out-degree is 0. We delete these nodes from X. Case 2: the resulting digraph contains at least one directed cycle. Then we delete the nodes of one of these cycles from X. This cycle will belong to the exchange. We continue this procedure, until X becomes empty.

Theorem 2.1. *[12] The exchange found by TTC is strongly stable.*

Proof. Let the cycles of the exchange found by TTC be $C_1, C_2, ..., C_k$ in the order we added them to the exchange. Suppose a cycle C forms a blocking coalition. Suppose that i is the smallest subscript for which $C \cap C_i \neq \emptyset$. Let X be the set of nodes in the algorithm right before we deleted the nodes of C_i from it. Then X contains every node from C. However, the nodes of C_i get their most preferred node in X, therefore C has to contain C_i which is a contradiction. □

Remark 2.2. *The exchange found by TTC is strongly stable, therefore it is also stable and b-way (strongly) stable for any* $b \geq 2$.

2.2 Top Trading Cycles and Chains

We are given a set of nodes V, and a special node w. Each node in V corresponds to a patient-donor pair, and w corresponds to the deceased donor waiting list. Every $v \in V$ has a strictly ordered preference list containing v, the nodes $u \in V$ such that the kidney of the donor corresponding to u is suitable for the patient corresponding to v, and it may contain w. (If w is not on v's preference list, it means that the waiting list option is not acceptable to v.) On v's preference list v is the last item for every $v \in V$. The Top Trading Cycles and Chains (TTCC) algorithm [11] described below finds an exchange consisting of disjoint directed cycles and w-chains (chains ending in w). The chains only intersect in w, and they are disjoint from the cycles. Let $u, v \in V$. Suppose uv is an arc of the exchange. If $v = u$, then u will not participate in the exchange. If $v \neq u$, then the patient corresponding to u will receive the kidney of the donor corresponding to v. If uw is an arc in the exchange, the patient corresponding to u receives high priority for the next compatible kidney on the deceased donor waiting list. If u is the tail of a w-chain, the donor corresponding to u will donate his kidney to the deceased donor waiting list.

During the TTCC algorithm, we will add directed cycles and w-chains to the exchange. At each step of the TTCC algorithm, some nodes from V will be active, and the others will be passive. When we add a w-chain to the exchange, the nodes of the chain aside from w will become passive, and the assignment along the chain will be fixed, however, the kidney at the tail of the chain will remain available. The set of active nodes is denoted by A and the set of passive nodes is denoted by P. At first $A = V$, $P = \emptyset$ and $X := \emptyset$. We create a digraph by taking the nodes of A, and from each node drawing an arc to its most preferred node in $A \cup X \cup \{w\}$. Each node in P points to his fixed assigned node. Case 1: The resulting digraph contains a directed cycle. The directed cycles contained in the digraph are disjoint. We delete the nodes of the directed cycles from A, and add the deleted cycles to the exchange. Case 2: The resulting digraph does not contain a directed cycle. Then there is a chain from every node to w in the resulting digraph. We select a w-chain with a specified chain selection rule (e.g. if the nodes are prioritized in a single list, choose the w-chain starting with the highest priority node). We add the selected w-chain to the exchange and fix the assignment along the chain. We delete the nodes of the chain from A and add them to P (except for w). If the selected w-chain contained a node from X, we delete it from X, and we add the tail of the selected w-chain to X. We repeat this procedure until A becomes empty.

Theorem 2.3. *[11] The TTCC algorithm implemented with the above described chain selection rule is strategy proof, and it finds a Pareto efficient solution.*

2.3 Stable partition

In an instance of SRI, we are given a graph $G = (V, E)$, and each node has strict preferences over its neighbours. Now we are interested in finding a complete stable matching. We have already mentioned is section 1.1 that a modified version of Irving's algorithm finds a (not necessarily complete) stable matching if there is one, or reports that none exists. The covered nodes are the same in any stable matching, therefore this algorithm can also decide whether an instance of SRI admits a complete stable matching. However, if the instance does not admit a complete stable matching, Irving's algorithm does not provide a simple evidence for why not. Tan gave a necessary and sufficient condition for the existence of a complete stable matching which we will describe later on. He defined a new structure called **stable partition**, and proved that an instance of the stable roommates problem always admits one.

Definition 2.4. *For $A \subseteq V$ a cyclic permutation $\pi(A) =< a_1, a_2, ...a_k >$ of the nodes in A is called a **semi-party permutation** if*

$|A| = 1$, or

$|A| = 2$ and $a_1a_2 \in E$, or

$|A| \geq 3$ and $a_ia_{i+1} \in E$ and a_i prefers a_{i+1} over a_{i-1} for $i = 1, ..., k$, subscripts modulo k.

Definition 2.5. *A stable partition π consists of a partition of V, and a specified semi-party permutation for each set in the partition, such that the following stability condition holds:*

Let A and B be two (not necessarily distinct) sets of the partition with specified semi-party permutations $\pi(A) =< a_1, ... >$ and $\pi(B) =< b_1, ... >$. Let $a_i \in A$, $b_j \in B$. If $|A| = 1$ or a_i prefers b_j over a_{i-1}, then, (unless $A = B$ and $|A| = 1$), $|B| \neq 1$ and b_j prefers b_{j-1} over a_i (if $|B| \geq 3$, $b_{j-1} = a_i$ is also possible).

*A set A of the partition is called a **party** in π, and the associated semi-party permutation $\pi(A)$ is called a **party permutation** for A.*

Definition 2.6. *A subset A of V is called a **party**, if there exists a stable partition π, such that A is a party in π. A party with odd cardinality is called an **odd party**.*

Another modified version of Irving's algorithm proposed by Tan always finds a stable partition in an instance of SRI. Moreover, Tan proved that the odd parties and the corresponding party permutations are the same in any two stable partitions. This provides a necessary and sufficient condition for the existence of a complete stable matching, namely the nonexistence of odd parties. These results can be found in [15].

Let π be a stable partition in G, with party permutations $\pi(A_i) =< a_1, ...a_{k_i} >$. If we replace the edges of G with bidirected edges, in the arising digraph the exchange defined by the directed cycles $(a_1, ..., a_{k_i})$ is 2-way stable. (This is straightforward from the definitions.) Therefore the above mentioned algorithm for finding a stable partition also gives an algorithm for finding a 2-way stable exchange, which will be later referred to as stable partition algorithm.

3 Stable exchanges with restrictions

First we state a few straightforward observations that can be found in [2].

- If an exchange is strongly stable, than it is also stable.
- An l-way exchange is also an $(l+1)$-way exchange.
- A b-way (strongly) stable (l-way) exchange is also a $(b-1)$-way (strongly) stable (l-way) exchange.

We will prove that the b-way stable l-way exchange problem is NP-complete for any $b \geq 2$, $l \geq 3$. (For $b = l = 3$, this was proved in [1].) For sake of simplicity first we prove the special case where $b = 2$ and $l = 3$.

Theorem 3.1. *The decision problem of finding a 2-way stable 3-way exchange is NP-complete.*

Proof. We reduce from the k-clique in k-partite graph problem. Given an instance $G = (X_1 \cup X_2 \cup ... \cup X_k, E)$ of the k-clique in k-partite graph problem, we create an instance of the 2-way stable 3-way exchange problem. By adding isolated nodes we may assume that $|X_i| = n_i$ is odd and $n_i \geq 5$ for $i = 1, ..., k$.

First we define an undirected graph, which then we turn into a digraph by replacing each edge with a bidirected edge. For every $i = 1, ..., k$ we define a circuit $U_i = (u_{i,1}, u_{i,2}, ..., u_{i,n_i})$, and for each $u_{i,j}$ we add a new edge $u_{i,j}v_{i,j}$. Let $(v_{i,j}, w_{i,j}, z_{i,j})$ be a circuit, with two new nodes, $w_{i,j}$ and $z_{i,j}$. For every $x \in X_i$ there is a distinct corresponding node in $V_i = \{v_{i,1}, ..., v_{i,n_i}\}$. For $x \in X_i$, $y \in X_j$, $i \neq j$, there is an edge between the corresponding nodes if and only if $xy \notin E$.

The preference lists are shown in the following table. Let the neighbours of $v_{i,j}$ in $\bigcup_{l \neq i} V_l$ be denoted by $N_{i,j}$. A set in the preference list means the nodes of the set in arbitrary order.

node	preference list			
$u_{i,j},\ i \in [k],\ j \in [n_i]$	$u_{i,j+1}$	$v_{i,j}$	$u_{i,j-1}$	
$v_{i,j},\ i \in [k],\ j \in [n_i]$	$w_{i,j}$	$N_{i,j}$	$u_{i,j}$	$z_{i,j}$
$w_{i,j},\ i \in [k],\ j \in [n_i]$	$z_{i,j}$	$v_{i,j}$		
$z_{i,j},\ i \in [k],\ j \in [n_i]$	$v_{i,j}$	$w_{i,j}$		

See figure 1. We will prove that the constructed instance admits a 2-way stable 3-way exchange if and only if there is a k-clique in G. First suppose that the constructed instance admits a 2-way stable 3-way exchange.

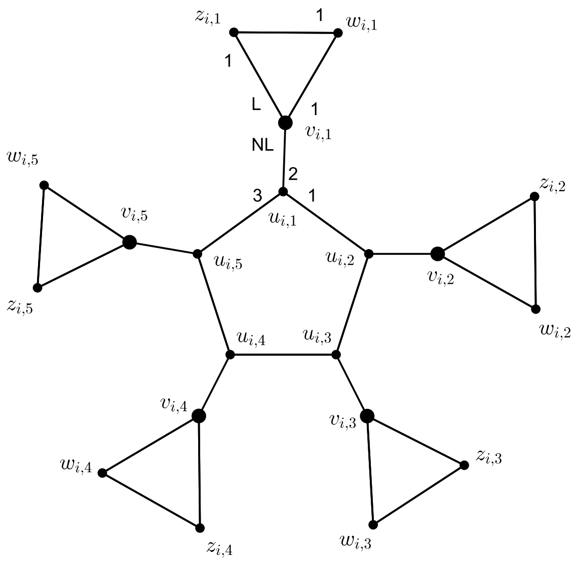

Figure 1: All the edges of the graph are bidirected edges. L:=last, NL:=next to last.

