Invariants of Knots from Grid Homologies

Master's Thesis

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Introduction

The aim of this thesis is to develop a homology theory using grid diagrams of knots, and to introduce a knot invariant, called Υ using these grid homologies.

Knot theory has been occupying scientists of various disciplines since the ancient times [20]. Although their tools have gone under great development, the main problem remained the same: we would like to distinguishing knots, and find out whether two knots are the same (that is, they can be moved to each other via certain operations), or one can be untied until it becomes an unknot.

The idea of grid diagrams is not new. Former versions have been studied for more than a century as convenient tools for understanding knots and links [3], [4], [9], [11], [5]. Since every oriented link can be represented by a grid diagram (see Chapter 2 for the basic definitions), the advantage of this theory is that knots and links can be examined using combinatorical methods. In the second half of the twentieth century, knot theory was connected with more and more different areas of mathematics, which lead to outstanding results.

In recent years, knot theory has undergone significant progress, mostly because of algebraic concepts, such as knot Floer homology, Heegaard Floer homology and the generalizations of knot polynomials. Grid diagrams have also been reconsidered [10], [12], [19] to build up grid homology theories which enabled mathematicians to approach problems algebraically. Several knot invariants were introduced, that may be improved in the future to turn into more effective ones.

This work is based on the book Grid Homology for Knots and Links [16] of Ozsváth, Stipsicz and Szabó, where they used grid homologies to develop the concordance invariant τ first defined in [15]. Υ is a one-parameter family of invariants, in addition, an upgraded version of τ. It was first introduced by Ozsváth, Stipsicz and Szabó in [13] using knot Floer homology. The significance of this thesis is that we introduce the Υ invariant in a different approach by building up a grid homology theory. This gives us the opportunity to examine questions combinatorically, which may be an effective way of solving open problems in the future.

1. Knots and links

1.1 Basic notions

Definition 1.1. *A* knot K *is a smooth embedding of* S^1 *into* S^3. *An* ℓ*-component* link L*, is the disjoint union of* ℓ *knots.*

If $L = K_1 \cup K_2 \cup ... \cup K_\ell$, we call K_i the i^{th} component of L. We use the notation $\vec{L}$ if the link L is oriented. The *trivial knot* shown in Figure 1.1 is also called the *unknot*. $\mathcal{U}_d(L)$ denotes the link obtained by adding d unlinked unknots to the link L.

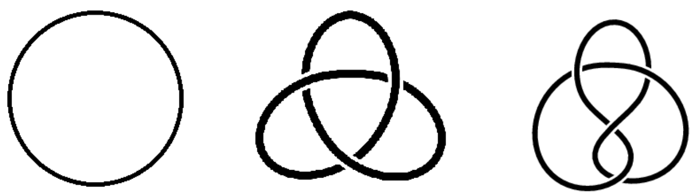


Figure 1.1: The unknot, the trefoil knot and the figure-eight knot

Definition 1.2. *The links* L_1 *and* L_2 *are* ambiently isotopic *if there exists a smooth map* $h : S^3 \times [0,1] \to S^3$ *such that* $h_t = h|_{S^3 \times \{t\}}$ *is a diffeomorphism for every* $t \in [0,1]$, $h_0 = id_{S^3}$, $h_1(\vec{L}_1) = \vec{L}_2$*, and* h_1 *preserves the orientation on the components.*

Definition 1.3. *We say that the two links* L_1 *and* L_2 *are* equivalent *if they are ambiently isotopic. We denote their equivalence by* $L_1 \sim L_2$. *The equivalence class of a link under this equivalence relation is called the* type *of the link.*

Definition 1.4. *Two* ℓ*-component links* $\vec{L}_1$ *and* $\vec{L}_2$ *defined by the smooth maps* $f_1, f_2 : \bigcup_{j=1}^{\ell} S^1 \to S^3$ *respectively, are* isotopic *if* f_1 *and* f_2 *are isotopic, that is, there exists a smooth map* $F : (\bigcup_{j=1}^{\ell} S^1) \times [0,1] \to S^3$ *with the property that for each* $t \in [0,1]$ $F_t = F|_{(\cup_{j=1}^{\ell} S^1) \times \{t\}}$ *are* ℓ*-component links with* $F_0 = f_1$ *and* $F_1 = f_2$.

It is known by the isotopy extension theorem [6] that two links are ambiently isotopic if and only if they are isotopic.

Remark 1.5. *We can also define the equivalence of links* $\vec{L}_1$ *and* $\vec{L}_2$ *by the existence of an orientation-preserving diffeomorphism* $f : S^3 \to S^3$ *with* $f(\vec{L}_1) = \vec{L}_2$.
Note that this gives the same equivalence relation since the group of orientation-preserving diffeomorphisms of S^3 *is connenced.*

Definition 1.6. *The* mirror image $m(L)$ *of a link* L *is obtained by reflecting* L *through a plane in* $\mathbb{R}^3$*. Reversing orientations of all the components of* $\vec{L}$ *gives* $-\vec{L}$.

Remark 1.7. *Instead of* S^3 *we can consider knots and links in* $\mathbb{R}^3 = S^3 \setminus \{p\}$*. The theory is the same in this case: two knots in* $\mathbb{R}^3$ *are equivalent if and only if they are equivalent when viewed in* S^3.

1.2 Invariants of knots and links

The main problem of knot theory is to distinguish knots from each other. For this reason it is very useful to introduce quantities that stay unchanged under isotopy. Since these invariants seem to be the best tool of the classification, creating new ones is a huge goal of modern researches.

Definition 1.8. *A function* f *is a link invariant if it is defined on the set of links and has the property that* $f(L_1) = f(L_2)$ *for* $L_1 \sim L_2$*.* f *is a complete invariant of links if* $L_1 \sim L_2 \Leftrightarrow f(L_1) = f(L_2)$.

We distinguish many types of knot invariants including numerical invariants, polynomial invariants, algebraic knot invariants. In the following section we define knot diagrams, then later in this work we introduce an algebraic invariant using grid homology theory (see Chapter 4 for the details).

1.2.1 Knot diagrams

Knot diagrams serve as fundamental example: not only they play an important role in visualizing knots, but also there are many invariants that are defined by using knot diagrams, such as the unknotting number, the crossing number, the bridge number, knot polynomials (for example the Alexander-polynomial [1], the Kauffman-polynomial [8], the Jones-polinomial [7]), or some algebraic invariants.

Shortly, a knot diagram is a picture of a projection of a knot onto a plane, where we allow at most double points. Consider a link $L \subset \mathbb{R}^3$, and an oriented plane $P \subset \mathbb{R}^3$. Let $pr_P : \mathbb{R}^3 \to P$ be the orthogonal projection to P. For a generic choice of P the projection pr_P restricted to L is an immersion with finitely many double points. At the double points, we illustrate the strand passing under as an interrupted curve segment. If L is oriented, we place an arrow on the diagram tangent to each component of L. These arrows determine the orientation of $\vec{L}$. The resulting diagram $\mathcal{D}$ is called a *knot or link diagram* of $\vec{L}$. It is obvious that a link diagram determines the link type.

Definition 1.9. *The* sign *of a crossing in the diagram of an oriented link is determined by a neighbourhood of the crossing being homeomorphic to one of the figures below:*

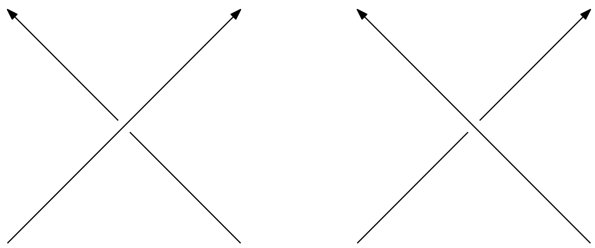


Figure 1.2: A neighbourhood of a positive (on the left) and a negative (on the right) crossing

Definition 1.10. *A link diagram $\mathcal{D}$ is called* alternating *if the crossings alternate between positive and negative ones as we traverse each component of the link. A link admitting an alternating diagram is called an* alternating link*.*

Modifying the sign of a crossing changes the knot itself. Despite of this, sometimes it is useful to apply the following operation. A *crossing change* in a knot is a modification when we move around the knot in the three-space and allow two different strands of it to pass through one another transversely. Every knot can be transformed into the unknot after a finite sequence of crossing changes.

Definition 1.11. *The minimal number of crossing changes required to convert a knot K into the unknot is called the* unknotting number *of K and is denoted by $u(K)$.*

Now we introduce some local modifications of a link diagram: the Reidemeister moves, first described by Alexander and Briggs in 1926, [2] then independently by Reidemeister [18] in 1927. There are three types of them illustrated in Figure 1.3, 1.4 and 1.5:

(R-1) Twisting or untwisting a strand,

(R-2) Moving a loop over another strand or removing it from that,

(R-3) Sliding a string over or under a crossing.

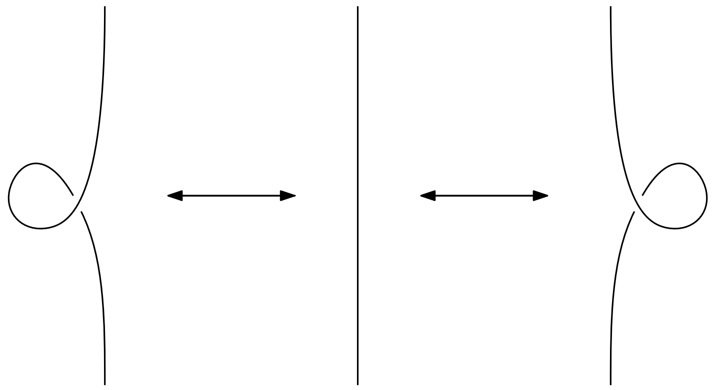


Figure 1.3: Reidemeister moves of type (R-1): twisting or untwisting a strand

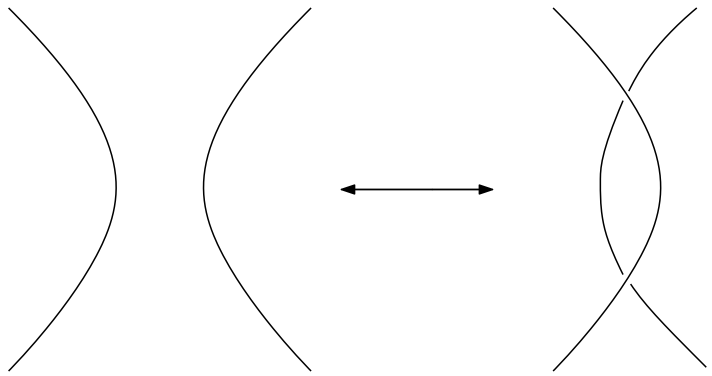


Figure 1.4: Reidemeister moves of type (R-2): moving a loop over another strand or removing it from that

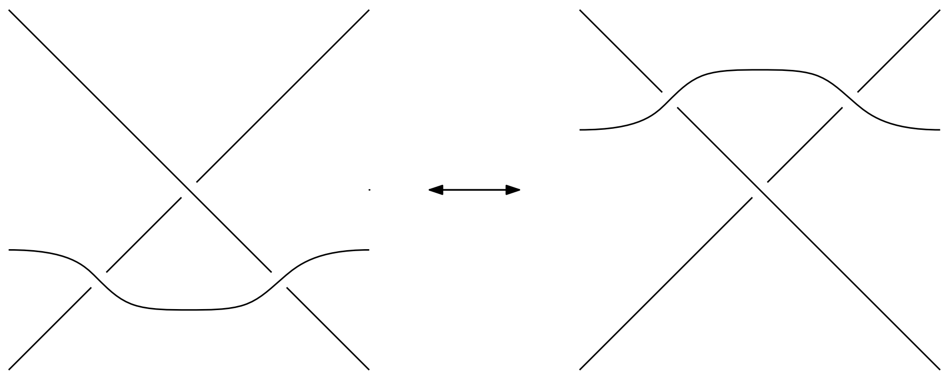


Figure 1.5: Reidemeister moves of type (R-3): Sliding a string over or under a crossing

The other parts of the diagram stay unchanged. It is easy to see that the Reidemeister moves preserve the link type.

The importance of these local changes comes from the following theorem of Reidemeister [17]:

Theorem 1.12. *Two link diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ represent equivalent links if and only if they can be transformed to each other by a finite sequence of Reidemeister moves and planar isotopies.*

Using this theorem it becomes much easier to decide whether a function defined on links is a link invariant or not: it is enough to check its invariance under the Reidemeister-moves.

1.3 The slice genus

Another important knot invariant is the slice genus defined in the following way:

Definition 1.13. *The* slice surface *of the knot K is an oriented surface F smoothly embedded into D^4 such that its boundary is K, that is, if $(F, \partial F) \subset (D^4, \partial D^4 = S^3)$ by a C^∞ embedding, with $\partial F = K$.*

Definition 1.14. *The* slice genus *or* four-ball genus *of a knot K is the minimal genus of a slice surface of K, i.e. the integer*

$$g_s(K) = g_4(K) = \min\{g(F) | (F, \partial F) \subset (D^4, S^3) \text{ is a slice surface for } K\}.$$

A knot K is a slice knot *if $g_s(K) = 0$, that is, if it admits a slice disk.*

The slice genus is related to the unknotting number by the inequality $g_s(K) \leq u(K)$ (see Section 2.6 in [16]). However, there are knots K for which the difference $u(K) - g_s(K)$ are arbitrarily large. Although there are known estimates, similarly to the unknotting number, the slice genus is also poorly understood.

Remark 1.15. *The slice genus also provides a connection between knot theory and 4-dimensional topology. For the details see Section 8.6 in [16].*

Now let us rephrase the notion of the slice genus using the following definition, to get an interesting result.

Definition 1.16. *Let $\vec{L}_0, \vec{L}_1 \subset S^3$ be two oriented links. A* cobordism *between $\vec{L}_0$ and $\vec{L}_1$ is a smoothly embedded, compact, oriented surface-with-boundary $W \subset S^3 \times [0,1]$ such that $W \cap (S^3 \times \{0\}) = \vec{L}_0 \times \{0\}$ and $W \cap (S^3 \times \{1\}) = \vec{L}_1 \times \{1\}$; on the two boundary components the orientation of W induces the orientation of $\vec{L}_0$ and the negative of the orientation of $\vec{L}_1$.*

The slice genus of a knot K can be determined as the minimal genus of a cobordism connecting K with the unknot. One of the main applications of the Υ invariant defined later in this work (see Chapter 4) is an inequality (Theorem 4.21), which also gives a lower bound for the slice genus, see Corollary 1.23 in the next section.

1.4 In advance to the Υ knot invariant

The main goal of this thesis is to introduce a knot invariant, called Υ (see Chapter 4). For a knot K and a parameter $t \in [0, 2]$ this invariant $\Upsilon_K(t)$ gives a real number. Later we will verify that this stays unchanged under isotopies.

The significance of this work is that it gives a new approach to the introduction of Υ. Altough Ozsváth, Stipsicz and Szabó [13], [14] have already defined it using knot Floer homology, this new bulid-up of the theory is very effective because it gives the opportunity to examine problems combinatorically.

In this section we state some properties and theorems about the Υ invariant. For this we need the following definition:

Definition 1.17. *Let $K_0, K_1 \subset S^3$ be two knots. A* concordance *between K_0 and K_1 is a smooth function $f : S^1 \times [0, 1] \to S^3 \times [0, 1]$ such that $f(S^1 \times \{0\}) = K_0 \times \{0\}$ and $f(S^1 \times \{1\}) = K_1 \times \{1\}$.*

Note that concordance is a special kind of cobordism, where the surface is an annulus.

Applying this notion, we can introduce an equivalence relation on the set of knots: if K_1 and K_2 are two knots, let us say that $K_1 \sim K_2$ if there is a concordance between them. Consider the connected sum operation on knots, which creates one knot from K_1 and K_2 in the following way: Suppose, that K_1 and K_2 can be separated by an S^2. Take an embedded rectangle in S^3 that has two opposite sides incident to K_1 and K_2, then cut out these two arcs from the two knots and replace them by the other two sides of the rectangle. This gives us a knot $K_1 \# K_2$, the *connected sum* of K_1 and K_2. This can be generalized to links, so that we take an embedded rectangle in S^3 that has two opposite sides incident to two arbitrary arcs of the link, then do the same replacement as above. This operation is called a *saddle move*. If we take k such disjoint, embedded rectangles, and repeat this procedure k times, then we say that we perform k simultaneous saddle moves. Observe that while the connected sum of two knots is independent from the choice of the rectangle, the result of a saddle move depends on the chosen rectangle.

It can be shown that concordance classes of knots equipped with the connected sum operation form a group, called the concordance group.

The following two topological properties of $\Upsilon_K(t)$ are proved in [13] of Ozsváth, Stipsicz and Szabó:

Proposition 1.18. *Υ_K is additive under connected sum of knots:*

$$\Upsilon_{K_1 \# K_2}(t) = \Upsilon_{K_1}(t) + \Upsilon_{K_2}(t).$$

Proposition 1.19. *For the mirror image $m(K)$ of the knot K, $\Upsilon_{m(K)}(t) = -\Upsilon_K(t)$.*

Using the above propositions, the following theorem can be verified:

Theorem 1.20. *If the knots K_1 and K_2 can be connected with a concordance, then $\Upsilon_{K_1}(t) = \Upsilon_{K_2}(t)$ for a fixed $t \in [0,2]$. Furthermore, $[K] \mapsto \Upsilon_K(t)$ is a group homomorphism from the concordance group to $\mathbb{R}$.*

For some classes of knots, the Υ_K invariant can be explicitly computed.

Theorem 1.21. *For an alternating knot K,*

$$\Upsilon_K(t) = (1 - |t-1|) \cdot \frac{\sigma}{2},$$

where σ is the signature of the knot K, defined in Chapter 2 of [16].

For torus knots, the Alexander polynomial [1] determines $\Upsilon_K(t)$. The proof of these above claims can be found in [13].

Finally, the next inequality is an important application of the Υ invariant, which will be proved in Chapter 4, as Theorem 4.21. The corollary of this is a bound for the slice genus, mentioned in the previous section.

Theorem 1.22. *Suppose that K_1 and K_2 are two knots that can be connected by a genus g cobordism in $[0,1] \times S^3$. Then for $t \in [0,1]$*

$$|\Upsilon_{K_1}(t) - \Upsilon_{K_2}(t)| \leq t \cdot g.$$

Corollary 1.23. *For a knot K and a parameter $t \in [0,1]$ the invariant $\Upsilon_K(t)$ bounds the slice genus of K:*

$$|\Upsilon_K(t) \leq t \cdot g_s(K)|.$$

2. Grid diagrams

2.1 Planar and toric grid diagrams

Definition 2.1. *A* planar grid diagram $\mathbb{G}$ *with* grid number n *is a square on the plane with n rows and n columns of small squares (i.e. an $n \times n$ grid), marked with X's and O's in a way that no square contains both X and O, and each row and each column contains exactly one X and one O.*

We use the notation $\mathbb{X}$ for the set of squares marked with an X, and $\mathbb{O}$ for the ones containing an O. Sometimes we label the markings: $\{O_i\}_{i=1}^n$, $\{X_i\}_{i=1}^n$. Every grid diagram $\mathbb{G}$ determines a diagram of an oriented link in the following way: In each row connect the O-marking to the X-marking, and in each column connect the X-marking to the O-marking with an oriented line segment, such that the vertical segments always pass over the horizontal ones. This way we get closed curves: the diagram of the oriented link $\vec{L}$ given by the grid diagram. We call $\mathbb{G}$ a *grid diagram for* $\vec{L}$.

The converse is also true:

Proposition 2.2. *Every oriented link in S^3 can be represented by a grid diagram.*

Proof. It is easy to see that every oriented link $\vec{L}$ is isotopic to a piecewise linear oriented link $\vec{L}'$, and we can also assume that the diagram of $\vec{L}'$ contains only horizontal and vertical segments. Applying the local modification indicated in Figure 2.1 we can achieve that the vertical segments become over-crossings at every crossing. Move the link into general position so that different segments are not collinear, and mark the turns by O's and X's in a way that each vertical segment points from X to O, while the horizontal ones point from O to X. This way we created a grid diagram representing $\vec{L}$.

□

Definition 2.3. *A* toroidal grid diagram *can be obtained by identifying the opposite sides of a planar grid diagram: its top boundary segment with its bottom one and its left boundary segment with its right one.*

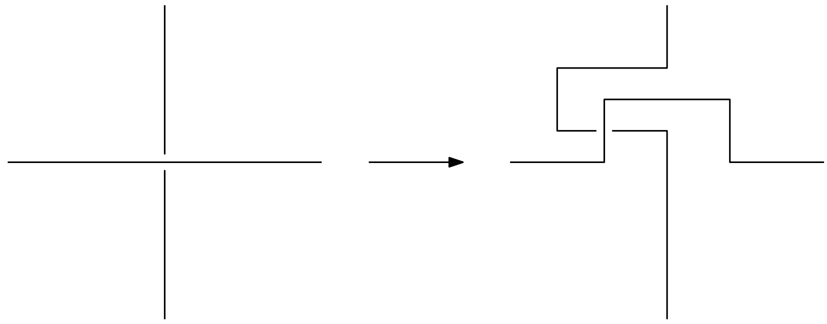


Figure 2.1: A local modification at a crossing

This way we transfer the planar grid diagram onto a torus, on which the horizontal and vertical segments separating the rows and columns of the squares become horizontal and vertical circles. The torus inherits its orientation from the plane.

The converse of this procedure also makes sense:

Definition 2.4. *A* planar realization *of a toroidal grid diagram is a planar grid diagram obtained by cutting up the toroidal grid diagram along a horizontal and a vertical circle.*

It is easy to see that there are n^2 different ways to do this, and that two different planar realizations of the same grid diagram represent isotopic links.

Definition 2.5. *A* cyclic permutation *of a planar grid diagram* $\mathbb{G}$ *is a process in which we permute the rows or the columns of* $\mathbb{G}$ *cyclically.*

Note that a cyclic permutation has no effect on the induced toroidal grid diagram, and that two different planar realizations of a toroidal grid diagram can always be connected by a sequence of cyclic permutations.

Regarding the oriented coordinate axes on the plane, the induced toroidal grid diagram inherits orientations on the horizontal and vertical circles. As a result, we get four preferred directions at every point in the torus, which we can think of as *North*, *South*, *East* and *West*. This way we can assign a northern, a southern, an eastern and a western edge to each of the squares of the toroidal grid diagram.

2.2 Grid moves

In 1995 Cromwell [4] introduced a list of alterations of grid diagrams that are the equivalent of Reidemeister moves for knot diagrams, and do not change the knot type. First we consider the two main types of moves on planar grid diagrams, then we extend these to toroidal grid diagrams. We will use the notations of [16].

Definition 2.6. *In a grid diagram $\mathbb{G}$ every column determines a closed interval of real numbers that connects the height of its O-marking with the height of its X-marking. Consider a pair of consecutive columns in $\mathbb{G}$, and suppose that the two intervals associated to them are either disjoint, or one is contained in the interior of the other (Figure 2.2). We say that the diagram $\mathbb{G}'$ differs from $\mathbb{G}$ by a* column commutation, *if it can be obtained by interchanging two consecutive columns of $\mathbb{G}$ that satisfy the above condition. A* row commutation *is defined analogously, using rows in place of columns. Column or row commutations collectively are called* commutation moves.

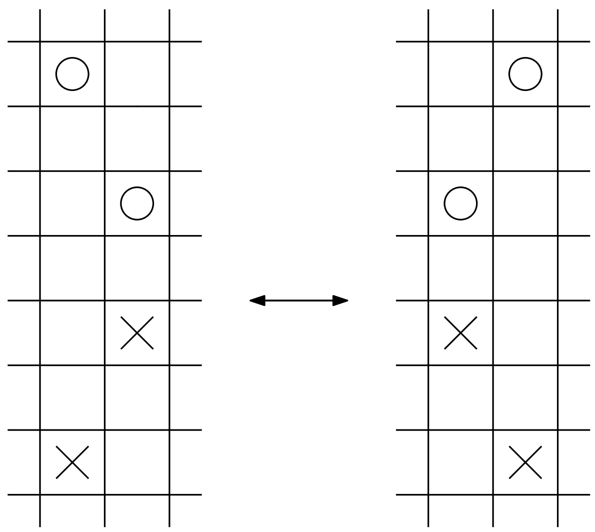


Figure 2.2: The commutation move

While commutation moves preserve the grid number, the second type of grid moves will change it.

Definition 2.7. *Let $\mathbb{G}$ be an $n \times n$ grid diagram. We say that the $(n+1) \times (n+1)$ grid diagram $\mathbb{G}'$ differs from $\mathbb{G}$ by a* stabilization, *(or that $\mathbb{G}'$ is the* stabilization *of $\mathbb{G}$) if it can be obtained from $\mathbb{G}$ in the following way: Choose a marked square in $\mathbb{G}$, and erase the marking in it, in the other marked square in its row and in the other marked square in its column. Then split the row and the column of the chosen marking in $\mathbb{G}$ into two, that is, add a new horizontal and a new vertical line to get an $(n+1) \times (n+1)$ grid. There are four ways to insert markings in the two new rows and columns to have a grid diagram, see Figure 2.3 for the case where the initial square in $\mathbb{G}$ was marked with an X. The new grid diagram, $\mathbb{G}'$ can be any of these four.*

We distinguish these types of stabilizations with notations depending on which marking it modifies and what direction the empty cell is: First, notice that the original marked square got subdivided into four cells, creating a 2×2 block of squares. We marked exactly three of these new cells. The first letter X or O of the code of the

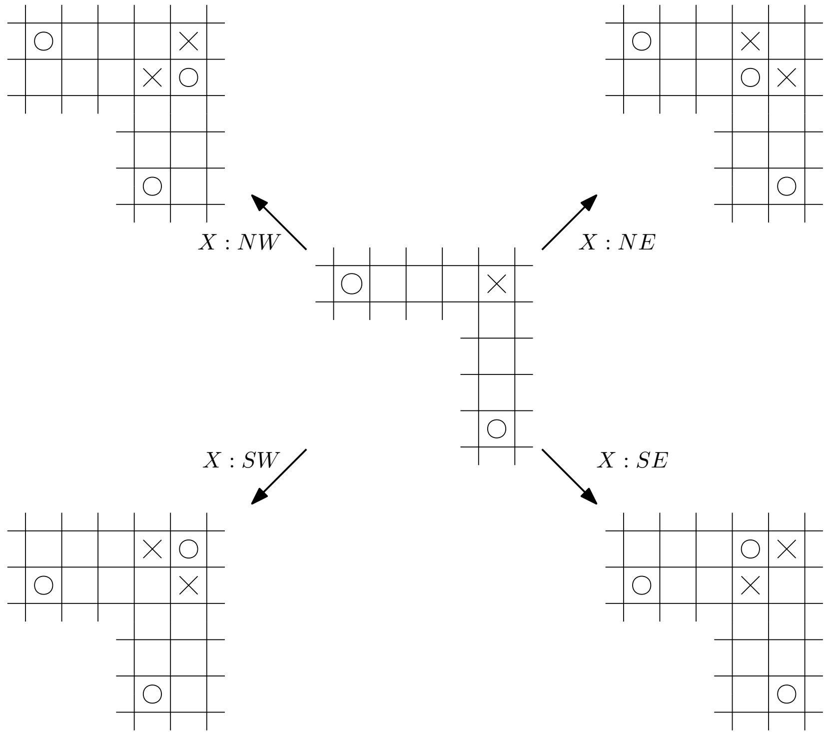


Figure 2.3: The four types of stabilization

stabilization type depends on which marking was included by the original square that we had chosen. (This letter appears twice in the new 2×2 block.) The second part of the notation signifies the position of the small square in the 2×2 block which remained empty. This we indicate by a direction: northwest (NW), southwest (SW), southeast (SE) or northeast (NE). So if the initial square contained an X-marking, the four possible types of stabilization (also shown in Figure 2.3) are denoted by $X : NW$, $X : SW$, $X : SE$ and $X : NE$. It is easy to see that a stabilization changes the projection either by a planar isotopy or by a Reidemeister move of type (R-1), i.e. a twist.