Lemma 3.2. *For every i, if the arcs $u_{i,j-1}u_{i,j}$ and $u_{i,j}u_{i,j+1}$ are not in the exchange, then $u_{i,j}v_{i,j}$ must be in the exchange for every $j = 1, ..., n_i$ (subscripts modulo n_i).*

Proof. Suppose, that $u_{i,j-1}u_{i,j}$ and $u_{i,j}u_{i,j+1}$ are not in the exchange. Notice that $u_{i,j}u_{i,j-1}$ cannot be in the exchange either, because the arcs of U_i are not in any cycle of length 3. $u_{i,j}$ is first on $u_{i,j-1}$'s preference list and $u_{i,j}$ does not get the first item on his list, therefore if $u_{i,j}v_{i,j}$ would not be in the exchange, that is $u_{i,j}$ would not get the second item on his list either, then $\{u_{i,j-1}, u_{i,j}\}$ would be a blocking pair. □

$|U_i| = n_i$ is odd, and it is not possible to have two consecutive arcs of U_i in the exchange (because the arcs of U_i are not in any cycle of length 3). Therefore it follows from the above lemma, that for every i there is a node u_{i,j_i} in U_i, such that $u_{i,j_i}v_{i,j_i}$ is in the exchange. Since this arc is not in any cycle of length 3, $v_{i,j_i}u_{i,j_i}$ has to be in the exchange as well.

Now we prove that the nodes corresponding to v_{i,j_i} for $i = 1, ..., k$ form a k-clique. Suppose there are two which are not connected. Then there is a bidirected edge between the nodes corresponding to them. But then these two nodes form a blocking pair, because they prefer each other to their current partner.

Now suppose there is a k-clique in G. Without loss of generality, we may assume that the nodes corresponding to the nodes of the k-clique are $v_{1,1}, v_{2,1}, ..., v_{k,1}$.

Lemma 3.3. *The 3-way exchange defined by the 2-cycles:* (u_{i,j_i}, u_{i,j_i+1}), $j_i = 2, 4, ..., n_i - 1$, $(u_{i,1}, v_{i,1})$, $(w_{i,1}, z_{i,1})$ *and the 3-cycles:* $(v_{i,j_i}, w_{i,j_i}, z_{i,j_i})$, $j_i = 2, 3, .., n_i$ *is stable (thus 2-way stable).*

Proof. In the 3-cycles everyone gets his first choice, so none of them can belong to a blocking coalition. The same applies to the nodes $u_{i,2}, u_{i,4}, ..., u_{i,n_i-1}$ and $w_{i,1}$. $u_{i,1}$ gets the second item on his list, therefore he would get his first item $u_{i,2}$ in a blocking coalition, but $u_{i,2}$ cannot belong to a blocking coalition. $u_{i,3}, u_{i,5}, ..., u_{i,n_i}$ get their third choice, so in a blocking coalition they would get the first or second item on their list, but we have seen that these nodes cannot belong to a blocking coalition, therefore they cannot either. $z_{i,1}$ gets his second choice therefore he would get his first choice $v_{i,1}$ in a blocking coalition, but $\{z_{i,1}, v_{i,1}\}$ is not a blocking pair, and in a bigger coalition only $w_{i,1}$ could get $z_{i,1}$, but he does not belong to a blocking coalition. Therefore the nodes that could belong to a blocking coalition are only $v_{i,1}$ for $i = 1, ..., k$, but these correspond to the nodes of a k-clique in G, thus they are not even connected. □

□

We used in the proof that the exchange we are looking for only contains cycles of length at most three, when we showed that certain arcs cannot belong to a cycle of length at least three. If we wish to extend this proof to l-way exchanges, where $l > 3$, we need to modify the construction.

Theorem 3.4. *The decision problem of finding a stable l-way exchange is NP-complete for any $l \geq 3$. The same holds for b-way stable l-way exchanges for any $b \geq 2$.*

Proof. For the first part of the theorem, we must show that the problem is in NP. Suppose we are given an l-way exchange. To check that it is stable, first we delete every arc uv such that u does not prefer v over what he got in the exchange. These arcs cannot belong to a blocking coalition. (Note that we deleted all the arcs of the exchange). The exchange was stable if and only if the remaining digraph does not contain a directed cycle.

For the NP-hardness proof, we modify the construction of the previous theorem. Let $t = l-2$ if l is odd, $t = l-1$ if l is even. We replace the bidirected edges $u_{i,j}u_{i,j+1}$ for $i = 1, ..., k,\ j = 1, ..., n_i$ subscripts modulo n_i with a bidirected $u_{i,j}u_{i,j+1}$ path of length t. The new nodes prefer the node succeeding them in the cycle over the one preceding them.

Lemma 3.5. *If there are two consecutive arcs of the cycle* $(u_{i,1}, ..., u_{i,2}, ..., u_{i,n_i-1}, ..., u_{i,n_i})$ *which are not in the exchange, then there is a subscript j for which $(u_{i,j}, v_{i,j})$ is a 2-cycle in the exchange.*

Proof. The proof is very similar to the proof of lemma 3.2. We suppose there are two consecutive arcs not in the exchange. The endpoints of the first arc form a blocking pair, unless the common endpoint of the arcs gets his second choice. Since the length of the cycle $(u_{i,1},...,u_{i,2},...,u_{i,n_i-1},...,u_{i,n_i})$ is odd and the nodes of the cycle are not in any cycle of length at most l and at least 3, there must be two consecutive arcs of the cycle not in the exchange, hence $u_{i,j}v_{i,j}$ is in the exchange for some j. This arc does not belong to a cycle of length at most l and at least 3, therefore $(u_{i,j},v_{i,j})$ is a 2-cycle in the exchange. □

From the above lemma, there is a node v_{i,j_i} for every i, who gets his next to last item in the exchange. The nodes corresponding to these form a k-clique in G, from the same proof as in theorem 3.1.

Now we prove that if there is a k-clique in G, the instance of SE admits a stable 3-way (and thus l-way) exchange. Without loss of generality, we may assume, that the nodes corresponding to the nodes of the k-clique are $v_{1,1},v_{2,1},...,v_{k,1}$. We modify the 3-way exchange defined in lemma 3.3, such that in the cycle $(u_{i,1},...,u_{i,2},...,u_{i,n_i-1},...,u_{i,n_i})$ every second arc from $u_{i,2}$ should belong to a 2-cycle. This 3-way exchange is still stable. (The proof is very similar to the proof of lemma 3.3.) □

Remark 3.6. *This construction does not work for b-way strongly stable l-way exchanges, because the cycles $(v_{i,1},w_{i,1},z_{i,1})$ are blocking in the strongly stable sense.*

Theorem 3.7. *The decision problem of finding a b-way strongly stable l-way exchange is NP-complete for any $b\geq 2$, $l\geq 3$.*

Proof. Given an instance of the k-clique in k-partite graph problem, we create an instance of the b-way strongly stable l-way exchange problem by modifying the construction of theorem 3.1. First we make the modifications we made in the proof of theorem 3.4 but with t being the smallest odd number at least $\max\{b-2,l-2\}$. Let $b'=b-1$ if b is odd, and $b'=b$ if b is even. We add new bidirected paths of length $b'+1$: $w_{i,j}y_{i,j}^1...,y_{i,j}^{b'}z_{i,j}$ for $i=1,...,k,\ j=1,...,n_i$. The new nodes prefer the node succeeding them over the one preceding them. We modify $w_{i,j}$ and $z_{i,j}$'s preference list:

$$w_{i,j}:\ y_{i,j}^1,\ z_{i,j},\ v_{i,j},$$

$$z_{i,j}:\ v_{i,j},\ y_{i,j}^{b'},\ w_{i,j}$$

.

We prove that this instance of SE admits a b-way strongly stable l-way exchange if and only if there is a k-clique in G. Suppose the instance admits a b-way strongly stable l-way exchange. Then it is also a b-way stable l-way exchange, and lemma

3.5 clearly holds in this case too. Just like before, it follows from the lemma that there is a k-clique in G.

Now suppose there is a k-clique in G. Without loss of generality, we may assume that the nodes corresponding to the nodes of the k-clique are $v_{1,1}, v_{2,1}, ..., v_{k,1}$. We prove that the 3-way (and thus l-way) exchange defined by the following 2-cycles and 3-cycles is b-way strongly stable.

$(u_{i,1}, v_{i,1})$,
the 2-cycles defined by every second arc of the cycle $(u_{i,1}, ..., u_{i,2}, ..., u_{i,n_i-1}, ..., u_{i,n_i})$ starting from $u_{i,2}$,
$(w_{i,1}, y^1_{i,1})$, $(y^2_{i,1}, y^3_{i,1})$ $(y^4_{i,1}, y^5_{i,1}), ..., (y^{b'-2}_{i,1}, y^{b'-1}_{i,1})$, $(y^{b'}_{i,1}, z_{i,1})$,
$(y^1_{i,j}, y^2_{i,j}), (y^3_{i,j}, y^4_{i,j}), ..., (y^{b'-1}_{i,j}, y^{b'}_{i,j})$,
$(v_{i,j}, w_{i,j}, z_{i,j})$ $i = 1, ..., k,\ j = 2, ..., n_i$.