The inverse of a stabilization is a *destabilization*. Commutation, stabilization and destabilization together are called *grid moves* or Cromwell moves.

The following theorem of Cromwell [4] shows us that grid moves and grid diagrams are really effective tools for creating knot invariants:

Theorem 2.8. *Two toroidal grid diagrams represent equivalent links if and only if there is a finite sequence of grid moves that transform one into the other.*

Grid moves can be transmitted naturally for the case of toroidal grid diagrams: two toroidal grid diagrams differ by a commutation move/a stabilization, if they have planar realizations that differ by a commutation move/a stabilization. The classification of the types of stabilizations works the same way as in the planar case.

In the definition of commutation move we required the two consecutive columns or rows to be either disjoint or one of them to be contained in the interior of the other. But there are two other possibilities, and considering these, we can introduce two further moves between grid diagrams:
Suppose that the grid diagram $\mathbb{G}$ has two consecutive columns, such that the X-marking in one of them occurs in the same row as the O-marking in the other. We say that $\mathbb{G}'$ is related to $\mathbb{G}$ by a *switch*, if it can be obtained from $\mathbb{G}$ by swapping a pair of such consecutive columns. Similarly, if two consecutive rows have an X- and an O-marking in the same column, interchanging them is also called a switch (Figure 2.4). It is easy to see that grid diagrams that differ by a switch determine the same link type. Even though a switch can be expressed as a sequence of commutations, stabilizations and destabilizations, some later proofs will become shorter and easier using this operation.

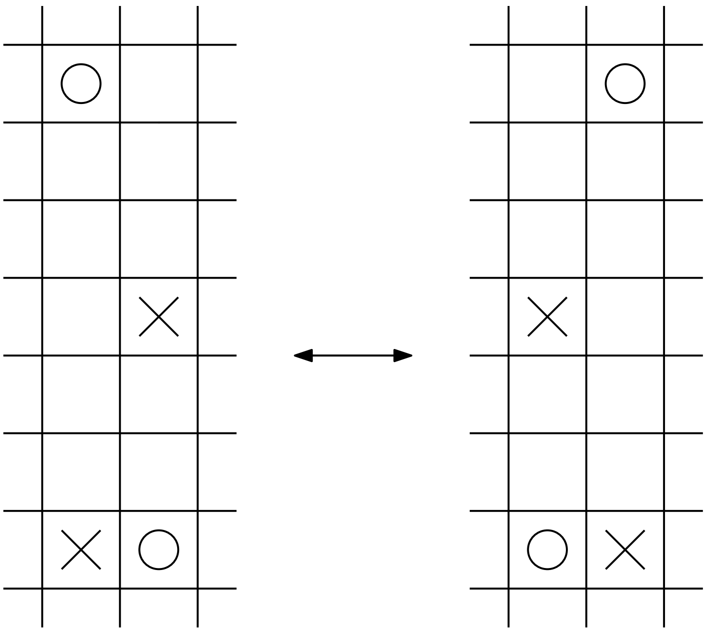


Figure 2.4: A switch of two columns

If we consider two consecutive columns in the grid diagram $\mathbb{G}$, such that the interior of their corresponding intervals are not disjoint, but neither is contained by the other, and interchange these columns to get grid diagram $\mathbb{G}'$, then $\mathbb{G}$ and $\mathbb{G}'$ are related by a *cross-commutation*. We use the same name for the procedure interpreted on rows in place of columns. If $\vec{L}$ and $\vec{L}'$ are two links whose grid diagrams $\mathbb{G}$ and $\mathbb{G}'$ differ by a cross-commutation, then $\vec{L}$ and $\vec{L}'$ are related by a crossing change.

On the torus there are some extra possibilities to express certain grid moves in terms of others: any stabilization can be expressed as a stabilization of type $X : SW$ followed by a sequence of switches and commutations. Moreover, it can be shown that a cyclic permutation is equivalent to a sequence of commutations in the plane, stabilizations and destabilizations of types $X : NW$, $X : SE$, $O : NW$ and $O : SE$.

3. The t-modified grid homology

Our goal in this chapter is to define a version of grid homology. For this, we will introduce the notion of grid states, and construct a chain complex. Furthermore, we need to show that the obtained grid homology is invariant under commutation and stabilization.

3.1 Algebraic background

First of all, we recall some basic notions that will play an important role afterwards.

Definition 3.1. *A set $S \subset \mathbb{R}$ is called* well-ordered*, if every subset of S has a minimal element.*

From this point on we will use the notation

$$\mathcal{R}' = \{\sum_{\alpha \in A} U^\alpha | A \subset \mathbb{R}_{\geq 0},\ A \text{ is well} - \text{ordered }\}$$

for the *long power series ring over* $\mathbb{F}_2$. (We denote by $\mathbb{F}_2$ the field with two elements.)

$\mathcal{R}'$ is indeed a ring with the following operations: For A, $B \subset \mathbb{R}_{\geq 0}$ well-ordered sets let $a = \sum_{\alpha \in A} U^\alpha$ and $b = \sum_{\beta \in B} U^\beta$. Define the sum on $\mathcal{R}'$ in a way that $a + b = \sum_{\gamma \in C} U^\gamma$, where $C = A \cup B \setminus A \cap B$ is the symmetric difference of the index sets.
Notice that C is well-ordered, since the union, the intersection and the difference of well-ordered sets are also well-ordered.
The multiplication on $\mathbb{R}'$ is defined as the Cauchy-product of the elements, that is,

$$\left(\sum_{\alpha \in A} U^\alpha\right) \cdot \left(\sum_{\beta \in B} U^\beta\right) = \sum_{\gamma \in A+B} \left(\sum_{\substack{\alpha \in A, \beta \in B \\ \alpha + \beta = \gamma}} U^\gamma\right).$$

Here $A+B$ is the sumset of A and B. This definition makes sense, because every γ can be written as the sum of some α and β finitely many ways: if we suppose the opposite, then there exist an infinite, strictly decreasing sequence of either the α's or the β's, which contradicts the fact that A and B are well-ordered sets.

Definition 3.2. *An $\mathcal{R}'$-module M is a* graded $\mathcal{R}'$-module *if it admits a splitting* $M = \bigoplus_d M_d$ *over* $\mathbb{F}_2$*, such that* $U^\alpha \cdot M_d \subset M_{d-\alpha}$ *for each* $d \in \mathbb{R}$ *and* $\alpha \in \mathbb{R}_{\geq 0}$*.*

A graded $\mathcal{R}'$-module homomorphism *is a homomorhism* $f : M \to M'$ *between two graded* $\mathcal{R}'$*-modules that preserves the grading, i.e. that sends* M_d *to* M'_d *for every* $d \in \mathbb{R}$*. An* $\mathcal{R}'$*-module homomorphism is called* homogeneous of degree s*, if it maps* M_d *to* M'_{d+s} *for all* $d, s \in \mathbb{R}$*.*

Definition 3.3. *A* graded chain complex *over* $\mathcal{R}'$ *is a pair* (C, ∂)*, where* C *is a graded* $\mathcal{R}'$*-module equipped with the* $\mathcal{R}'$*-module homomorphism* $\partial : C \to C$*. The map* ∂ *is homogeneous of degree -1 and satisfies* $\partial \circ \partial = 0$*.*

Definition 3.4. *Let* (C, ∂) *and* (C', ∂') *be two graded chain complexes over* $\mathcal{R}'$*. A* chain map $f : (C, \partial) \to (C', \partial')$ *is an* $\mathcal{R}'$*-module homomorphism with the property* $\partial' \circ f = f \circ \partial$*. If the chain map* f *is also a graded homomorphism, then* f *is a* graded chain map*.*

Definition 3.5. *An* isomorphism *of graded chain complexes is a graded chain map* $f : (C, \partial) \to (C', \partial')$ *for which there exists another graded chain map* $g : (C', \partial') \to (C, \partial)$ *satisfying the properties* $f \circ g = \mathrm{Id}_{C'}$ *and* $g \circ f = \mathrm{Id}_C$*. Two graded chain complexes are called* isomorphic*, if there is an isomorphism connecting them.*

Considering graded chain complexes, the interpretation of homology is the following:

Definition 3.6. *Suppose that* (C, ∂) *is a graded chain compex. Split* C *into homogeneous submodules as* $C = \bigoplus_d C_d$*. For each* $d \in \mathbb{R}$ *consider the homology module* $H_d = {C_d \cap \operatorname{Ker} \partial} \big/ {C_d \cap \operatorname{Im} \partial}$*. The* homology *of* (C, ∂) *is the* $\mathcal{R}'$*-module* $H(C) = \bigoplus_d H_d$*.*

A graded chain map $f : C \to C'$ between two graded chain complexes over $\mathcal{R}'$ induces a well-defined graded map on homology, $H(f) : H(C) \to H(C')$. If the induced homomorphism is an isomorphism, then f is called a *quasi-isomorphism.*

Definition 3.7. *Let* $f, g : (C, \partial) \to (C', \partial')$ *be two graded chain maps between graded chain complexes over* $\mathcal{R}'$*. An* $\mathcal{R}'$*-module homomorphism* $h : C \to C'$ *is called a* chain homotopy from g to f *if it is homogeneous of degree 1, and satisfies the equality*

$$f - g = \partial' \circ h + h \circ \partial.$$

We say that f *and* g *are* chain homotopic *if there exists a chain homotopy between them.*

It is easy to show that chain homotopic maps induce the same map on homology.

Definition 3.8. *A chain map $f : C \to C'$ is a* chain homotopy equivalence *if there exists a chain map $\phi : C' \to C$ with the property that $f \circ \phi$ and $\phi \circ f$ are both chain homotopic to the respective identity maps. In this case, ϕ is called a* chain homotopy inverse *to f. C and C' are* chain homotopy equivalent *complexes if there is a chain homotopy equivalence connecting them.*

Now we introduce a notion that will be useful in some proofs of Section 3.5.

Definition 3.9. *Let (C, ∂) and (C', ∂') be two chain complexes over $\mathcal{R}$. The* mapping cone *of a chain map $f : C \to C'$ is the chain complex $\mathrm{Cone}(f) = (C \oplus C', \partial_{\mathrm{Cone}})$, where the differential ∂_{Cone} for an element $(c, c') \in C \oplus C'$ is defined as*

$$\partial_{\mathrm{Cone}}(c, c') = (-\partial(c), \partial(c') + f(c)).$$

Proposition 3.10. *Consider the following short exact sequence of chain complexes.*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Then, there exists the long exact sequence of homologies:

$$\cdots \longrightarrow H_{n+1}(C) \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_N(C) \longrightarrow \cdots.$$

Corollary 3.11. *For C, C' chain complexes, a chain map $f : C \to C'$ is a quasi-isomorphism if and only if $H(\mathrm{Cone}(f)) = 0$.*

Proof. There exists a short exact sequence of chain complexes:

$$0 \longrightarrow C' \xrightarrow{\varphi} \mathrm{Cone}(f) \xrightarrow{\psi} C \longrightarrow 0.$$

Consider the associated long exact sequence

$$... \xrightarrow{H_{n+1}(\varphi)} H_{n+1}(\mathrm{Cone}(f)) \xrightarrow{H_n(\psi)} H_n(C) \xrightarrow{H_n(f)} H_n(C') \xrightarrow{H_n(\varphi)} H_n(\mathrm{Cone}(f)) \longrightarrow ....$$

If $H(\mathrm{Cone}(f)) = 0$, it is easy to see that $H(f)$ is both a monomorphism and an epimorphism, that is, $H(f)$ is an isomorphism.