Those nodes of the new paths added to the construction who got their first choice in the exchange (namely $y^m_{i,j}$ for $i = 1, ..., k,\ j = 2, ..., n_i, m = 1, 3, ..., b' - 1$ and $y^m_{i,1}$ for $i = 1, ..., k,\ m = 2, 4, ..., b'$), cannot belong to a blocking coalition of size at most b, because they are in a 2-cycle with their first choice in the exchange and they do not belong to any longer cycle of length at most b. For $j \geq 2$, $z_{i,j}$ gets his first choice, $v_{i,j}$, therefore he must get it in every blocking coalition he might belong to. $(z_{i,j}, v_{i,j})$ and $(z_{i,j}, v_{i,j}, w_{i,j})$ are the only cycles of length at most b which contain the arc $z_{i,j}v_{i,j}$, but these are not blocking, therefore $z_{i,j}$ cannot belong to a blocking coalition of size at most b. Those nodes of the new paths added to the construction who got their second choice in the exchange (namely $y^m_{i,j}$ for $i = 1, ..., k,\ j = 2, ..., n_i, m = 2, 4, ..., b'$ and $y^m_{i,1}$ for $i = 1, ..., k,\ m = 1, 3, ..., b' - 1$), cannot belong to a blocking coalition of size at most b, because they are in a 2-cycle with their second choice in the exchange, we have seen that their first choice cannot belong to a blocking coalition, and they do not belong to any longer cycle of length at most b. For $j \geq 2$, $w_{i,j}$ gets his second choice. He cannot belong to a blocking coalition of size at most b, since his first choice $y^1_{i,j}$, and his second choice $z_{i,j}$ cannot either. For $j \geq 2$, $v_{i,j}$ gets his first choice, $w_{i,j}$ in the exchange, who cannot belong to a blocking coalition of size at most b, therefore $v_{i,j}$ cannot either. Now we show that the nodes of the cycle $(u_{i,1}, ..., u_{i,2}, ..., u_{i,n_i-1}, ..., u_{i,n_i})$ cannot belong to a blocking coalition of size a most b. These nodes are in 2-cycles in the exchange, and they do not belong to any longer cycle of length at most b. It follows from this, that the nodes of the cycle that get their first choice cannot belong to a blocking coalition. The ones who get their second choice cannot belong to a blocking coalition either, because they would get their first choice in it, but we have just seen that these nodes cannot belong to a blocking coalition. $w_{i,1}$ is in a 2-cycle with his first choice, and this arc does not belong to any longer cycles of length at most b, which means that $w_{i,1}$ cannot belong to a blocking coalition of size at most b. This also holds for $z_{i,1}$, because he is in a

2-cycle with his second choice thus he would get his first choice $v_{i,1}$ in any blocking coalition he might be in, but the only cycles of length at most b which contain this arc are $(z_{i,1}, v_{i,1})$ and $(z_{i,1}, v_{i,1}, w_{i,1})$, and these are not blocking. We have shown from every node that it does not belong to a blocking coalition of size at most b except for $v_{i,1},\ i = 1, ..., k$. But these nodes correspond to the nodes of a k-clique thus they are not connected, therefore they cannot belong to a blocking coalition either. □

Now we return to 2-way stable 3-way exchanges, and prove that the problem remains NP-complete even in complete digraphs. If we added the missing arcs to the end of the preference lists in our construction in theorem 3.1, this would not work for proving this, because we used that certain arcs do not belong to a cycle of length 3, which would not be true in the modified version. Therefore we need a new construction.

Theorem 3.8. *The decision problem of finding a 2-way stable 3-way exchange in a complete digraph is NP-complete.*

Proof. The reduction is from the k-clique in k-partite graph problem. Given an instance $G = (X_1 \cup X_2 \cup ... \cup X_k, E)$ of the k-clique in k-partite graph problem, we create an instance of the 2-way stable 3-way exchange problem. By adding isolated nodes we may assume that $|X_i| = n_i$ is odd for $i = 1, ..., k$, and $n_i \geq 5$. For every i, we create a set of nodes $C_i = \{v_{i,1}, v_{i,2}, ..., v_{i,n_i}\}$. The digraph of the 2-way stable 3-way exchange problem is the complete digraph defined on these nodes. For every node in X_i, there is a distinct corresponding node in C_i. For $v_{i,j}$ we denote the set of nodes in $\bigcup_{t \neq i} C_t$ for which the corresponding node in $\bigcup_{t \neq i} X_t$ and the node corresponding to $v_{i,j}$ are not connected with $N_{i,j}$. Let $T_{i,j} = (\bigcup_{t \neq i} C_t) - N_{i,j}$. (These are the nodes that the corresponding node in G is connected with the node corresponding to $v_{i,j}$ in G.)

Now we describe the preference lists. A set in the preference list means the nodes of the set in arbitrary order. For $i \in [k],\ j \in [n_i]$ the preference list of $v_{i,j}$ is shown in the following table. (subscripts modulo n_i).

node					preference list
$v_{i,j}$	$v_{i,j+1}$	$v_{i,j-1}$	$N_{i,j}$	$C_i - v_{i,j}$	$T_{i,j}$

We prove that there is a k-clique in G if and only if the constructed instance admits a 2-way stable 3-way exchange.

First suppose that there is a k-clique in G. Without loss of generality we may assume that the nodes corresponding to the nodes of the k-clique are $v_{1,3}, v_{2,3}, ..., v_{k,3}$.

Lemma 3.9. *The 3-way exchange defined by the directed 3-cycles* $(v_{i,1}, v_{i,2}, v_{i,3})$ *and 2-cycles* $(v_{i,j}, v_{i,j+1})$ *for* $i = 1, 2, ..., k$; $j = 4, 6, ..., n_i - 1$ *is 2-way stable.*

Proof. The nodes $v_{i,j}$ for $j = 1, 2, 4, 6, ..., n_i - 1$ get the first item on their preference list, therefore they cannot belong to a blocking pair. For $5 \leq 2t + 1 \leq n_i$, $v_{i,2t+1}$ cannot belong to a blocking pair either, because he gets the second item on his list, so the only way he could belong to a blocking pair is if $\{v_{i,2t+1}, v_{i,2t+2}\}$ would be blocking, but v_{2t+2} cannot belong to a blocking pair. Finally a pair $\{v_{i,3}, v_{j,3}\}$ cannot be blocking because $v_{i,3}$ and $v_{j,3}$ correspond to two nodes of a k-clique in G so they prefer what they got in the exchange over each other. □

Now suppose that the constructed instance admits a 2-way stable 3-way exchange.

Lemma 3.10. *Suppose we have an odd bidirected cycle* $C = (c_1, c_2, ..., c_t)$ *in the digraph, such that each node of the cycle has the successive node in the cycle as his first choice and the preceding node as his second choice. If there is a 2-way stable exchange, then either there are two consecutive arcs of* C *in that exchange, or the whole backwards cycle is in the exchange.*

Proof. Suppose that there are two consecutive arcs of C: c_ic_{i+1} and $c_{i+1}c_{i+2}$ so that neither of them is in the exchange. If $c_{i+1}c_i$ is not in the exchange, then $\{c_i, c_{i+1}\}$ is a blocking pair, because c_{i+1} is first on c_i's preference list, c_i is second on c_{i+1}'s preference list, and $c_{i+1}c_{i+2}$ and $c_{i+1}c_i$ are not in the exchange, which means that c_{i+1} cannot get the first or the second item on his list in the exchange. If $c_{i+1}c_i$ is in the exchange, then $c_{i-1}c_i$ cannot be in the exchange, because there is only one arc entering c_i in the exchange. Therefore if c_ic_{i-1} is not in the exchange, then it follows from the same argument, that $\{c_{i-1}, c_i\}$ is a blocking pair, so c_ic_{i-1} has to belong to the exchange. Repeating the same argument yields that the whole backwards cycle has to be in the exchange.

If there are no two consecutive arcs of C such that neither of them is in the exchange, there have to be two consecutive arcs of C which are in the exchange, since C is odd. □

The cycles $C_i = (v_{i,1}, v_{i,2}, ..., v_{i,n_i})$ meet the conditions of the above lemma for every $i \in [k]$, $j \in [n_i]$, and since $n_i \geq 5$, the backwards cycles cannot belong to the 3-way exchange. Therefore there exist two consecutive arcs $v_{i,j_i}v_{i,j_i+1}$ and $v_{i,j_i+1}v_{i,j_i+2}$ in each C_i, which are in the exchange. Thus the cycles $(v_{i,j_i}, v_{i,j_i+1}, v_{i,j_i+2})$ are in the exchange.

Now we prove that the nodes corresponding to v_{i,j_i+2} for $i = 1, ..., k$ form a k-clique. Suppose there are two, which are not connected in G. Then there exist i, l such that $v_{i,j_i+2} \in N_{l,j_l+2}$ and $v_{l,j_l+2} \in N_{i,j_i+2}$. For $t = i, l$, v_{t,j_t+2} gets v_{t,j_t} in the exchange over which he prefers the nodes of N_{t,j_t+2}, therefore v_{i,j_i+2} and v_{l,j_l+2} prefer each other, which means that $\{v_{i,j_i+2}, v_{l,j_l+2}\}$ is a blocking pair, thus we have reached a contradiction. □

Remark 3.11. *It follows from the same proof, that the decision problem of finding a (b-way) stable 3-way exchange in a complete digraph is NP-complete for any* $b \geq 2$.

Remark 3.12. *It can be proved in a similar way, that the decision problem of finding a (b-way) stable 4-way exchange in a complete digraph is NP-complete for any* $b \geq 2$. *The same proof does not work for l-way exchanges if* $l \geq 5$, *(because we cannot guarantee that the cycle containing the two consecutive arcs from* C_i *does not contain nodes from other* C_j*'s).*

Remark 3.13. *All the NP-completeness theorems in this section apply for the special cases when the digraph is symmetric.*

4 Maximizing the number of covered nodes

4.1 NP-hardness

An instance might admit more than one stable exchanges, therefore in is a natural goal to maximize the number of covered nodes in the exchange. It follows from the theorem below that this problem is NP-hard.