Now suppose that f is a quasi-isomorphism. Then, $H(f)$ is a monomorphism, thus, because of the exactness, $\mathrm{Ker}(H(f)) = \mathrm{Im}(H(\psi)) = 0$. This also means that $H(\varphi)$ is an epimorphism, since $\mathrm{Ker}(H(\psi)) = \mathrm{Im}(H(\varphi)) = H(\mathrm{Cone}(f))$.
But we also know that $H(f)$ is an epimorphism, therefore $\mathrm{Im}(H(f)) = \mathrm{Ker}(H(\varphi)) = H(C')$. Since $H(\varphi)$ maps $H(C')$ to 0, and it is an epimorphism at the same time, we get that $H(\mathrm{Cone}(f)) = 0$. ☐

Proposition 3.12. *Suppose that C is a free, graded chain complex over $\mathcal{R}$ that is bounded above. Then $H(C) \neq 0$ if and only if $H(C/_{U^\alpha} \cdot C) \neq 0$ for a fixed $\alpha \in \mathbb{R}_{>0}$.*

Proof. We assumed that C is free, thus there exists a short exact sequence

$$0 \longrightarrow C \xrightarrow{\cdot U^\alpha} C \longrightarrow C/_{U^\alpha} \cdot C \longrightarrow 0$$

Considering the associated long exact sequence, it is easy to see that if $H(C) = 0$, then $H(C/_{U^\alpha} \cdot C) = 0$.

Now suppose that $H(C) \neq 0$. Since C is bounded above, $H(C)$ has a homogeneous, non-zero element x with maximal grade. Then x cannot be of the form $y \cdot U^\alpha$ for any $y \in H(C)$, else the grade of x was not maximal. Therefore x must inject to $H(C/_{U^\alpha} \cdot C)$, and this way we got a non-zero element of $H(C/_{U^\alpha} \cdot C)$. □

3.2 Grid states and gradings

Since we would like to define a grid homology, we need a chain complex. To have one, first we introduce the concept of grid states, then, for the boundary map, we take rectangles connecting grid states.

Definition 3.13. *A* grid state $\mathbf{x}$ *for a grid diagram $\mathbb{G}$ with grid number n consists of n points in the torus such that each horizontal and each vertical circle contains exactly one element of $\mathbf{x}$. The set of grid states for $\mathbb{G}$ is denoted by $\mathbf{S}(\mathbb{G})$.*

Definition 3.14. *Consider two grid states $\mathbf{x}, \mathbf{y} \in \mathbf{S}(\mathbb{G})$ that overlap in $n-2$ points in the torus. The four points left out are the corners of an embedded rectangle r, that inherits an orientation from the torus. The boundary of r is the union of oriented arcs, two of which are on the horizontal and two on the vertical circles. We say that the rectangle r* goes from $\mathbf{x}$ to $\mathbf{y}$ *if the horizontal segments in ∂r point from the components of $\mathbf{x}$ to the components of $\mathbf{y}$, while the vertical segments point from the components of $\mathbf{y}$ to the components of $\mathbf{x}$.*

For $\mathbf{x}, \mathbf{y} \in \mathbf{S}(\mathbb{G})$, we use the notation $\mathrm{Rect}(\mathbf{x}, \mathbf{y})$ for the set of rectangles going from $\mathbf{x}$ to $\mathbf{y}$. Observe that $\mathrm{Rect}(\mathbf{x}, \mathbf{y})$ is either empty, or it consists of two rectangles. In this latter case $\mathrm{Rect}(\mathbf{y}, \mathbf{x})$ also consists of two rectangles, see Figure 3.1.

The rectangle $r \in \mathrm{Rect}(\mathbf{x}, \mathbf{y})$ may contain elements of $\mathbf{x}$ in its interior $\mathrm{Int}(r)$, and $\mathbf{x} \cap \mathrm{Int}(r) = \mathbf{y} \cap \mathrm{Int}(r)$. Mostly we will work with special rectangles of the following type:

Definition 3.15. *We call a rectangle $r \in Rect(\mathbf{x}, \mathbf{y})$ an* empty rectangle *if*

$$\mathbf{x} \cap \mathrm{Int}(r) = \mathbf{y} \cap \mathrm{Int}(r) = \emptyset.$$

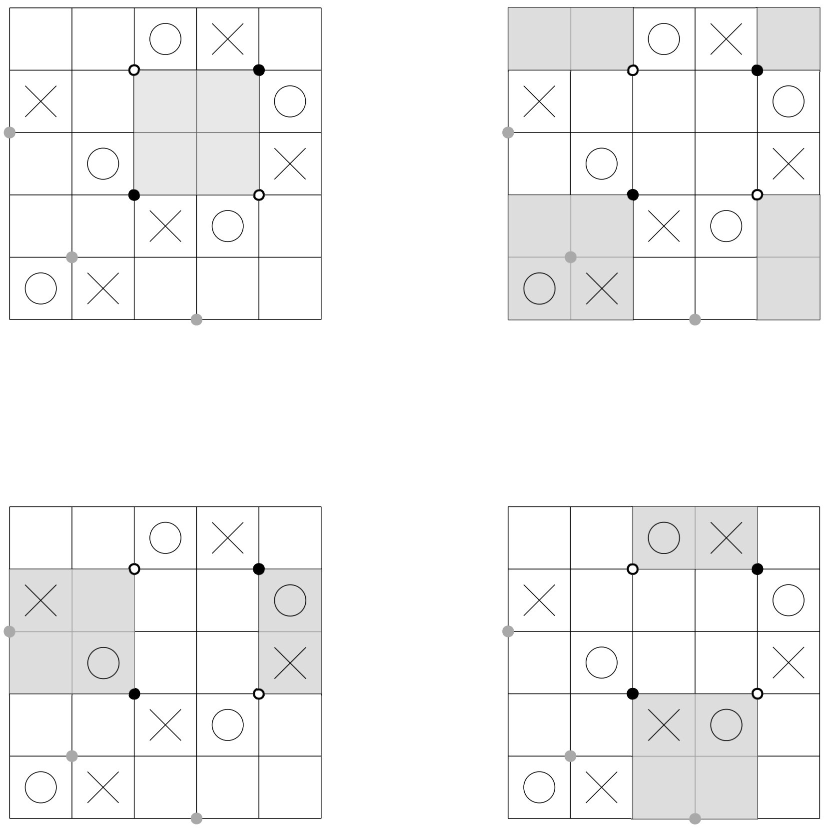


Figure 3.1: Rectangles connecting grid states

The set of empty rectangles from $\mathbf{x}$ *to* $\mathbf{y}$ *is denoted by* $\mathrm{Rect}^\circ(\mathbf{x},\mathbf{y})$.

We will need a generalization of the rectangle idea:

Definition 3.16. *For* $\mathbf{x},\mathbf{y} \in \mathbf{S}(\mathbb{G})$ *a (positive)* domain ψ *from* $\mathbf{x}$ *to* $\mathbf{y}$ *is a formal linear combination of the small squares of* $\mathbb{G}$*, where the coefficients are non-negative integers, and the horizontal boundary segments of* ψ *point from the components of* $\mathbf{x}$ *to the components of* $\mathbf{y}$*. The set of domains from* $\mathbf{x}$ *to* $\mathbf{y}$ *is denoted by* $\pi(\mathbf{x},\mathbf{y})$.

To introduce grid homology, it is necessary to equip grid states with gradings. First we introduce two integral-valued functions, called the Maslov and the Alexander grading. The proof of the following propositions can be found in Chapter 4 of [16].

Proposition 3.17. *For any toroidal grid diagram* $\mathbb{G}$*, there exists a function*

$$M_{\mathbb{O}} : \mathbf{S}(\mathbb{G}) \to \mathbb{Z},$$

called the Maslov function on grid states*, which is uniquely characterized by the following two properties:*

- *Let* $\mathbf{x}^{NW\ O}$ *be the grid state whose components are the upper left corners of the squares marked with* O*. Then,*

$$M_{\mathbb{O}}(\mathbf{x}^{NW\ O}) = 0.$$

- *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are two grid states that can be connected by some rectangle* $r \in \mathrm{Rect}(\mathbf{x}, \mathbf{y})$*, then*

$$M_{\mathbb{O}}(\mathbf{x}) - M_{\mathbb{O}}(\mathbf{y}) = 1 - 2 \cdot |r \cap \mathbb{O}| + 2 \cdot |\mathbf{x} \cap \mathrm{Int}(r)|. \tag{3.1}$$

Another function, $M_{\mathbb{X}}$ can be defined similarly, but using X-markings in place of the O-markings. Unless stated otherwise, the Maslov function on grid states refers to $M_{\mathbb{O}}$.

There is an explicit way to compute the Maslov grading. For this we introduce some notations:

Consider the partial ordering on points in the plane $\mathbb{R}^2$ specified by $(p_1, p_2) < (q_1, q_2)$ if $p_1 < q_1$ and $p_2 < q_2$. Let P and Q be two finite sets of points in the plane, and denote by $\mathcal{I}(P, Q)$ the number of pairs $p \in P$, $q \in Q$ with $p < q$. By symmetrizing this function, we define

$$\mathcal{J}(P, Q) = \frac{\mathcal{I}(P, Q) + \mathcal{I}(Q, P)}{2}.$$

Now consider a fundamental domain $[0, n) \times [0, n)$ for the torus in the plane, included its left and bottom edges. We can think of a grid state $\mathbf{x} \in \mathbf{S}(\mathbb{G})$ as a collection of points with integer coordinates in this. Likewise, $\mathbb{O} = \{O_i\}_{i=1}^{n}$ can be viewed as a collection of points with half-integer coordinates in the fundamental domain. $M_{\mathbb{O}}$ can be given by the formula:

$$M_{\mathbb{O}}(\mathbf{x}) = \mathcal{J}(\mathbf{x}, \mathbf{x}) - 2 \cdot \mathcal{J}(\mathbf{x}, \mathbb{O}) + \mathcal{J}(\mathbb{O}, \mathbb{O}) + 1.$$

Definition 3.18. *The* Alexander function *on grid states is defined in terms of the Maslov functions by the following formula:*

$$A(\mathbf{x}) = \frac{1}{2} \cdot (M_{\mathbb{O}}(\mathbf{x}) - M_{\mathbb{X}}(\mathbf{x})) - \left(\frac{n-1}{2}\right).$$

Proposition 3.19. *Let* $\mathbb{G}$ *be a toroidal grid diagram for a knot, and* $\mathbf{x}, \mathbf{y} \in \mathbf{S}(\mathbb{G})$*. For any rectangle* $r \in \mathrm{Rect}^{\circ}(\mathbf{x}, \mathbf{y})$*, the Alexander function is integral valued and*

$$A(\mathbf{x}) - A(\mathbf{y}) = |r \cap \mathbb{X}| - |r \cap \mathbb{O}|. \tag{3.2}$$

3.3 The grid complex

Definition 3.20. *Let $\mathcal{R} \leq \mathcal{R}'$ be the subring generated by the elements U^t and U^{2-t}.*

Now we are ready to define the chain complex which will give us the desired homologies.

Definition 3.21. *For each $t \in [0,2]$ we define the* t-modified grid complex $tGC^-(\mathbb{G})$ *as the free module over $\mathcal{R}$ generated by $\mathbf{S}(\mathbb{G})$, equipped with the $\mathcal{R}$-module endomorphism ∂_t^-, whose value on any $\mathbf{x} \in \mathbf{S}(\mathbb{G})$ is given by*

$$\partial_t^-(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbf{S}(\mathbb{G})} \left(\sum_{r\in\mathrm{Rect}^\circ(\mathbf{x},\mathbf{y})} U^{t\cdot|\mathbb{X}\cap r|+(2-t)\cdot|\mathbb{O}\cap r|} \right) \cdot \mathbf{y}. \tag{3.3}$$

Theorem 3.22. *The operator $\partial_t^- : tGC^-(\mathbb{G}) \to tGC^-(\mathbb{G})$ satisfies $\partial_t^- \circ \partial_t^- = 0$.*

Proof. We follow the proof of Lemma 4.6.7 in [16]. Consider grid states $\mathbf{x}$ and $\mathbf{z}$. For a fixed $\psi \in \pi(\mathbf{x},\mathbf{z})$ denote by $N(\psi)$ the number of ways we can decompose ψ as a juxtaposition of two empty rectangles $r_1 * r_2$. Notice that if $\psi = r_1 * r_2$ for some $r_1 \in \mathrm{Rect}^\circ(\mathbf{x},\mathbf{y})$ and $r_2 \in \mathrm{Rect}^\circ(\mathbf{y},\mathbf{z})$, then

$$|\mathbb{X}\cap\psi| = |\mathbb{X}\cap r_1| + |\mathbb{X}\cap r_2| \quad \text{and} \quad |\mathbb{O}\cap\psi| = |\mathbb{O}\cap r_1| + |\mathbb{O}\cap r_2|.$$

It follows that the endomorphism $\partial_t^- \circ \partial_t^-$ can be expressed on $\mathbf{x}$ as

$$\partial_t^- \circ \partial_t^-(\mathbf{x}) = \sum_{\mathbf{z}\in\mathbf{S}(\mathbb{G})} \left(\sum_{\psi\in\pi(\mathbf{x},\mathbf{z})} N(\psi)\cdot U^{t\cdot|\mathbb{X}\cap\psi|+(2-t)\cdot|\mathbb{O}\cap\psi|} \right) \cdot \mathbf{z}.$$

Take a pair of empty rectangles $r_1 \in \mathrm{Rect}^\circ(\mathbf{x},\mathbf{y})$ and $r_2 \in \mathrm{Rect}^\circ(\mathbf{y},\mathbf{z})$ so that $N(\psi) > 0$ holds for the domain $r_1 * r_2 = \psi$. There are three basic cases (see Figure 4.4 in [16] for an illustrarion):

- $\mathbf{x} \setminus (\mathbf{x}\cap\mathbf{z})$ consists of 4 elements. Then, r_1 and r_2 do not share any common corners. In this case, ψ can only be decomposed as $r_1 * r_2$ or $r_2 * r_1$. Therefore, $N(\psi) = 2$.