Theorem 4.1. *It is NP-complete to decide if an instance of the stable exchange problem admits a complete stable exchange.*

Proof. We reduce from the k-clique in k-partite graph problem. For every k-partite graph $G = (V_1, ..., V_k, E)$, we define an instance of the stable exchange problem. First we describe an undirected graph, which we then turn into a digraph, by replacing each edge with a bidirected edge. For every $i = 1, ..., k$ we define a node w_i, and n_i circuits: $(w_i, x_{i,j}, y_{i,j}, z_{i,j})$ $j = 1, ..., n_i$. Let $x_{i,j}z_{i,j}$ be an edge for every $i \in [k]$, $j \in [n_i]$. Let $X_i = \{x_{i,1}, ..., x_{i,n_i}\}$. Fore every $v \in V_i$ there is a distinct corresponding node in X_i. For $v \in V_i, v' \in V_j$ there is an edge between the corresponding nodes if and only if $vv' \notin E$. Let the neighbours of $x_{i,j}$ in $\bigcup_{l \neq i} X_l$ be denoted by $N_{i,j}$. The preference lists are shown in the following table. A set in the preference list means the nodes of the set in arbitrary order.

node	preference list			
$z_{i,j},\ i \in [k],\ j \in [n_i]$	$y_{i,j}$	w_i		
$y_{i,j},\ i \in [k],\ j \in [n_i]$	$x_{i,j}$	$z_{i,j}$		
$x_{i,j},\ i \in [k],\ j \in [n_i]$	$z_{i,j}$	$N_{i,j}$	w_i	$y_{i,j}$
$w_i,\ i \in [k]$	arbitrary	X_i		

See figure 2.

Lemma 4.2. *In every complete exchange there is a node in every X_i who gets one of the last two items on his list.*

Proof. If there is a complete stable exchange, then someone gets w_i in the exchange, for every i. If someone from X_i gets w_i, then this person gets the next to last item on his list. Otherwise someone from Z_i must get w_i, suppose $z_{i,j}$ does. Then $z_{i,j}$ cannot get $y_{i,j}$ and $y_{i,j}$ must be covered, therefore $x_{i,j}$ must get him, which means that $x_{i,j}$ gets the last item on his list. □

Take a node form every X_i who gets his last or next to last choice. These nodes prefer each other to what they get in the exchange, therefore no two of them are connected in the digraph, therefore the corresponding nodes form a k-clique in G.

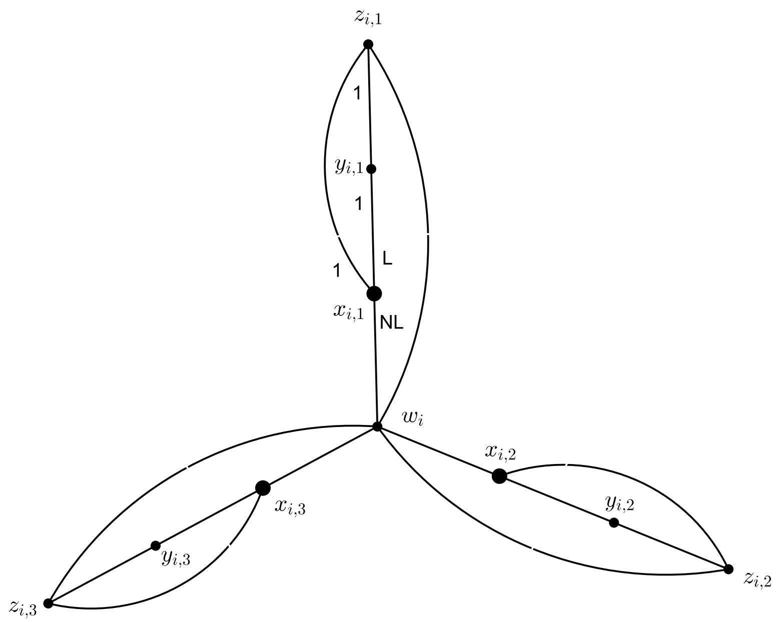

Figure 2: All the edges of the graph are bidirected edges. L:=last, NL:=next to last.

Suppose there is a k-clique in G. We may assume that the nodes corresponding to the nodes of the k-clique are $x_{1,1}, x_{2,1}, ..., x_{k,1}$. We prove that the complete exchange defined by the following directed cycles is stable:

$$(x_{i,1}, w_i, z_{i,1}, y_{i,1}), \quad (x_{i,j}, z_{i,j}, y_{i,j}), \quad i = 1, ..., k, \quad j = 2, ..., n_i.$$

The nodes of the cycles $(x_{i,j}, z_{i,j}, y_{i,j})$, $i = 1, ..., k$, $j = 2, ..., n_i$ get their first choice thus cannot belong to a blocking coalition. The same holds for $z_{i,1}$ and $y_{i,1}$ for $i = 1, ..., k$. We have seen that all the nodes other then $x_{i,1}$ connected to w_i cannot belong to a blocking coalition, therefore w_i cannot either. This leaves us with only $x_{1,1}, ..., x_{k,1}$. But these correspond to the nodes of a k-clique, meaning that they are not connected in the digraph, thus no subset of them forms a blocking coalition. □

For proving that there is a k-clique in G, we only used that the instance admits a 2-way stable exchange. Therefore the same proof applies to the following theorem.

Theorem 4.3. *It is NP-complete to decide if an instance of the stable exchange problem admits a complete b-way stable exchange for any $b \geq 2$.*

Roth and Postlewaite [12] proved that the exchange found by the TTC algorithm is the only strongly stable solution. However, there might be more then one b-way stable exchanges.

Theorem 4.4. *It is NP-complete to decide if an instance of the stable exchange problem admits a complete b-way strongly stable exchange for any $b \geq 2$.*

Proof. We modify our previous construction. Let us replace the bidirected $x_{i,j}z_{i,j}$ path of length 2 with a bidirected path of length t, where $t = b$ if b is even, and $t = b + 1$ otherwise. The new nodes prefer the neighbour which is closer to $x_{i,j}$. Now we show that lemma 4.2 remains true for this construction too. If there is a complete stable exchange, then someone gets w_i in the exchange. If it is $x_{i,j}$ for some $j \in [n_i]$, then he gets his next to last choice, so we are done. Otherwise $z_{i,j}$ gets w_i for some $j \in [n_i]$. This $z_{i,j}$ does not get the next node on the path from $z_{i,j}$ to $x_{i,j}$, so its other neighbour has to get it. If this arc is in a cycle of length at least 3 then since $z_{i,j}$ gets w_i, it could only be in the cycle $(w_i, x_{i,j}, ..., z_{i,j})$. Therefore in that case $x_{i,j}$ gets his last choice. So we only need to cover the case when the first and second node after $z_{i,j}$ on the path from $z_{i,j}$ to $x_{i,j}$ are switched in the exchange. In that case, the nodes of the $z_{i,j}x_{i,j}$ path cannot belong to a cycle of length at least 3 and every one of them is covered in the exchange, therefore every second arc of the $z_{i,j}x_{i,j}$ path belongs to a 2-cycle, and since the length of the path is even $x_{i,j}$ is paired with the previous node of the $z_{i,j}x_{i,j}$ path, thus gets his last choice.

Just like in theorem 4.1, the lemma implies that there is a k-clique in G.

If there is a k-clique in G, without loss of generality, we may assume that the nodes corresponding to the nodes of the k-clique are $x_{1,1}, x_{2,1}, ..., x_{k,1}$. We prove that the complete exchange defined by the following directed cycles is strongly stable:

$$(w_i, z_{i,1}, ..., x_{i,1}), \quad (z_{i,j}, ..., x_{i,j}), \quad i = 1, ..., k, \quad j = 2, ..., n_i.$$

The nodes of the following cycles of length at least b get their first choice thus cannot belong to a blocking coalition: $(z_{i,j}, ..., x_{i,j})$, $i = 1, ..., k$, $j = 2, ..., n_i$. The nodes of the cycles $(w_i, z_{i,1}, ..., x_{i,1})$ except for the $x_{i,1}$'s and w_i's get their first choice in the exchange, therefore they get the same item in any blocking coalition which they belong to. The only cycles of length at most b which contain one of these nodes are 2-cycles, but the nodes of the cycles $(w_i, z_{i,1}, ..., x_{i,1})$ prefer the item succeeding them to the one preceding them, therefore these pairs are not blocking, which implies that the nodes of the cycles $(w_i, z_{i,1}, ..., x_{i,1})$ except for the $x_{i,1}$'s and w_i's cannot belong to a blocking pair. We have seen that all the nodes other than $x_{i,1}$ connected to w_i cannot belong to a blocking coalition, and $\{w_i, x_{i,1}\}$ is not a blocking pair, therefore w_i cannot belong to a blocking coalition either. This leaves us with only $x_{1,1}, ..., x_{k,1}$. But these correspond to nodes of a k-clique, meaning that they are not connected in the digraph, and thus no subset of them forms a blocking coalition. □

Remark 4.5. *All the NP-completeness theorems in this subsection apply for the special cases when the digraph is symmetric.*

4.2 Approximation algorithms

We have seen that it is NP-hard to maximize the number of covered nodes in a 2-way stable exchange. Now we will check how well the known algorithms for finding a 2-way stable exchange approximate the problem.

Definition 4.6. *An algorithm for a maximization problem is called an* ***α-approximation algorithm*** *for $\alpha < 1$, if it runs in polynomial time, and for every input it finds a solution whose value is at least α times the optimum. α is called the* ***approximation ratio*** *of the algorithm.*

First we do not assume that the digraph is symmetric.

Claim 4.7. *There does not exist an $\alpha < 1$ for which the Top Trading Cycles algorithm is an α-approximation algorithm for finding a 2-way stable exchange which covers the maximum number of nodes.*

Proof. Suppose there is such an α. Take an integer k, such that $\frac{2}{k} < \alpha$. Take two directed cycles of length k: $(u_1, u_2, ..., u_k)$ and $(v_1, v_2, ..., v_k)$ and two additional arcs u_2v_1 and v_2u_1. u_2 prefers v_1 over u_3 and v_2 prefers u_1 over v_3, all the other nodes have only one acceptable node on their lists. TTC takes the cycle (v_1, v_2, u_1, u_2) in the exchange in the first step and then terminates. Therefore it finds a 2-way stable exchange which covers 4 nodes. However, the 2-way stable exchange defined by the cycles $(u_1, u_2, ..., u_k)$ and $(v_1, v_2, ..., v_k)$ covers all the $2k$ nodes. Since $\alpha > \frac{2}{k}$, $4 < \alpha(2k)$ thus TTC is not an α-approximation. □

However, if we do assume that the digraph is symmetric, the following claim holds.

Claim 4.8. *If the digraph is symmetric, TTC is a $\frac{1}{2}$-approximation algorithm, and the approximation ratio is sharp.*

Proof. Let U be the set of nodes that are not covered in a 2-way stable exchange E_{TTC} given by TTC. The nodes of U are independent in the digraph, because if any two were connected the pair would block E_{TTC}. Let E_{OPT} be the optimal 2-way stable exchange. Suppose that some nodes from U are covered by E_{OPT}. Since U is an independent set, these nodes are given to some nodes outside of U in E_{OPT} and each of them is given to a different node. Therefore E_{OPT} covers at most twice as many nodes as E_{TTC} which means that TTC is a $\frac{1}{2}$-approximation algorithm.

Now we present a sharp example. Take a bidirected circuit $(u_1, ..., u_k)$ and additional bidirected edges $u_i v_i$ for $i = 1, ..., k$. The first item on u_i's preference list is u_{i+1}, the second is v_i and the third is u_{i-1} for $i = 1, ..., k$ subscripts modulo k. TTC takes the cycle $(u_1, u_2, ..., u_k)$ in the exchange in the first step, and then terminates, so the exchange covers half of the nodes. However, the 2-way stable exchange defined by the 2-cycles (u_i, v_i) $i = 1, ..., k$ covers all the nodes. □

We still assume that the digraph is symmetric. In this case, another way of finding a 2-way stable exchange is by finding a stable partition. (See subsection 2.3 for details.)

Claim 4.9. *The stable partition algorithm is a $\frac{2}{3}$-approximation for finding a 2-way stable exchange and the approximation ratio is sharp.*

Proof. We denote the optimal 2-way stable exchange by E_{OPT}. Let U be the set of nodes that are not covered in the 2-way stable exchange given by the stable partition algorithm. The nodes of U are the parties of size 1 in the stable partition, therefore they are not connected. Suppose the nodes in $U' \subseteq U$ are covered by E_{OPT}. Just like in claim 4.8, the set of nodes X that get a node from U' in E_{OPT} are such that $X \cap U = \emptyset$ and each node from U' is given to a different node from X. (Thus $|X| = |U'|$.) We will show that the set of nodes Y that get a node from X in the stable partition algorithm are such that $Y \cap X = \emptyset$ (clearly $Y \cap U = \emptyset$ and $|X| = |Y|$). Suppose $x' \in Y \cap X$ and x' gets $x \in X$ in the stable partition algorithm. In E_{OPT}, x gets u and x' gets u', $u, u' \in U$. u and u' are parties of size one in the stable partition, therefore x prefers x' over u and x' prefers x over u'. But this means that $\{x, x'\}$ blocks E_{OPT} which is a contradiction. We proved that $Y \cap X = \emptyset$, from which it follows that the optimal solution covers at most $\frac{3}{2}$ times the number of nodes covered by the 2-way stable exchange found by the stable partition algorithm.

Now we present a sharp example. The digraph consists of k distinct bidirected circuits of length three: (u_i, v_i, w_i). The preference lists are shown in the table below.

node	preference list	
$u_i,\ i \in [k]$	v_i	w_i
$v_i,\ i \in [k]$	u_i	w_i
$w_i,\ i \in [k]$	u_i	v_i

The stable partition algorithm finds the 2-way stable exchange defined by the 2-cycles (u_i, v_i), $i = 1, ..., k$, which covers $2k$ nodes, however, the optimal solution covers all the $3k$ nodes: the exchange defined by the cycles (u_i, v_i, w_i), $i = 1, ..., k$ is 2-way stable. □

5 Maximizing the number of 2-cycles

Theorem 5.1. *The problem of finding a stable exchange containing maximum number of 2-cycles is NP-hard. The problem is also NP-hard for b-way stability for any* $b \geq 2$.

Proof. We reduce from the k-clique in k-partite graph problem. Given an instance $G = (X_1, X_2, ..., X_k, E)$ of the k-clique in k-partite graph problem, we create an instance of the stable exchange problem. We may assume that $|X_i| = n_i \geq 2$ for $i = 1, ..., k$. For every $i = 1, ..., k$ we define n_i bidirected circuits: $C_{i,j} = (u_{i,j}, v_{i,j}, w_{i,j}, y_{i,j}, z_{i,j}),\ j = 1, ..., n_i$. Let $w_{i,j}u_{i,j}$ be an arc for every i, j. We define a set of $n_i - 1$ new nodes: $T_i = \{t_{i,1}, ..., t_{i,n_i-1}\}$. There is an arc from $w_{i,j}$ to every node in T_i and from every node in T_i there is an arc to $u_{i,j}$ for $i = 1, .., k,\ j = 1, ..., n_i$. Let $W_i = \{w_{i,1}, ..., w_{i,n_i}\}$. Fore every $x \in X_i$ there is a distinct corresponding node in W_i. For $x \in X_i, x' \in X_j,\ i \neq j$, there is an bidirected edge between the corresponding nodes if and only if $xx' \notin E$. Let the neighbours of $w_{i,j}$ in $\bigcup_{l \neq i} W_l$ be denoted by $N_{i,j}$. The preference lists are shown in the following table. A set in the preference list means the nodes of the set in arbitrary order.

node	preference list				
$u_{i,j},\ i \in [k],\ j \in [n_i]$	$v_{i,j}$	$z_{i,j}$			
$v_{i,j},\ i \in [k],\ j \in [n_i]$	$w_{i,j}$	$u_{i,j}$			
$w_{i,j},\ i \in [k],\ j \in [n_i]$	$y_{i,j}$	$v_{i,j}$	T_i	$N_{i,j}$	$u_{i,j}$
$y_{i,j},\ i \in [k],\ j \in [n_i]$	$z_{i,j}$	$w_{i,j}$			
$z_{i,j},\ i \in [k],\ j \in [n_i]$	$u_{i,j}$	$y_{i,j}$			
$t_{i,j},\ i \in [k],\ j \in [n_i - 1]$	arbitrary				

See figure 3.

Notice that each node in the odd cycles: $C_{i,j},\ i \in [k],\ j \in [n_i]$ has the successive node in the cycle as his first choice and the preceding node as his second choice, therefore it follows from lemma 3.10 that either there are two consecutive arcs of $C_{i,j}$ in the exchange, or the whole backwards cycle is in the exchange. From this it follows that at most two nodes belong to a 2-cycle in any stable exchange, and since none of the nodes of T_i belong to any 2-cycle, at most $n = \sum n_i$ 2-cycles can be in the exchange. Now we prove that there is a stable exchange with n 2-cycles if and only if there is a k-clique in G.

First suppose there is a stable exchange with n 2-cycles. Then each $C_{i,j}$ contains a 2-cycle for $i = 1, ..., k,\ j = 1, ..., n_i$. The nodes $y_{i,j}$ and $z_{i,j}$ do not belong to any cycle which contains 3 nodes from $C_{i,j}$, therefore the 2-cycle in $C_{i,j}$ is $(y_{i,j}, z_{i,j})$ and the arcs $u_{i,j}v_{i,j}$ and $v_{i,j}w_{i,j}$ are in the exchange. There are $n_i - 1$ nodes in T_i, therefore there is a j_i for which the nodes of C_{i,j_i} belong to cycles that do not contain any nodes from

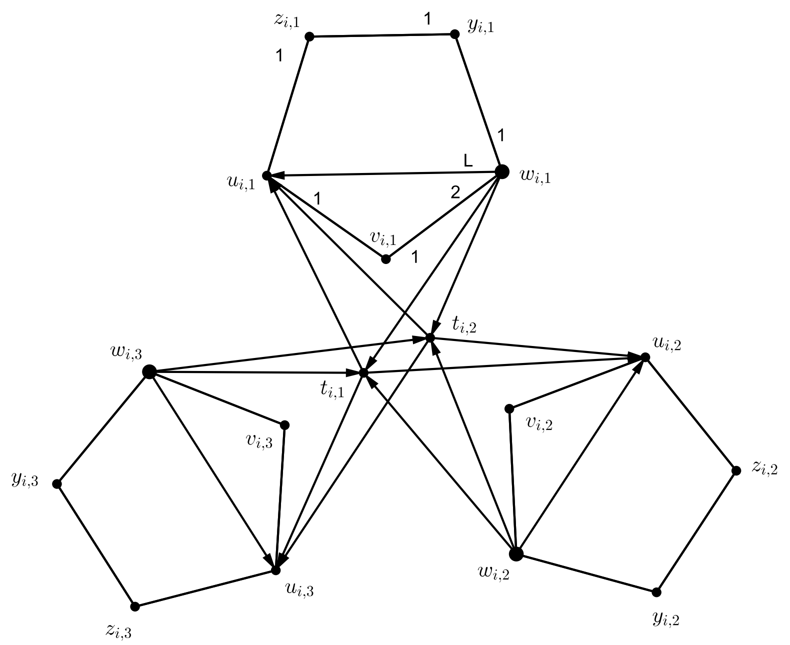

Figure 3: The undirected edges of the graph are bidirected edges. L:=last, NL:=next to last.

T_i. From the nodes of C_{i,j_i}, only w_{i,j_i} is connected with nodes outside of $T_i \cup C_{i,j_i}$, therefore the cycle $(u_{i,j_i}, v_{i,j_i}, w_{i,j_i})$ is in the exchange, which means that w_{i,j_i} gets the last item on his preference list. The nodes $w_{i,j_i},\ i = 1, ..., k,\ j = 1, ..., n_i$ are not connected (otherwise their would be a blocking pair), thus the nodes corresponding to them form a k-clique in G.