- $\mathbf{x} \setminus (\mathbf{x}\cap\mathbf{z})$ consists of 3 elements. In this case, the local multiplicities of ψ are all 0 or 1. The corresponding region in the torus is a hexagon with five corners of $90°$, and one of $270°$. Now $N(\psi) = 2$ again, and the two decompositions can be obtained by cutting ψ at the $270°$ corner in two different directions.

- $\mathbf{x} = \mathbf{z}$. In this case, r_1 and r_2 intersect along two edges, and ψ is an annulus. Since the rectangles are empty, the height or width of this annulus is 1 (called a thin annulus), and $N(\psi) = 1$. Hence $|\mathbb{X}\cap\psi| = |\mathbb{O}\cap\psi| = 1$.

Contributions from the first two cases cancel in pairs, because we are working modulo 2. So it is enough to consider the terms where ψ is a thin annulus. Note that every thin annulus is a proper choice for ψ. Now we have

$$\partial_t^- \circ \partial_t^-(\mathbf{x}) = \left(\sum_{\psi \text{ is a thin annulus}} N(\psi) \cdot U^2 \right) \cdot \mathbf{x} = 2n \cdot U^2 \cdot \mathbf{x} = 0.$$

□

Let us introduce a grading on the preferred basis of $tGC^-\mathbb{G}$.

Definition 3.23. *For* $\mathbf{x} \in \mathbf{S}(\mathbb{G})$ *the* t-grading *of* $U^\alpha \cdot \mathbf{x}$ *is defined as*

$$\mathrm{gr}_t(U^\alpha \cdot \mathbf{x}) = M(\mathbf{x}) - t \cdot A(\mathbf{x}) - \alpha,$$

where M *is the Maslov and* A *is the Alexander function.*

Proposition 3.24. *The differential* ∂_t^- *drops the* t*-grading by one.*

Proof. Let $\mathbf{x} \in \mathbf{S}(\mathbb{G})$, and consider a non-zero term $U^{t\cdot|\mathbb{X}\cap r|+(2-t)\cdot|\mathbb{O}\cap r|} \cdot \mathbf{y}$ of the sum in the definition of $\partial_t^-(\mathbf{x})$ in (3.3). Since $r \in \mathrm{Rect}^\circ(\mathbf{x}, \mathbf{y})$, from (3.1) and (3.2) we know that

$$M(\mathbf{y}) = M(\mathbf{x}) - 1 + 2 \cdot |\mathbb{O} \cap r| \quad \text{and}$$

$$A(\mathbf{y}) = A(\mathbf{x}) - |\mathbb{X} \cap r| + |\mathbb{O} \cap r|.$$

Therefore,

$$\mathrm{gr}_t(U^{t\cdot|\mathbb{X}\cap r|+(2-t)\cdot|\mathbb{O}\cap r|} \cdot \mathbf{y}) = M(\mathbf{y}) - t \cdot A(\mathbf{y}) - t \cdot |\mathbb{X} \cap r| - (2-t) \cdot |\mathbb{O} \cap r| =$$

$$= (M(\mathbf{x}) - 1 + 2 \cdot |\mathbb{O} \cap r|) - t \cdot (A(\mathbf{x}) - |\mathbb{X} \cap r| + |\mathbb{O} \cap r|) - t \cdot |\mathbb{X} \cap r| - (2-t) \cdot |\mathbb{O} \cap r| =$$

$$= M(\mathbf{x}) - t \cdot A(\mathbf{x}) - 1 = \mathrm{gr}_t(\mathbf{x}) - 1.$$

□

Definition 3.25. *Let* $tGC_d^-(\mathbb{G})$ *denote the vector space over* $\mathbb{F}_2$ *generated by those monomials* $U^\alpha \cdot \mathbf{x}$*, where* $\mathbf{x} \in \mathbf{S}(\mathbb{G})$ *and* $\mathrm{gr}_t(U^\alpha \cdot \mathbf{x}) = d$*. If* $x \in tGC^-(\mathbb{G})$ *lies in* $tGC_d^-(\mathbb{G})$ *for some* d*, we say that* x *is* homogeneous*.*

We can define the homology of $(tGC^-(\mathbb{G}), \partial_t^-)$:

Definition 3.26.

$$tGH_d^-(\mathbb{G}) = {\mathrm{Ker}(\partial_t^-) \cap tGC_d^-(\mathbb{G})} \big/ {\mathrm{Im}(\partial_t^-) \cap tGC_d^-(\mathbb{G})}$$

$$tGH^-(\mathbb{G}) = \bigoplus_d tGH_d^-(\mathbb{G}).$$

3.4 Commutation invariance

The aim of this section is to prove that tGH^- is invariant under commutation. We follow the lines and now introduce the same notations as Section 5.1 in [16].

Consider two toroidal grid diagrams $\mathbb{G}$ and $\mathbb{G}'$ with grid number n, such that this latter can be obtained from $\mathbb{G}$ by a commutation move. Draw these two diagrams onto the same torus in a way that the X- and O-markings are fixed, and two of the vertical circles are curved according to Figure 3.2. Denote the horizontal circles of $\mathbb{G}$ by $\alpha = \{\alpha_1, ...\alpha_n\}$ and its vertical circles by $\beta = \{\beta_1, ...\beta_n\}$. Then the set of horizontal circles of $\mathbb{G}'$ is also α, but its vertical circles are different: $\gamma = \{\beta_1, ..., \beta_{i-1}, \gamma_i, \beta_{i+1}, ..., \beta_n\}$. Here we use the labelling compatible with the cyclic ordering of the toroidal grid, namely, β_{k+1} for $k = 1, ..., n-1$ is the vertical circle immediately to the east of β_k.

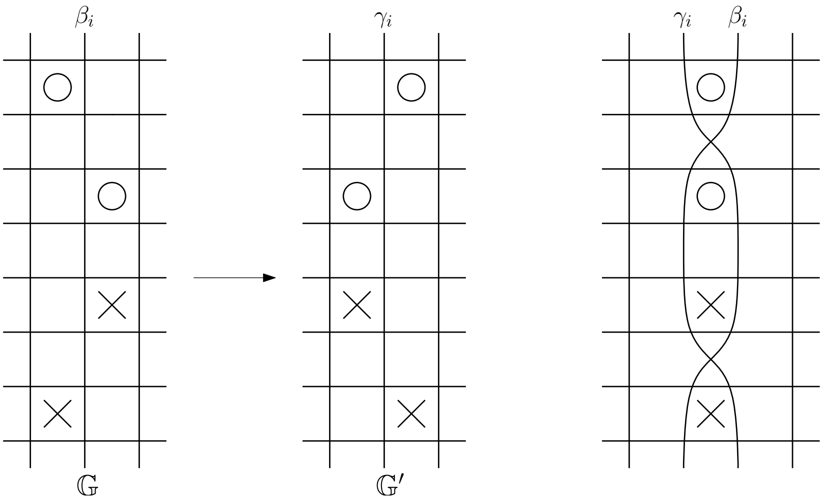


Figure 3.2:

Draw these vertical circles so that β_i meet γ_i perpendicularly in two points, that do not lie on any of the horizontal circles. Denote these points by a and b due to the following: Take the complement of $\beta_i \cup \gamma_i$ in the grid. This consists of two bigons intersecting in a and b. Consider the one, of which the western boundary is a part of β_i, and the eastern boundary is a part of γ_i. Let a be the southern and b the northern vertex of this bigon.

To make a connection between the chain complex of $\mathbb{G}$ and the chain complex of $\mathbb{G}'$, suitable pentagons, defined below will do a good service.

Definition 3.27. *Consider two grid states* $\mathbf{x} \in \mathbf{S}(\mathbb{G})$ *and* $\mathbf{y}' \in \mathbf{S}(\mathbb{G}')$. *A* pentagon p from $\mathbf{x}$ to $\mathbf{y}'$ *is an embedded disk in the torus that satisfies the following conditions:*

- *The boundary of p is the union of five arcs lying in α_j, β_j or γ_i for i and for some j.*
- *Four corners of p are in $\mathbf{x} \cup \mathbf{y}'$.*
- *If we consider any corner point of p, it is the intersection of two curves of $\{\alpha_j, \beta_j, \gamma_i\}_{j=1}^{n}$. These two curves divide a small disk on the torus into four quadrants, and p intersects exactly one of them.*
- *The horizontal segments in ∂p point from the components of $\mathbf{x}$ to the components of $\mathbf{y}'$.*

We use the notation $\mathrm{Pent}(\mathbf{x}, \mathbf{y}')$ for the set of pentagons going from $\mathbf{x}$ to $\mathbf{y}'$ (Figure 3.3).

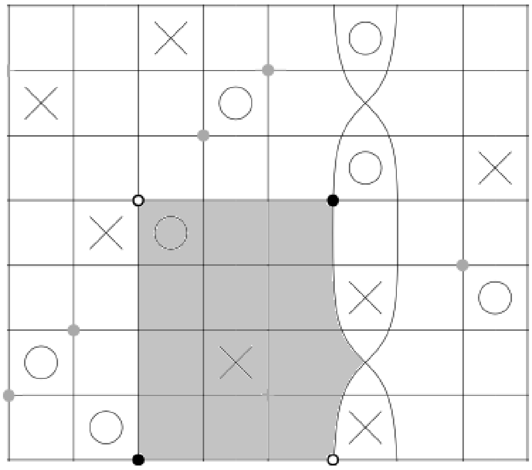


Figure 3.3: An empty pentagon

Observe that the set $\mathrm{Pent}(\mathbf{x}, \mathbf{y}')$ consists of at most one element, and it is empty unless $\mathbf{x}$ and $\mathbf{y}'$ share exactly $n-2$ elements. From the above properties of a pentagon follows that its fifth corner point is the distinguished point a.

Pentagons from $\mathbf{y}'$ to $\mathbf{x}$ are defined similarly. The only difference is in the forth condition: the horizontal boundary segments point from the components of $\mathbf{y}'$ to the components of $\mathbf{x}$. The fifth vertex of such pentagons is b.

Definition 3.28. *We call a pentagon $p \in \mathrm{Pent}(\mathbf{x}, \mathbf{y}')$ an* empty pentagon *if*

$$\mathbf{x} \cap \mathrm{Int}(p) = \mathbf{y}' \cap \mathrm{Int}(p) = \emptyset.$$

The set of empty pentagons from $\mathbf{x}$ to $\mathbf{y}'$ is denoted by $\mathrm{Pent}^{\circ}(\mathbf{x}, \mathbf{y}')$. (See Figure 3.3.)

Define the $\mathcal{R}$-module map $P : tGC^-(\mathbb{G}) \to tGC^-(\mathbb{G}')$ by the formula:

$$P(\mathbf{x}) = \sum_{\mathbf{y}' \in \mathbf{S}(\mathbb{G}')} \left(\sum_{p \in \text{Pent}^\circ(\mathbf{x},\mathbf{y}')} U^{t \cdot |\mathbb{X} \cap p| + (2-t) \cdot |\mathbb{O} \cap p|} \right) \cdot \mathbf{y}' \quad \text{for each } \mathbf{x} \in \mathbf{S}(\mathbb{G}).$$

Lemma 3.29. *For any* $\mathbf{x} \in \mathbf{S}(\mathbb{G})$*, the map* P *preserves the* t*-grading, i.e.* $\text{gr}_t(P(\mathbf{x})) = \text{gr}_t(\mathbf{x})$.

Proof. Consider one term of the sum. Let $\mathbf{y}' \in \mathbf{S}(\mathbb{G}')$ such that there exists a pentagon $p \in \text{Pent}^\circ(\mathbf{x}, \mathbf{y}')$. It is easy to verify the following formulae, for the details see Lemma 5.1.3. in [16]:

$$M(\mathbf{x}) - M(\mathbf{y}') = -2 \cdot |p \cap \mathbb{O}| + 2 \cdot |\mathbf{x} \cap \text{Int}(p)|$$

$$A(\mathbf{x}) - A(\mathbf{y}') = |p \cap \mathbb{X}| - |p \cap \mathbb{O}|$$

Using these we conclude that

$$\text{gr}_t(U^{t \cdot |\mathbb{X} \cap p| + (2-t) \cdot |\mathbb{O} \cap p|} \cdot \mathbf{y}') = M(\mathbf{y}') - t \cdot A(\mathbf{y}') - (t \cdot |\mathbb{X} \cap p| + (2-t) \cdot |\mathbb{O} \cap p|) =$$

$$= M(\mathbf{x}) + 2 \cdot |\mathbb{O} \cap p| - t \cdot (A(\mathbf{x}) - |\mathbb{X} \cap p| + |\mathbb{O} \cap p|) - (t \cdot |\mathbb{X} \cap p| + (2-t) \cdot |\mathbb{O} \cap p|) =$$

$$= M(\mathbf{x}) - t \cdot A(\mathbf{x}) = \text{gr}_t(\mathbf{x}).$$

Hence, $\text{gr}_t(P(\mathbf{x})) = \text{gr}_t(\mathbf{x})$. □

Proposition 3.30. *The map* P *is a chain map.*

Proof. We need to show that $(P \circ \partial_t^-)(\mathbf{x}) = (\partial_t^- \circ P)(\mathbf{x})$ for every $\mathbf{x} \in \mathbf{S}(\mathbb{G})$, which means the same as $(P \circ \partial_t^- + \partial_t^- \circ P)(\mathbf{x}) = 0$.