Now suppose there is a k-clique in G. We may assume that the nodes corresponding to the nodes of the k-clique are $w_{1,n_1}, w_{2,n_2}, ..., w_{k,n_k}$. We prove that the exchange defined by the following directed cycles is stable (and contains n 2-cycles):

$$(y_{i,j}, z_{i,j}),\ i = 1, ..., k,\ j = 1, ..., n_i,$$

$$(u_{i,n_i}, v_{i,n_i}, w_{i,n_i}),\ i = 1, ..., k,$$

$$(u_{i,j}, v_{i,j}, w_{i,j}, t_{i,j}),\ i = 1, ..., k,\ j = 1, ..., n_{i-1}.$$

The following nodes get their first choice in the exchange and thus cannot belong to a blocking coalition: $u_{i,j}, v_{i,j}, y_{i,j},\ i = 1, ..., k,\ j = 1, ..., n_i$. $z_{i,j}$ gets his second choice and we have just seen that his first choice $u_{i,j}$ cannot belong to a blocking coalition, therefore $z_{i,j}$ cannot belong to a blocking coalition either, for every $i \in [k],\ j \in [n_i]$. Suppose $w_{i,j}$ is in a blocking coalition for some $i \in [k],\ j \in [n_i - 1]$. He cannot get $y_{i,j}$, $v_{i,j}$ or $u_{i,j}$ in it, because we have already seen, that these cannot belong to a blocking coalition. He cannot get something from $N_{i,j}$ either, because he prefers what he got in the exchange over these nodes. So he must get someone from T_i in the blocking coalition, but the nodes of T_i could only get $u_{i,j}$ for some $i \in [k]$, $j \in [n_i]$, which is a contradiction. This also means that the nodes of T_i, $i \in [k]$ cannot belong to a blocking coalition either. We are left with w_{i,n_i}, $i = 1, ..., k$, but these are not even connected.

Notice that in the first part of the proof we only used that the exchange is 2-way stable, which proves the NP-hardness of the problem for b-way stability for any $b \geq 2$. □

6 2-way stable pairwise exchanges with chains

A recent innovation in kidney exchange is allowing chains. There are altruists who are willing to donate one of their kidneys to any patient who needs it. In an exchange with chains we allow chains ending in an altruist besides cycles. The cycles should be short in practice, however, the chains might be longer since the operations along a chain do not necessarily have to be carried out simultaneously (although it is desirable). Chains are currently being used in kidney exchanges in the US. Most of the models ignore the differences between suitable kidneys and try to find a solution where the maximum number of nodes are covered, or define weights on the arcs and try to find a solution with maximum total weight. (See e.g. in [4].)

In this section we study the problem of finding 2-way stable pairwise exchanges with chains. Pairwise means that besides the chains only 2-cycles are allowed. (The 2-way stable l-way exchange with chains problem is NP-hard since its special case, where there are no altruist nodes is NP-complete.) The altruist nodes are sink nodes in the digraph, therefore they cannot belong to a blocking pair in the original sense. However, in this case this might not be the best notion of stability. We define three types of stability according to when does a pair (v, a) where a is an altruist block the exchange.
Type 1: never.
Type 2: if and only if v prefers a over what he got in the exchange, and a is not covered in the exchange.
Type 3: if and only if v prefers a over what he got in the exchange.

In subsection 6.1 and theorem 6.6 we do not make a difference between the three types of stability since these problems are NP-complete for all three types.

6.1 NP-completeness, and restriction on the number of altruists

Theorem 6.1. *The decision problem of finding a 2-way stable pairwise exchange with chains is NP-complete, even if the number of altruists is restricted to 1.*

Proof. We reduce from the k-clique in k-partite graph problem. Given an instance $G = (X_1 \cup X_2 \cup ... \cup X_k, E)$ of the k-clique in k-partite graph problem, we create an instance of the 2-way stable pairwise exchange with chains problem. By adding isolated nodes we may assume that $|X_i| = n_i$ is even for $i = 1, ..., k$.

First we define an undirected graph, which then we turn into a digraph by replacing each edge with a bidirected edge. For every $i = 1, ..., k$ we define a circuit $T_i = (t_{i,1}, t_{i,2}, ..., t_{i,n_i})$, and for every $i = 1, ..., k + 1$, we define a circuit

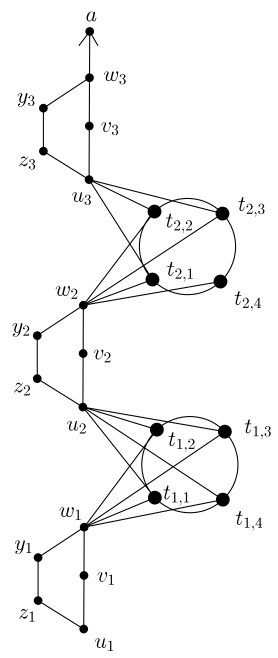

Figure 4: The undirected edges of the graph are bidirected edges. L:=last, NL:=next to last.

$C_i = (u_i, v_i, w_i, y_i, z_i)$. From w_i and u_{i+1} there is an edge to every node of T_i. For every $x \in X_i$ there is a distinct corresponding node in T_i. For $x \in X_i$, $y \in X_j$, $i \neq j$, there is an edge between the corresponding nodes, if and only if $xy \notin E$. We turn this into a digraph and we add the only altruist node a, and an arc from w_{k+1} to a.

Let the neighbours of $t_{i,j}$ in $\bigcup_{l \neq i} T_l$ be denoted by $N_{i,j}$. The preference lists are shown in the following table. A set in the preference list means the nodes of the set in arbitrary order.

node					preference list
u_1	v_1	z_1			
$u_i,\ i \geq 2$	v_i	z_i	T_{i-1}		
$v_i,\ i \in [k]$	w_i	u_i			
$w_i,\ i \in [k]$	y_i	v_i	T_i		
w_{k+1}	y_{k+1}	v_{k+1}	a		
$y_i,\ i \in [k+1]$	z_i	w_i			
$z_i,\ i \in [k+1]$	u_i	y_i			
$t_{i,j},\ i \in [k],\ j \in [n_i]$	$t_{i,j+1}$	$t_{i,j-1}$	$N_{i,j}$	u_{i+1}	w_i

See figure 4.

We will prove that the constructed instance admits a 2-way stable pairwise exchange with chains if and only if there is a k-clique in G. First suppose that the constructed instance admits a 2-way stable pairwise exchange with chains. Notice that each node in the odd cycles: $C_i,\ i \in [k+1]$ has the successive node in the cycle as his first choice and the preceding node as his second choice, therefore it follows from lemma 3.10 that either there are two consecutive arcs of C_i in the exchange, or the whole backwards cycle is in the exchange. The latter is not possible, because we do not allow cycles of length more than two in the exchange. Since there is only one altruist node, there could be at most 1 chain in the exchange, and we have just seen that two consecutive arcs from every $C_i,\ i \in [k+1]$ belongs to this chain. There is only one node, w_1, where the chain can enter or leave C_1, thus the chain must start in C_1. The chain ends in a, therefore it must enter and leave each C_i for $i = 2, ..., k+1$. At least 3 nodes are in the chain from C_i, therefore the chain must enter and leave through different nodes of C_i. The only nodes of C_i where the chain could enter or leave are u_i and w_i, therefore either the chain enters through u_i and leaves through w_i or the other way around. The latter is not possible, because since two consecutive arcs of C_i are in the exchange, this would mean that w_iy_i, y_iz_i and z_iu_i are in the chain and v_i remains uncovered, therefore $\{u_i, v_i\}$ would block the exchange.

The chain enters C_{i+1} through u_{i+1}, therefore there is a j_i such that $t_{i,j_i}u_{i+1}$ is in the exchange for every $i \in [k]$. The nodes corresponding to t_{i,j_i} for $i = 1, ..., k$ form a k-clique in G. Suppose there are two which are not connected. Then there is a bidirected edge between the nodes corresponding to them. But then these two nodes form a blocking pair, because they prefer each other to their current partner.

Now suppose there is a k-clique in G. Without loss of generality, we may assume that the nodes corresponding to the nodes of the k-clique are $t_{1,2}, t_{2,2}, ..., t_{k,2}$.

Lemma 6.2. *The pairwise exchange with chains defined by the following chain and pairs is 2-way stable.*
$u_1v_1w_1t_{1,1}t_{1,2}u_2v_2w_2t_{2,1}t_{2,2}u_3...w_kt_{k,1}t_{k,2}u_{k+1}v_{k+1}w_{k+1}a$,
$(t_{i,j}, t_{i,j+1}),\ i = 1, ..., k,\ j = 3, 5, ..., n_i - 1,$ *subscripts modulo* n_i.
$(y_i, z_i),\ i = 1, ..., k+1.$

Proof. The following nodes got their first choice in the exchange, thus cannot belong to a blocking pair: w_{k+1}, y_i, u_i, v_i, for $i = 1, ..., k+1$, $t_{i,1}, t_{i,j}$ for $i = 1, ..., k,\ j = 3, 5, ..., n_i - 1$. The following nodes got their second choice in the exchange, and the first item on their preference list got his first choice, therefore they cannot belong to a blocking pair: z_i, for $i = 1, ..., k+1$, $t_{i,j}$ for $i = 1, ..., k,\ j = 4, 6, ..., n_i$. The remaining nodes are w_i and $t_{i,2}$ for $i = 1, ..., k$. w_i is only connected with $t_{i,2}$ among these nodes, and $\{w_i, t_{i,2}\}$ is not a blocking pair since $t_{i,2}$ prefers u_{i+1} (what he has got in the exchange) over w_i. Therefore w_i cannot belong to a blocking pair for every $i \in [k]$. Now we are left with $t_{i,2}$ $i = 1, .., k$, but these nodes correspond to the nodes of a k-clique, hence are not connected. □

□

Theorem 6.3. *The decision problem of finding a 2-way strongly stable pairwise exchange with chains is NP-complete, even if the number of altruist is restricted to 1.*

Proof. We reduce from the k-clique in k-partite graph problem. We use the same construction as in theorem 6.1, but with modified preference lists. The new preference lists are shown in the table below.

node	preference list				
u_1	v_1	z_1			
$u_i,\ i \geq 2$	v_i	z_i	T_{i-1}		
$v_i,\ i \in [k]$	w_i	u_i			
$w_i,\ i \in [k]$	y_i	T_i	v_i		
w_{k+1}	y_{k+1}	a	v_{k+1}		
$y_i,\ i \in [k+1]$	z_i	w_i			
$z_i,\ i \in [k+1]$	u_i	y_i			
$t_{i,j},\ i \in [k],\ j \in [n_i]$	$t_{i,j+1}$	$N_{i,j}$	u_{i+1}	$t_{i,j-1}$	w_i

We will prove that the constructed instance admits a 2-way strongly stable pairwise exchange with chains if and only if there is a k-clique in G. First suppose that the constructed instance admits a 2-way strongly stable pairwise exchange with chains.