First we generalize the notion of a domain (Definition 3.16). By cutting the torus along the circles of α, of β and γ_i we obtain a union of triangles, rectangles and pentagons, call them elementary regions. Fix grid states $\mathbf{x} \in \mathbf{S}(\mathbb{G})$ and $\mathbf{y}' \in \mathbf{S}(\mathbb{G}')$. Let a domain ψ from $\mathbf{x}$ to $\mathbf{y}'$ be a formal sum of the closures of the elementary regions with non-negative integer coefficients, which satisfies the condition that the horizontal boundary segments of ψ point from the components of $\mathbf{x}$ to the components of $\mathbf{y}'$.

As in the proof of Theorem 3.22, we can easily see that

$$(P \circ \partial_t^- + \partial_t^- \circ P)(\mathbf{x}) = \sum_{\mathbf{z}' \in \mathbf{S}(\mathbb{G}')} \left(\sum_{\psi \in \pi(\mathbf{x},\mathbf{z}')} N(\psi) \cdot U^{t \cdot |\mathbb{X} \cap \psi| + (2-t) \cdot |\mathbb{O} \cap \psi|} \right) \cdot \mathbf{z}',$$

where $N(\psi)$ denotes the number of decompositions of ψ as a juxtaposition of an empty rectangle followed by an empty pentagon or as an empty pentagon followed by an empty rectangle.

Consider a domain ψ such that $N(\psi) > 0$. We examine three basic cases (for illustrarion see Figure 5.5 and 5.6 in [16]):

- $\mathbf{x} \setminus (\mathbf{x} \cap \mathbf{z}')$ consists of 4 elements. Then ψ can be obtained as the juxtaposition of a pentagon p and a rectangle r which do not share any common corner. In this case the only decompositions of ψ are $p * r$ and $r * p$. Therefore, $N(\psi) = 2$.

- $\mathbf{x} \setminus (\mathbf{x} \cap \mathbf{z}')$ consists of 3 elements. In this case either all of the local multiplicities of ψ are 0 and 1, or also 2 appears as a local multiplicity. Either way, ψ has seven corners, one of which is of degree 270°. We get two different decompositions of ψ for a pentagon and a rectangle by cutting it at this corner in two different directions. The order of the smaller domains is unique, thus $N(\psi) = 2$.

- $\mathbf{x} \setminus (\mathbf{x} \cap \mathbf{z}')$ consists of 1 element. Then ψ goes around the torus. This can happen in two ways: the juxtaposition is either a vertical thin annulus together with a triangle or a horizontal thin annulus minus a triangle. In the former case ψ decomposes uniquely into a thin pentagon and a thin rectangle, in the latter case $N(\psi) = 2$.

Since we work over $\mathbb{F}_2$, it is enough to consider the cases where $N(\psi) = 1$ (then ψ is a thin annulus with a triangle). It can be shown that the contributions coming from these domains cancel in pairs. For the details see Lemma 5.1.4. in [16]. □

Now we define an analogous $\mathcal{R}$-module homomorphism $P' : tGC^-(\mathbb{G}') \to tGC^-(\mathbb{G})$. For a grid state $\mathbf{x} \in \mathbf{S}(\mathbb{G}')$, let

$$P'(\mathbf{x}') = \sum_{\mathbf{y} \in \mathbf{S}(\mathbb{G})} \left(\sum_{p \in \mathrm{Pent}^\circ(\mathbf{x}', \mathbf{y})} U^{t \cdot |\mathbb{X} \cap p| + (2-t) \cdot |\mathbb{O} \cap p|} \right) \cdot \mathbf{y}.$$

Henceforth we will show that the two maps P and P' are homotopy inverses of each other. For this aim we need the following notion:

Definition 3.31. *Consider two grid states* $\mathbf{x}$ *and* $\mathbf{y} \in \mathbf{S}(\mathbb{G})$*. A* hexagon h *from* $\mathbf{x}$ *to* $\mathbf{y}$ *is an embedded disk in the torus that satisfies the following conditions:*

- *The boundary of h is the union of six arcs lying in α_j, β_j or γ_i for i and for some j.*

- *Four corners of h are in* $\mathbf{x} \cup \mathbf{y}$*, and the two further corners are a and b.*

- *If we consider any corner points of h, it is the intersection of two curves of $\{\alpha_j, \beta_j, \gamma_i\}_{j=1}^n$. These two curves divide the torus into four quadrants, and h lies in exactly one of them.*

- *The horizontal segments of ∂h point from the components of $\mathbf{x}$ to the components of $\mathbf{y}$.*

We use the notation $\mathrm{Hex}(\mathbf{x}, \mathbf{y})$ *for the set of hexagons going from* $\mathbf{x}$ *to* $\mathbf{y}$*.*

Definition 3.32. *We call a hexagon* $h \in \mathrm{Hex}(\mathbf{x}, \mathbf{y})$ *an* empty hexagon *if*

$$\mathbf{x} \cap \mathrm{Int}(h) = \mathbf{y} \cap \mathrm{Int}(h) = \emptyset.$$

The set of empty hexagons from $\mathbf{x}$ *to* $\mathbf{y}$ *is denoted by* $\mathrm{Hex}^\circ(\mathbf{x}, \mathbf{y})$*.*

Define the $\mathcal{R}$-module homomorphism $H : tGC^-(\mathbb{G}) \to tGC^-(\mathbb{G})$ for each $\mathbf{x} \in \mathbf{S}(\mathbb{G})$ by the formula:

$$H(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbf{S}(\mathbb{G})} \left(\sum_{h \in \mathrm{Hex}^\circ(\mathbf{x},\mathbf{y})} U^{t \cdot |\mathbb{X} \cap h| + (2-t) \cdot |\mathbb{O} \cap h|} \right) \cdot \mathbf{y}.$$

An analogous map $H' : tGC^-(\mathbb{G}') \to tGC^-(\mathbb{G}')$ can be defined by counting empty hexagons from $\mathbf{S}(\mathbb{G}')$ to itself.

Lemma 3.33. *The map* $H : tGC^-(\mathbb{G}) \to tGC^-(\mathbb{G})$ *provides a chain homotopy from the chain map* $P' \circ P$ *to the identity map on* $tGC^-(\mathbb{G})$*.*

Proof. First we show that H increases the t-grading by 1. Consider a non-zero term of the sum in the definition of $H(\mathbf{x})$. Let $\mathbf{y} \in \mathbf{S}(\mathbb{G})$ such that there exists a hexagon $h \in \mathrm{Hex}^\circ(\mathbf{x}, \mathbf{y})$. This h can be augmented to a rectangle $r \in \mathrm{Rect}(\mathbf{x}, \mathbf{y})$ by adding a bigon with vertices a and b, that contains exactly one X-marking and one O-marking. Applying Equations (3.1) and (3.2) we can derive that

$$M(U^{t \cdot |\mathbb{X} \cap h| + (2-t) \cdot |\mathbb{O} \cap h|} \cdot \mathbf{y}) = M(\mathbf{x}) + 1 \quad \text{and} \quad A(U^{t \cdot |\mathbb{X} \cap h| + (2-t) \cdot |\mathbb{O} \cap h|} \cdot \mathbf{y}) = A(\mathbf{x}).$$

To have that H is a chain homotopy from $P' \circ P$ to the identity map on $tGC^-(\mathbb{G})$, we have to verify that

$$\partial_t^- \circ H + H \circ \partial_t^- = \mathrm{Id} - P' \circ P, \quad \text{that is,}$$

$$(\partial_t^- \circ H + H \circ \partial_t^- + P' \circ P)(\mathbf{x}) = \mathbf{x} \quad \text{for any } \mathbf{x} \in \mathbf{S}(\mathbb{G}).$$

The idea of the proof is the same as in the proof of Theorem 3.22 and Proposition 3.30. For a domain $\psi \in \pi(\mathbf{x}, \mathbf{z})$ denote by $N(\psi)$ the number of ways ψ can be decomposed as one of the followings:

- $\psi = r * h$, where r is an empty rectangle and h is an empty hexagon;

- $\psi = h * r$, where h is an empty hexagon and r is an empty rectangle;
- $\psi = p * p'$, where p is an empty pentagon from $\mathbf{S}(\mathbb{G})$ to $\mathbf{S}(\mathbb{G}')$ and p' is an empty pentagon from $\mathbf{S}(\mathbb{G}')$ to $\mathbf{S}(\mathbb{G})$.

Obviously,

$$\partial_t^- \circ H + H \circ \partial_t^- + P' \circ P)(\mathbf{x}) = \sum_{\mathbf{z} \in \mathbf{S}(\mathbb{G})} \left(\sum_{\psi \in \pi(\mathbf{x},\mathbf{z})} N(\psi) \cdot U^{t \cdot |\mathbb{X} \cap \psi| + (2-t) \cdot |\mathbb{O} \cap \psi|} \right) \cdot \mathbf{z}. \quad (3.4)$$

For a domain ψ for which $N(\psi) > 0$, we have three basic cases again (for illustrarion see Figure 5.8 and 5.9 in [16]):

- $\mathbf{x} \setminus (\mathbf{x} \cap \mathbf{z})$ consists of 4 elements. Then ψ can be decomposed into the juxtaposition of a hexagon h and a rectangle r which do not share any common corner. In this case the only decompositions of ψ are $h * r$ and $r * h$. Therefore, $N(\psi) = 2$.
- $\mathbf{x} \setminus (\mathbf{x} \cap \mathbf{z})$ consists of 3 elements. Then ψ has eight vertices, and there are two possibilities: either seven corners are 90° and one is 270°, or five corners are 90° and three are 270° (then two of the 270° corners are at a and b). In the former case, we get two different decompositions of ψ by cutting at the 270° corner in two different directions. In the latter case $N(\psi) = 2$ again, since at one of the three corners we can cut in two different directions, while at the other two corners the direction for cutting is uniquely determined.
- $\mathbf{x} = \mathbf{z}$. Then ψ is an annulus enclosed by β_i and one of the consecutive vertical circles. In this case $N(\psi) = 1$.

There is a unique domain ψ which has non-zero contribution in Equation (3.4).
For this, $N(\psi) = 1$ and $\psi \in \pi(\mathbf{x}, \mathbf{x})$ contains no marking. Therefore the only term of the sum is $\mathbf{x}$. $\square$

Gathering the above propositions, we reach the main result of this section:

Theorem 3.34. *Let $\mathbb{G}$ and $\mathbb{G}'$ be two grid diagrams that differ by a commutation move. Then*

$$tGH^-(\mathbb{G}) \cong tGH^-(\mathbb{G}').$$

$\square$

3.5 Stabilization invariance

Our goal in this chapter is to check whether tGH^- is invariant under stabilization. In fact, we will need some modifications to make it invariant. For this, we will use the following notions and notations:

Definition 3.35. *Let $X = \bigoplus_d X_d$ and $Y = \bigoplus_d Y_d$ be graded vector spaces, such that in at least one of them, the set of grades appearing in it is bounded above. Their tensor product $X \otimes Y = \bigoplus_d (X \otimes Y)_d$ is the graded vector space with*

$$(X \otimes Y)_d = \bigoplus_{d_1+d_2=d} X_{d_1} \otimes Y_{d_2}.$$

Definition 3.36. *For a graded vector space X and $s \in \mathbb{R}$ the* shift of X by s *denoted by $X[\![s]\!]$, is the graded vector space that is isomorphic to X as a vector space and the grading on $X[\![s]\!]$ is given by the relation $X[\![s]\!]_d = X_{d+s}$.*

Consider the two-dimensional graded vector space W with one generator of grading 1-t and another of grading 0. Take any other graded vector space X. Then

$$X \otimes W \cong X [\![1-t]\!] \oplus X. \tag{3.5}$$

This procedure can be iterated, for example:

$$(X \otimes W) \otimes W = X \otimes W^{\otimes 2} \cong X [\![2-2t]\!] \oplus X [\![1-t]\!] \oplus X [\![1-t]\!] \oplus X$$

Now we can introduce the true invariant, which is an equivalence class of pairs consisting of a group and an integer. Concretely, for a grid diagram $\mathbb{G}$, consider the pair $(tGH^-(\mathbb{G}), n)$, where n is the grid number of $\mathbb{G}$. Let $\mathbb{G}'$ be another grid diagram with grid number n', such that $n \leq n'$. The two pairs $(tGH^-(\mathbb{G}), n)$ and $(tGH^-(\mathbb{G}'), n')$ are called equivalent if

$$tGH^-(\mathbb{G}') \cong tGH^-(\mathbb{G}) \otimes W^{\otimes(n'-n)}.$$

Henceforward in this section we will prove that ${(tGH^-(\mathbb{G}), n)}/{\sim}$ is indeed invariant under stabilization.