Lemma 6.4. *Every $w_i,\ i \in [k+1]$ belongs to a chain in the exchange.*

Proof. If there is a $w_i,\ i \in [k+1]$ which is not covered at all, than $\{w_i, v_i\}$ is a blocking pair since w_i is first on v_i's preference list. Therefore this case is not possible. Suppose there is a w_i which does not belong to a chain. We know it is covered, so it must be in a pair.

Case 1: $i \in [k]$ and $(w_i, t_{i,j})$ is a pair in the exchange, for some $j \in [n_i]$. Then $\{t_{i,j-1}, t_{i,j}\}$ is a blocking pair, because $t_{i,j}$ is first on $t_{i,j-1}$'s preference list, and $t_{i,j}$ prefers $t_{i,j-1}$ to w_i. Thus we have reached a contradiction.

Case 2: $i \in [k+1]$ and (w_i, y_i) is a pair in the exchange. Then z_i must get the first item on his preference list: u_i, since otherwise $\{y_i, z_i\}$ would be a blocking pair. The only nodes v_i is connected to are w_i and u_i, but these nodes are taken, therefore v_i remains uncovered. However, v_i is the first item on u_i's preference list which means that $\{u_i, v_i\}$ is a blocking pair, which is a contradiction.

Case 3: $i \in [k+1]$ and (w_i, v_i) is a pair in the exchange. Then y_i must get the first item on his preference list: z_i, since otherwise $\{w_i, y_i\}$ would be a blocking pair. But then $\{z_i, u_i\}$ is blocking pair, because u_i is first on z_i's preference list, and z_i is second on u_i's and u_i cannot get his first choice v_i in the exchange. □

Since there is only one altruist node, there could be at most 1 chain in the exchange. From the above lemma this chain contains w_i for $i = 1, ..., k+1$. The arc $t_{i,j}w_i$ cannot belong to the exchange for any $j \in [n_i]$ since in that case $t_{i,j-1}t_{i,j}$ would block the exchange. This means that the chain cannot enter C_i through w_i for any $i \in [k]$. The chain can only enter or leave C_1 through w_1, therefore the chain has to start in C_1. The chain ends in a, so it enters and leaves every C_i for $i = 2, .., k+1$. The only nodes of C_i where the chain could enter or leave are u_i and w_i. We have already seen that it cannot enter through w_i, therefore the chain enters C_{i+1} through u_{i+1} and leaves it through w_{i+1} for every $i \in [k]$, which means that there is a j_i such that $t_{i,j_i}u_{i+1}$ is in the exchange for every $i \in [k]$. The nodes corresponding to t_{i,j_i} for $i = 1, ..., k$ form a k-clique in G. Suppose there is two which are not connected. Then there is a bidirected edge between the nodes corresponding to them. But then these two nodes form a blocking pair, because they prefer each other to their current partner.

Now suppose there is a k-clique in G. Without loss of generality, we may assume that the nodes corresponding to the nodes of the k-clique are $t_{1,n_1}, t_{2,n_2}, ..., t_{k,n_k}$.

Lemma 6.5. *The pairwise exchange with chains defined by the following chain and pairs is 2-way strongly stable.*
$u_1v_1w_1t_{1,1}t_{1,2}...t_{1,n_1}u_2v_2w_2t_{2,1}t_{2,2}...t_{2,n_2}u_3...w_kt_{k,1}t_{k,2}...t_{k,n_k}u_{k+1}v_{k+1}w_{k+1}a,$
$(y_i, z_i),\ i = 1, ..., k+1.$

Proof. The nodes of the chain prefer the node succeeding them in the chain over the one preceding them, therefore none of the arcs of the chain determine a blocking pair. The following nodes of the chain get their first choice in the exchange, thus cannot belong to a blocking pair: $u_i,\ v_i$, for $i = 1, ..., k+1$, $t_{i,j}$ for $i = 1, ..., k,\ j = 1, 2, ..., n_i - 1$.
The nodes $y_i, z_i,\ i = 1, ..., k+1$ belong to pairs in the exchange, y_i gets his first choice and z_i gets his second choice and we have seen that the first choice u_i of z_i cannot belong to a blocking pair for every $i \in [k+1]$, therefore these nodes cannot belong to a blocking pair either. The remaining nodes are w_{k+1}, w_i and t_{i,n_i} for $i = 1, ..., k$. y_{k+1}, the first choice of w_{k+1}, cannot belong to a blocking pair, therefore w_{k+1} cannot either. For $i = 1, ..., k$, w_i is only connected with t_{i,n_i} among those nodes for whom we have not seen that they cannot belong to a blocking pair yet. $\{w_i, t_{i,n_i}\}$ is not a blocking pair since t_{i,n_i} prefers u_{i+1} (what he has got in the exchange) over w_i. Therefore w_i cannot belong to a blocking pair for every $i \in [k]$. Now we are left with

t_{i,n_i} $i = 1, .., k$, but these nodes correspond to the nodes of a k-clique, hence are not connected. □

□

6.2 Restrictions on the length of the chains

Theorem 6.6. *The decision problem of finding a 2-way stable pairwise exchange with chains, where the lengths of the chains are at most l is NP-complete for any given $l \geq 2$.*

Proof. We reduce from the k-clique in k-partite graph problem. Given an instance $G = (X_1 \cup X_2 \cup ... \cup X_k, E)$ of the k-clique in k-partite graph problem, we create an instance of the 2-way stable pairwise exchange with chains problem, where the length of the chains are at most l. By adding isolated nodes we may assume that $|X_i| = n_i$ is odd for $i = 1, ..., k$.

First we define an undirected graph, which then we turn into a digraph by replacing each edge with a bidirected edge. For every $i = 1, ..., k$ we define a circuit $U_i = (u_{i,1}, u_{i,2}, ..., u_{i,n_i})$ and a path $u_{i,j}v_{i,j}w^1_{i,j}w^2_{i,j}...w^{l-1}_{i,j}$ from $u_{i,j}$, for every $i \in [k],\ j \in [n_i]$. Let $V_i = \{v_{i,1}, ..., v_{i,n_i}\}$. For every $x \in X_i$ there is a distinct corresponding node in V_i. For $x \in X_i$, $y \in X_j$, $i \neq j$, there is an edge between the corresponding nodes, if and only if $xy \notin E$. We turn this into a digraph and we add altruist nodes $w^l_{i,j}$ and an arc from $w^{l-1}_{i,j}$ to $w^l_{i,j}$, for every $i \in [k],\ j \in [n_i]$.

Let the neighbours of $v_{i,j}$ in $\bigcup\limits_{l \neq i} V_l$ be denoted by $N_{i,j}$. The preference lists are shown in the following table. A set in the preference list means the nodes of the set in arbitrary order.

node	preference list		
$u_{i,j}\ i \in [k],\ j \in [n_i]$	$u_{i,j+1}$	$v_{i,j}$	$u_{i,j-1}$
$v_{i,j}\ i \in [k],\ j \in [n_i]$	$w^1_{i,j}$	$N_{i,j}$	$u_{i,j}$
$w^1_{i,j}\ i \in [k],\ j \in [n_i]$,	$w^2_{i,j}$	$v_{i,j}$	
$w^n_{i,j}\ i \in [k],\ j \in [n_i],\ 2 \leq n \leq l-1$	$w^{n+1}_{i,j}$	$w^{n-1}_{i,j}$	

See figure 5.

Suppose there is a 2-way stable pairwise exchange with chains of length at most l. The arcs of the cycles U_i, and the arcs $u_{i,j}v_{i,j}$ do not belong to any chain of length at most l which ends in an altruist node, for $i = 1, .., k,\ j = 1, .., n_i$. Therefore if the nodes of these cycles are covered, they must be covered by a 2-cycle. If $u_{i,j-1}u_{i,j}$ and $u_{i,j}u_{i,j+1}$ subscripts modulo n_i are two consecutive arcs of U_i which are not in the exchange, then $(u_{i,j}, v_{i,j})$ is a 2-cycle of the exchange, otherwise $\{u_{i,j-1}, u_{i,j}\}$ would block the exchange. Since U_i is an odd cycle for every i there must be a j_i for which

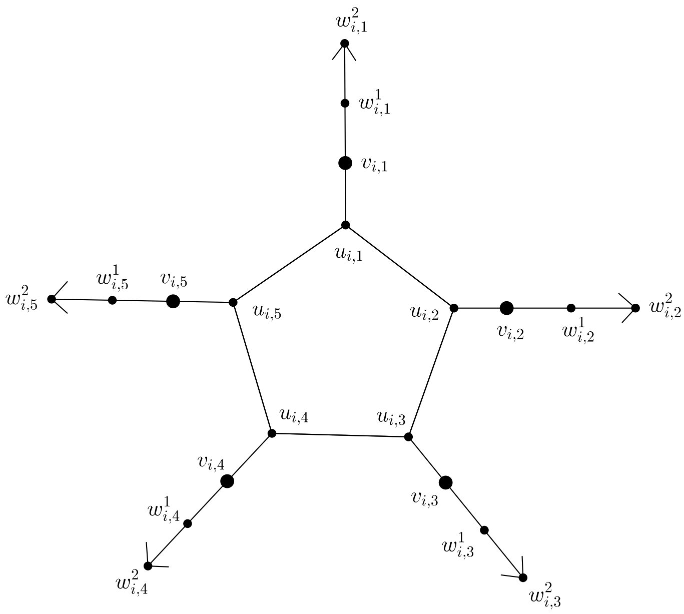

Figure 5: The undirected edges of the graph are bidirected edges. L:=last, NL:=next to last.

(u_{i,j_i}, v_{i,j_i}) is a 2-cycle of the exchange. (Same proof as in lemma 3.2.) The nodes corresponding to $v_{i,j_i},\ i = 1, ..., k$ form a k-clique in G.