Remark 3.37. ${(tGH^-(\mathbb{G}), n)}/{\sim}$ *is invariant under commutation, since we have seen in the previous section that commutation moves do not change $tGH^-(\mathbb{G})$, and the grid number n obviously remains the same.*

Let $\mathbb{G}$ be a grid diagram. By performing a stabilization of type $X : SW$, we get the diagram $\mathbb{G}'$. Number the markings in the way that O_1 is the newly-introduced

O-marking, O_2 is in the consecutive row below O_1, X_1 and X_2 lie in the same row as O_1 and O_2, respectively, i.e. $\begin{array}{c|c} X_1 & O_1 \\ \hline & X_2 \end{array}$.

Denote c the intersection point of the new horizontal and vertical circles in $\mathbb{G}'$. Considering this point, we can partition the grid states of the stabilized diagram $\mathbb{G}'$ into two parts, depending on whether or not they contain the intersection point c. Define the sets $\mathbf{I}(\mathbb{G}')$ and $\mathbf{N}(\mathbb{G}')$ so that $\mathbf{x} \in \mathbf{I}(\mathbb{G}')$ if c is included in $\mathbf{x}$, and $\mathbf{x} \in \mathbf{N}(\mathbb{G}')$ if c is not included in $\mathbf{x}$. Now $\mathbf{S}(\mathbb{G}') = \mathbf{I}(\mathbb{G}') \cup \mathbf{N}(\mathbb{G}')$ gives a decomposition of $tGC^-(\mathbb{G}') \cong I \oplus N$, where I and N denote the $\mathcal{R}$-modules spanned by the grid states of $\mathbf{I}(\mathbb{G}')$ and $\mathbf{N}(\mathbb{G}')$ respectively.

There is a one-to-one correspondence between grid states of $\mathbf{I}(\mathbb{G}')$ and grid states of $\mathbf{S}(\mathbb{G})$: Let

$$e : \mathbf{I}(\mathbb{G}') \to \mathbf{S}(\mathbb{G}), \qquad \mathbf{x} \cup \{c\} \mapsto \mathbf{x}.$$

From this point on, we will work with certain types of domains again:

Definition 3.38. *For grid states $\mathbf{x} \in \mathbf{S}(\mathbb{G}')$ and $\mathbf{y} \in \mathbf{I}(\mathbb{G}')$ a positive domain $p \in \pi(\mathbf{x}, \mathbf{y})$ is called* into L *or of type iL, if it is trivial, or if it satisfies the following conditions:*

- *At each corner point in $\mathbf{x} \cup \mathbf{y} \setminus \{c\}$ at least three of the four adjoining squares have local multiplicity 0.*
- *Three of the four squares attaching at the corner point c have the same local multiplicity k, and at the southwest square meeting c the local multiplicity of p is $k - 1$.*
- *The grid state $\mathbf{y}$ has $2k + 1$ components that are not in $\mathbf{x}$.*

The set of domains of type iL is denoted by $\pi^{iL}(\mathbf{x}, \mathbf{y})$ (see Figure 3.4).

Definition 3.39. *Analogously, for grid states $\mathbf{x} \in \mathbf{S}(\mathbb{G}')$ and $\mathbf{y} \in \mathbf{I}(\mathbb{G}')$ a positive domain $p \in \pi(\mathbf{x}, \mathbf{y})$ is called* into R *or of type iR, if it satisfies the following conditions:*

- *At each corner point in $\mathbf{x} \cup \mathbf{y} \setminus \{c\}$ at least three of the four adjoining squares have local multiplicity 0.*
- *Three of the four squares attaching at the corner point c have the same local multiplicity k, and at the southeast square meeting c the local multiplicity of p is $k + 1$.*
- *The grid state $\mathbf{y}$ has $2k + 2$ components that are not in $\mathbf{x}$.*

The set of domains of type iR is denoted by $\pi^{iR}(\mathbf{x}, \mathbf{y})$ (see Figure 3.4).

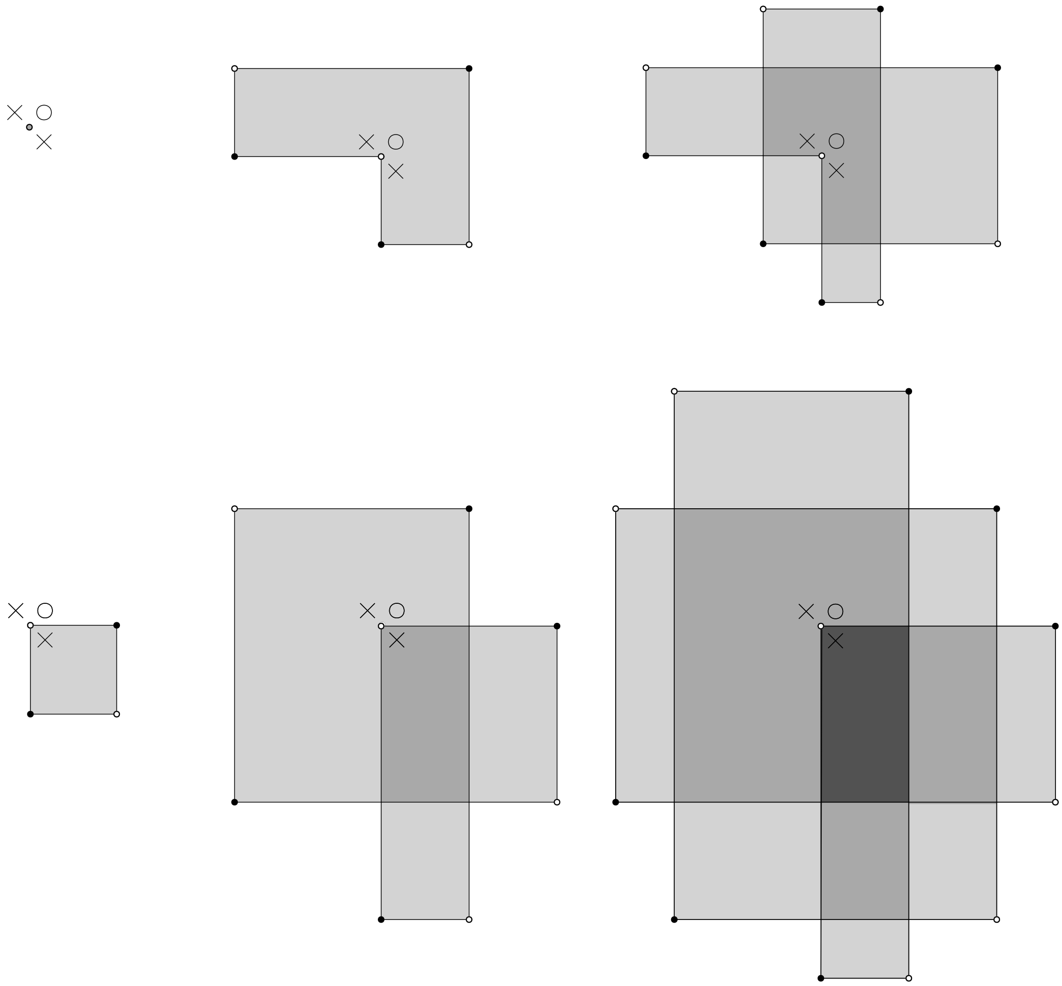


Figure 3.4: Destabilization domains of type iL (first row) and iR (second row)

The domains of type iL and iR are called *destabilization domains*. We use the notation $\pi^D = \pi^{iL} \cup \pi^{iR}$.

Definition 3.40. *Let the* complexity *of the trivial domain be one. For other destabilization domains the complexity tells the number of horizontal segments in the boundary of the domain.*

For example the upper right domain on Figure 3.4 has complexity 5. Observe that the domains of type iL are the destabilization domains with odd complexity, while the domains of type iR are the ones with even complexity.

Lemma 3.41. *Let* $\mathbf{x}, \mathbf{y} \in \mathbf{S}(\mathbb{G}')$ *be grid states such that there exists a destabilization domain* $p \in \pi(\mathbf{x}, \mathbf{y})$ *of complexity* k*. Then there is a sequence of grid states* $\mathbf{x}_1, ..., \mathbf{x}_k$ *such that* $\mathbf{x}_1 = \mathbf{x}$, $\mathbf{x}_k = \mathbf{y}$, *and empty rectangles* $r_1, ..., r_{k-1}$ *with* $r_i \in \mathrm{Rect}^\circ(\mathbf{x}_i, \mathbf{x}_{i+1})$, *so that* p *decomposes as* $r_1 * ... * r_{k-1}$*. Among such rectangle sequences, there is a unique one in which every rectangle* r_i *has an edge on the distinguished vertical circle going through* c*.*

The proof of this lemma can be found as the proof of Lemma 13.3.11. in [16].

Now we are ready to introduce the chain map which gives the isomorphism between $tGC^-(\mathbb{G}')$ and $tGC^-(\mathbb{G})\,[\![1-t]\!] \oplus tGC^-(\mathbb{G})$. For this, we will use the abbreviations $\bar{\mathbb{X}} := X_1 \cup X_3 \cup ... \cup X_n$ and $\bar{\mathbb{O}} := O_1 \cup O_3 \cup ... \cup O_n$.

Definition 3.42.
The $\mathcal{R}$-module homomorphisms $D^{iL} : tGC^-(\mathbb{G}') \to tGC^-(\mathbb{G})[\![1-t]\!]$ and $D^{iR} : tGC^-(\mathbb{G}') \to tGC^-(\mathbb{G})$ are defined on a grid state $\mathbf{x} \in \mathbf{S}(\mathbb{G})$ in the following way:

$$D^{iL}(\mathbf{x}) := \sum_{\mathbf{y} \in I(\mathbb{G}')} \sum_{p \in \pi^{iL}(\mathbf{x},\mathbf{y})} U^{t\cdot|\bar{\mathbb{X}}\cap p| + (2-t)\cdot|\bar{\mathbb{O}}\cap p|} \cdot e(\mathbf{y})$$

$$D^{iR}(\mathbf{x}) := \sum_{\mathbf{y} \in I(\mathbb{G}')} \sum_{p \in \pi^{iR}(\mathbf{x},\mathbf{y})} U^{t\cdot|\bar{\mathbb{X}}\cap p| + (2-t)\cdot|\bar{\mathbb{O}}\cap p|} \cdot e(\mathbf{y})$$

Proposition 3.43. *D^{iL} increases the grading by $(1-t)$, whereas D^{iR} preserves the grading.*

Proof. Let $\mathbf{x}, \mathbf{y} \in \mathbf{S}(\mathbb{G}')$. First observe that $\mathrm{gr}_t(e(\mathbf{y})) = \mathrm{gr}_t(\mathbf{y}) + 1 - t$. To get the result about D^{iL} our goal is to show that for $p \in \pi^{iL}(\mathbf{x},\mathbf{y})$

$$\mathrm{gr}_t(\mathbf{x}) = \mathrm{gr}_t(U^{t\cdot|\bar{\mathbb{X}}\cap p| + (2-t)\cdot|\bar{\mathbb{O}}\cap p|} \cdot \mathbf{y}).$$

Suppose that the complexity of p is $k \in \mathbb{N}$. Then by Lemma 3.41, p can be decomposed as the juxtaposition of $k-1$ rectangles. Consider the Maslov and the Alexander gradings. From Equations (3.1) and (3.2) we have that

$$M(\mathbf{x}) - M(\mathbf{y}) = (1 - 2\cdot|\mathbb{O} \cap r_1|) + (1 - 2\cdot|\mathbb{O} \cap r_2|) + ... + (1 - 2\cdot|\mathbb{O} \cap r_{k-1}|) =$$

$$= k - 1 - 2\cdot|\mathbb{O} \cap p| = -2\cdot|\bar{\mathbb{O}} \cap p|,$$

$$A(\mathbf{x}) - A(\mathbf{y}) = |\mathbb{X} \cap p| - |\mathbb{O} \cap p| = |\bar{\mathbb{X}} \cap p| - |\bar{\mathbb{O}} \cap p|$$

because an iL-type domain contains O_1 and X_2 with the same multiplicity.

$$\mathrm{gr}_t(\mathbf{x}) - \mathrm{gr}_t(\mathbf{y}) = (M(\mathbf{x}) - M(\mathbf{y})) - t\cdot(A(\mathbf{x}) - A(\mathbf{y})) = -t\cdot|\bar{\mathbb{X}} \cap p| - (2-t)\cdot|\bar{\mathbb{O}} \cap p|,$$

from which we get

$$\mathrm{gr}_t(\mathbf{x}) = \mathrm{gr}_t(U^{t\cdot|\bar{\mathbb{X}}\cap p| + (2-t)\cdot|\bar{\mathbb{O}}\cap p|} \cdot e(\mathbf{y})) - 1 + t.$$

Now let $p \in \pi^{iR}(\mathbf{x},\mathbf{y})$ with complexity k.

$$M(\mathbf{x}) - M(\mathbf{y}) = k - 1 - 2\cdot|\mathbb{O} \cap p| = 1 - 2\cdot|\bar{\mathbb{O}} \cap p|,$$

$$A(\mathbf{x}) - A(\mathbf{y}) = |\mathbb{X} \cap p| - |\mathbb{O} \cap p| = |\bar{\mathbb{X}} \cap p| - |\bar{\mathbb{O}} \cap p| + 1$$

because in an iR-type domain the multiplicity of X_2 is one greater than the multiplicity of O_1.

$$\mathrm{gr}_t(\mathbf{x}) - \mathrm{gr}_t(\mathbf{y}) = (M(\mathbf{x}) - M(\mathbf{y})) - t \cdot (A(\mathbf{x}) - A(\mathbf{y})) = 1 - t - t \cdot |\bar{\mathbb{X}} \cap p| - (2-t) \cdot |\bar{\mathbb{O}} \cap p|,$$

from which we get

$$\mathrm{gr}_t(\mathbf{x}) = \mathrm{gr}_t(U^{t \cdot |\bar{\mathbb{X}} \cap p| + (2-t) \cdot |\bar{\mathbb{O}} \cap p|} \cdot e(\mathbf{y})).$$

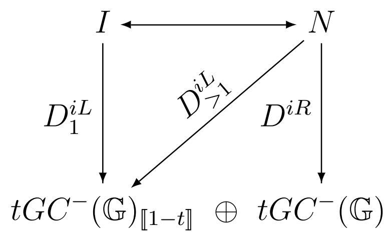
□

Let $C := tGC^-(\mathbb{G})[\![1-t]\!] \oplus tGC^-(\mathbb{G})$, and consider the map $\partial : C \to C$ such that $\partial(x, y) = (\partial_t^-(x), \partial_t^-(y))$ holds for any $(x, y) \in C$. Obviously, the pair (C, ∂) is a chain complex.