Now suppose there is a k-clique in G. We may assume that the nodes corresponding to the nodes of the k-clique are $v_{1,1}, ..., v_{k,1}$. It is easy to see that the exchange defined by the following 2-cycles and chains is 2-way stable.
$(u_{i,1}, v_{i,1}),\ (u_{i,j} u_{i,j+1}),\ i = 1, ..., k,\ j = 2, 4, ..., n_i - 1,$
$w_{i,1}^1 ... w_{i,1}^l,\ v_{i,j} w_{i,j}^1 ... w_{i,j}^l,\ i = 1, ..., k,\ j = 2, ..., n_i.$

□

Theorem 6.7. *The decision problem of finding a 2-way stable pairwise exchange with chains of length 1 is NP-complete for the first two types of stability.*

Proof. We follow the proof of the previous theorem, with a slight change of the construction. We have the construction of theorem 6.6 with $l = 1$, but this does not work for the second type of stability, because $\{v_{i,1}, w_{i,1}^1\}$ blocks the described exchange. So to each altruist we add a new arc with a new tail pointing to it. $((y_{i,j} w_{i,j}^1),\ i = 1, ..., k,\ j = 1, ..., n_i.)$ This modification does not change the first part of the proof, however, we need to change the exchange described in the second part of the proof. We keep what we had, and add new chains: $y_{i,1} w_{i,1}^1,\ i = 1, ..., k$ to get a stable exchange with chains for the second type of stability. □

Theorem 6.8. *There is a polynomial time algorithm that finds a 2-way stable pairwise exchange with chains of length 1 if there is one or reports that none exists, for the third type of stability.*

Proof. Algorithm: 1) Delete every arc for which its tail prefers an altruist over its head. 2) Delete the altruists. 3) Delete every non-bidirected edge. 4) Replace every bidirected edge with an undirected edge. 5) Run Irving's algorithm [9] for finding a (not necessarily complete) stable matching in an instance of SRI. If it reports that no stable matching exists report that no 2-way stable pairwise exchange with chains of length 1 exists. If it finds a stable matching, consider the altruists and the arcs pointing to them after step 1. If the subgraph determined by the uncovered nodes of the stable matching and the altruists consists of independent arcs and isolated nodes, this and the matching defines a 2-way stable pairwise exchange with chains of length 1. If not there is none.

The arcs we deleted in step 1 cannot belong to a stable exchange, because the tail and the preferred altruist would form a blocking pair. These arcs cannot belong to a blocking pair either, because every node gets his most preferred altruist or better in a stable exchange. Therefore what we get after step one admits a stable solution if and only if the original does.

Suppose that after step 1 there is a 2-way stable pairwise exchange with chains of length 1. If we delete the altruist nodes, the 2-cycles determine a 2-way stable 2-way exchange in the remaining graph. This is because after step 1, if there is an arc from a node to an altruist, that altruist is the last item on that node's preference list. There are no two uncovered nodes in this 2-way stable 2-way exchange, that there is an arc from each of them to the same altruist, because then the original exchange would not be stable. The 2-way stable 2-way exchange problem is equivalent to SRI, and we know that the uncovered nodes are the same in every stable matching [9]. From this the correctness of the rest of the algorithm is straightforward. □

Claim 6.9. *There is a polynomial time algorithm that finds a 2-way stable pairwise exchange with chains if there is one, or reports that none exists, if the number of altruists and the length of the chains in the exchange are restricted by given constants.*

Proof. Suppose there are some given disjoint chains ending in altruists. We show that we can decide in polynomial time, if we can complete the given chains with 2-cycles such that we obtain a 2-way stable pairwise exchange with chains.

First for every node that belongs to a chain, we delete every arc leaving this node with an endpoint over whom he prefers the node he gets in the chain. These arcs cannot belong to a blocking pair. If we delete the arcs of the chains and the altruist nodes too, there is a 2-way stable 2-way exchange in the remaining digraph so that the nodes of the chains are uncovered, if and only if we can complete the given chains with 2-cycles such that we obtain a 2-way stable pairwise exchange with chains. (The uncovered nodes are the same in every 2-way stable 2-way exchange [9].)

There are only constant many ways for the possible chains in the exchange, therefore if we check every possible way, the algorithm is still polynomial. □

Remark 6.10. *All the NP-completeness theorems in this section apply for the special cases when all the edges are bidirected edges, except for the ones with an altruist endpoint.*

7 Parameterized complexity

Problems with some fixed parameter k are called **parameterized problems**. A parameterized problem is **fixed parameter tractable (fpt)** if it can be solved in time $f(k)|x|^{O(1)}$ where k is the fixed parameter, f is a function only depending on k and $|x|$ is the input size. **FPT** is the set of fixed parameter tractable problems.

There are some parameterized problems, for example independent set and k-clique for which no fpt algorithm is known. The established way in complexity theory of comparing the hardness of problems is by formulating appropriate notions of reducibility and completeness.

A **parameterized reduction** from a parameterized problem P to a parameterized problem Q maps an instance x with parameter k to an instance x' with parameter k' in $f(k)|x|^{O(1)}$ time, such that x is a yes instance of P if and only if x' is a yes instance of Q, and $k' \leq g(k)$ for some function g only depending on k.

For the notion of completeness, Downey and Fellows [5] defined the W-hierarchy of parameterized problem classes. To describe that, we need a few definitions.

A **Boolean circuit** consists of input gates, END gates, OR gates, negation gates and one output gate. The **depth** of a circuit is the maximum length of a path from an input to the output. The **weft** of a circuit is the maximum number of gates having in-degree greater than two in any path from an input to the output. The **weighted circuit satisfiability** problem is defined as follows.
Instance: A Boolean circuit C
Parameter: $k \in \mathbb{N}$
Question: Is there an assignment with exactly k true values such that the output is true.
The set of circuits having weft at most t and depth at most d is denoted by $\mathbf{C[t, d]}$. $\mathbf{W[t]}$ is the class of parameterized problems P, such that there exists a constant d and a parameterized reduction from P to the weighted circuit satisfiability problem of $C[t, d]$.

$$FPT \subseteq W[1] \subseteq W[2] \subseteq ...$$

A parameterized problem P is $W[t]$-**complete** if it is in $W[t]$ and there is a parameterized reduction from every problem in $W[t]$ to P. A parameterized problem P is $W[t]$-**hard** if there is a parameterized reduction from a $W[t]$-complete problem to P. It has been proved in [6] that the k-clique in k-partite graph problem is $W[1]$-complete.

Theorem 7.1. *The 2-way stable 3-way exchange problem with the number of 3-cycles in the exchange as a parameter is $W[1]$-hard even in complete digraphs.*

Proof. The reduction described in the NP-hardness proof the 2-way stable 3-way exchange problem (theorem 3.8) is a parameterized reduction from the W[1]-complete

k-clique in k-partite graph problem since the number of 3-cycles in the constructed 2-way stable 3-way exchange is k. □

Remark 7.2. *The (b-way) stable l-way exchange problem is $W[1]$-hard for any $b \geq 2$ and $l = 3, 4$, because the proof of the NP-hardness is from the same reduction as in theorem 3.8 (remarks 3.11 and 3.12).*

Theorem 7.3. *The 2-way stable pairwise exchange with chains problem is $W[1]$-hard even if the number of altruists is restricted to 1, if the parameter is the length of the longest chain in the exchange.*

Proof. The reduction described in the NP-hardness proof the 2-way stable pairwise exchange with chains problem (theorem 6.1) is a parameterized reduction from the W[1]-complete k-clique in k-partite graph problem since the length of the only chain of the constructed 2-way stable pairwise exchange with chains is at most $6k$. □

References

[1] P. Biró, E. McDermid, Three-Sided Stable Matchings with Cyclic Preferences, Algorithmica Volume 58, Issue 1 (2010), 5-18

[2] P. Biró, The stable matching problem and its generalizations: an algorithmic and game theoretical approach, PhD thesis, Budapest University of Technology and Economics, 2007

[3] P. Biró, D. F. Manlove, R. Rizzi. "Maximum weight cycle packing in directed graphs, with application to kidney exchange programs."Discrete Mathematics, Algorithms and Applications 1.04 (2009): 499-517.

[4] J. P. Dickerson, A. D. Procaccia, T. Sandholm, Optimizing Kidney Exchange with Transplant Chains: Theory and Reality. Proceedings of the 11th International Conference on Autonomous Agents and Multiagent Systemsb(AAMAS 2012)

[5] R. Downey, M. Fellows, Fixed-parameter tractability and completeness I: Basic theory, SIAM J. Comput. 24 (1995), 873–921

[6] M. R. Fellows, D. Hermelinb, F. Rosamonda, S. Vialette, On the parameterized complexity of multiple-interval graph problems. Theoretical Computer Science 410 (2009) 53–61

[7] D. Gale, L. S. Shapley. College Admissions and the Stability of Marriage. Amer. Math. Monthly, 69(1):9–15, 1962.

[8] R.W. Irving, An efficient algorithm for the stable roommates" problem. Journal of Algorithms (1985) 6

[9] R. W. Irving, D. F. Manlove, (2002) The stable roommates problem with ties. Journal of Algorithms 43(1):85-105.

[10] R.W. Irving. The cycle-roommates problem: a hard case of kidney exchange. Informa- tion Processing Letters, 103:1–4, 2007.

[11] A. Roth, T. Sonmez, M. U. Unver. Kidney exchange. No. w10002. National Bureau of Economic Research, 2003.

[12] A. Roth, A. Postlewaite, Weak versus strong domination in a market with indivisible goods. Economics Letters (1977) 9

[13] C. Ng, D. S. Hirschberg. Three-dimensional stable matching problems. SIAM J. Discrete Math., 4(2):245–252, 1991.

[14] L. Shapley, H. Scarf, On cores and indivisibility. Journal of Mathematical Economics (1974) 1/1

[15] J.J.M. Tan. A necessary and sufficient condition for the existence of a complete stable matching. Journal of Algorithms, 12:154 178, 1991.