Definition 3.44.
Let $D : tGC^-(\mathbb{G}') \to C$ be the destabilization map defined for any $x \in tGC^-(\mathbb{G}')$ as

$$D(x) := (D^{iL}(x), D^{iR}(y)) \in C.$$

The map D^{iL} can be decomposed as $D^{iL} = D_1^{iL} + D_{>1}^{iL}$, where the subscript indicates the restriction on the complexity of the destabilization domains. Using this, we can draw the following schematic picture for D, where the top row represents $tGC^-(\mathbb{G}')$ with its decomposition as $I \oplus N$, and the bottom row shows C. The map D can be seen in the arrows connecting the two rows.



Proposition 3.45. *The destabilization map D is a chain map.*

The proof of this proposition is based on counting regions on the grid diagram, and it is the analogous of the proof of Lemma 13.3.13. in [16].

Proposition 3.46. *Suppose that C and D are graded chain complexes over $\mathcal{R}$, such that they are free modules, and the grades appearing in them is bounded above. Let $\alpha \in \mathbb{R}_{\geq 0}$ and $f : C \to D$ be a graded chain map. Then f a quasi-isomorphism if and only if it induces a quasi-isomorphism $\bar{f} : {}^{C}\!/_{U^\alpha \cdot C} \to {}^{D}\!/_{U^\alpha \cdot D}$.*

Proof. First observe that ${\mathrm{Cone}(f)}\big/{U^\alpha \cdot \mathrm{Cone}(f)} \cong \mathrm{Cone}(\bar{f})$.

From Corollary 3.11 we know that the map f is a quasi-isomorphism if and only if $H(\mathrm{Cone}(f)) = 0$. According to Proposition 3.12, this is equivalent to $H({\mathrm{Cone}(f)}\big/{U^\alpha \cdot \mathrm{Cone}(f)}) \neq 0$. By our observation, this holds if and only if $H(\mathrm{Cone}(\bar{f})) = 0$, which, by Corollary 3.11 again, is equivalent to $\bar{f}$ being a quasi-isomorphism. □

Now let us use the following notations.

$$C_1 := tGC^-(\mathbb{G}') \quad \text{and} \quad D_1 := tGC^-(\mathbb{G}) \oplus tGC^-(\mathbb{G})\,[\![1-t]\!]\,,$$

$$C_2 := {C_1}\big/{U^t \cdot C_1} \quad \text{and} \quad D_2 := {D_1}\big/{U^t \cdot D_1},$$

$$C_3 := {C_2}\big/{U^{2-t} \cdot C_2} \quad \text{and} \quad D_3 := {D_2}\big/{U^{2-t} \cdot D_2}.$$

Furthermore, define $f = D : C_1 \to D_1$ and let $\bar{f} : C_2 \to D_2$ be the map induced by f. Finally, define $\bar{\bar{f}} : C_3 \to D_3$, the map induced by $\bar{f}$. Apply Proposition 3.46 for $\alpha = 2-t$. Then we get that if $\bar{\bar{f}}$ is a quasi-isomorphism, then $\bar{f}$ is also a quasi-isomorphism. Now apply Proposition 3.46 again, to have that if $\bar{f}$ is a quasi-isomorphism, then f is also a quasi-isomorphism. To prove the stabilization invariance, we need to verify that $f = D$ is a quasi-isomorphism. For this, it is enough to show that $\bar{\bar{f}}$ is a quasi-isomorphism between C_3 and D_3.

Proposition 3.47. $\bar{\bar{f}}$ *is a quasi-isomorphism.*

Proof. Recall the notion of the fully blocked grid homology ([16]), which is the simplest version of grid homology.

Definition 3.48. *The* fully blocked grid chain complex $\widetilde{GC}(\mathbb{G})$ *associated to the grid diagram* $\mathbb{G}$ *is a free* $\mathbb{F}_2$*-module generated by the grid states of* $\mathbb{G}$ *with the differential*

$$\tilde{\partial}_{\mathbb{O},\mathbb{X}}(\mathbf{x}) = \sum_{\mathbf{y}\mathbf{S}(\mathbb{G})} \sum_{\{r \in \mathrm{Rect}^\circ(\mathbf{x},\mathbf{y}) | r \cap \mathbb{O} = r \cap \mathbb{X} = \emptyset\}} \mathbf{y}.$$

Observe that $C_3 \cong \widetilde{GC}(\mathbb{G}')$ and $D_3 \cong \widetilde{GC}(\mathbb{G}) \oplus \widetilde{GC}(\mathbb{G})_{[\![1-t]\!]}$. From the proof of Proposition 5.2.2 of [16], it follows that there is a quasi-isomorphism from C_3 to D_3, which is in fact $\bar{\bar{f}}$. □

4. The Υ knot invariant

4.1 The definition of Υ

Definition 4.1. *Let M be a module over $\mathcal{R}$. The torsion submodule* $\text{Tors}(M)$ *of M is*

$$\text{Tors}(M) = \{m \in M \mid \text{ there is a non} - \text{zero } p \in \mathcal{R} \text{ with } p \cdot m = 0\}.$$

Definition 4.2. *For $t \in [1, 2]$,*

$$\Upsilon_{\mathbb{G}}(t) := \max\{\text{gr}_t(x) | x \in tGH^-(\mathbb{G}),\ x \text{ is homogeneous and non} - \text{torsion}\}$$

Theorem 4.3. *Let $\mathbb{G}$ and $\mathbb{G}'$ be two grid diagrams such that $\mathbb{G}'$ can be obtained from $\mathbb{G}$ by a grid move. If $t \in [1, 2]$, then $\Upsilon_{\mathbb{G}}(t) = \Upsilon_{\mathbb{G}'}(t)$.*

Proof. If $\mathbb{G}'$ can be obtained from $\mathbb{G}$ by a commutation, then from Theorem 3.34 we know that tGH^- is invariant under commutation, thus $\Upsilon_{\mathbb{G}}(t) = \Upsilon_{\mathbb{G}'}(t)$.

Suppose that $\mathbb{G}'$ can be obtained from $\mathbb{G}$ by a stabilization, and let d denote the maximal grade which appears among homogeneous non-torsion elements of $tGH^-(\mathbb{G})$. Then the maximal grade of homogeneous non-torsion elements of $tGH^-(\mathbb{G})_{[\![1-t]\!]}$ is $d + 1 - t$. In case of $t \in [1, 2]$, $d + 1 - t \leq d$, thus the maximal grade of homogeneous non-torsion elements of $tGH^-(\mathbb{G}') \cong tGH^-(\mathbb{G}) \oplus tGH^-(\mathbb{G})_{[\![1-t]\!]}$ is also d. ☐

In case of $t \in [0, 1]$ the above proof does not work, but the Υ invariant can be defined by extending it to $t \in [0, 2]$ symmetrically. For $t \in [0, 1]$, let $\Upsilon_G(t) = \Upsilon_G(2-t)$. Obviously, $\Upsilon_G(t)$ is also a knot invariant for $t \in [0, 1]$.

Consequently, for a knot K and $t \in [0, 2]$ we can define $\Upsilon_K(t)$ by taking a grid diagram G representing K:

$$\Upsilon_K(t) := \Upsilon_G(t).$$

According to Theorem 4.3, this is well-defined, and the following corollary is immediate:

Corollary 4.4. *For $t \in [1, 2]$, $\Upsilon_K(t)$ is a knot invariant, that is, if K_1 and K_2 are isotopic knots, then $\Upsilon_{K_1}(t) = \Upsilon_{K_2}(t)$.*

Now let us recite some relevant properties of the Υ invariant. The following results are obtained by examining $\Upsilon_K(t)$ as a function of t. These propositions are proved in [13].

Proposition 4.5. *For a knot K, $\Upsilon_K(0) = 0$.*

From this and the symmetry of Υ, it is easy to see that $\Upsilon_K(2) = 0$ also holds.

Proposition 4.6. *For $m, n \in \mathbb{N}_{>0}$, the quantity $\Upsilon_K(\frac{m}{n})$ lies in $\frac{1}{n}\mathbb{Z}$.*

Proposition 4.7. *For any knot K the function $\Upsilon_K : t \to \Upsilon_K(t)$ is piecewise linear, and its derivative has finitely many discontinuities. Each slope is an integer.*

Proposition 4.8. *The slope of $\Upsilon_K(t)$ at $t = 0$ is given by $-\tau(K)$, where τ is a knot invariant, defined in Chapter 6 of [16].*

4.2 Crossing changes

In this section we examine how Υ changes under crossing changes.

Let K_+ and K_- be two knots which differ only in a crossing.

Proposition 4.9. *There are $\mathcal{R}$-module maps*

$$C_- : tGH^-(K_+) \to tGH^-(K_-) \quad \text{and} \quad C_+ : tGH^-(K_-) \to tGH^-(K_+)$$

where C_- is homogeneous and preserves the grading, and C_+ is homogeneous of degree $-2+t$. $C_- \circ C_+$ and $C_+ \circ C_-$ are both the multiplication by U^{2-t}.

Before proving this proposition we introduce some notations: Represent the knots K_+ and K_- by the grid diagrams $\mathbb{G}_+$ and $\mathbb{G}_-$ differing by a cross-commutation of columns. Again, we draw these two diagrams onto the same torus so that the X- and the O-markings are fixed. Using the same notations as in Section 3.4, let $\alpha = \{\alpha_1, ..., \alpha_n\}$ be the horizontal circles of the diagrams, $\beta = \{\beta_1, ..., \beta_n\}$ the vertical circles of $\mathbb{G}_+$, and $\gamma = \{\beta_1, ..., \beta_{i-1}, \gamma_i, \beta_{i+1}, ...\beta_n\}$ the vertical circles of $\mathbb{G}_-$. Draw β_i and γ_i so that they meet perpendicularly in four points, and these intersection points do not lie on any of the horizontal circles. This way the complement of $\beta_i \cup \gamma_i$ has five components, four of which are bigons marked by a single X or a single O. Label the O-markings so that O_1 is above O_2. Now the bigon marked by O_2 has a common vertex with one of the X-marked bigons; denote this point by s. The two X-labelled bigons share a vertex on $\beta_i \cap \gamma_i$, since $\mathbb{G}_+$ can be obtained from $\mathbb{G}_-$ by a cross-commutation. Call this common point t (Figure 4.1).

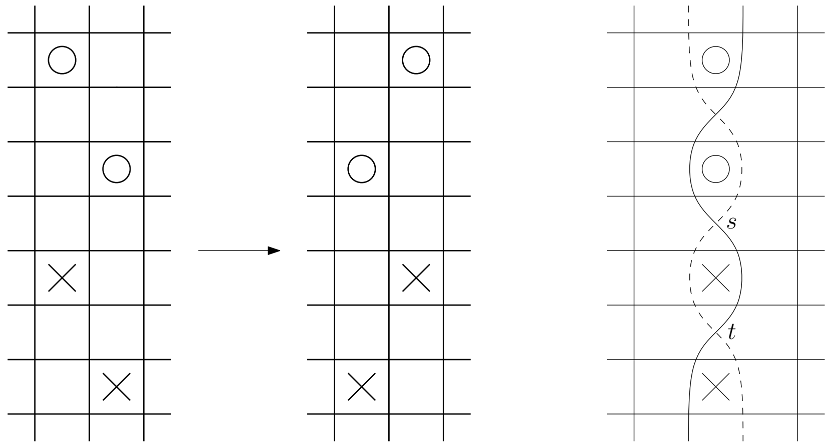


Figure 4.1:

Proof. Fix grid states $\mathbf{x}_+ \in \mathbf{S}(\mathbb{G}_+)$ and $\mathbf{x}_- \in \mathbf{S}(\mathbb{G}_-)$. We use the notation $\mathrm{Pent}^\circ{}_s(\mathbf{x}_+, \mathbf{x}_-)$ for the set of empty pentagons from $\mathbf{x}_+$ to $\mathbf{x}_-$ containing s as a vertex, and similarly, $\mathrm{Pent}^\circ{}_t(\mathbf{x}_-, \mathbf{x}_+)$ for the set of empty pentagons from $\mathbf{x}_-$ to $\mathbf{x}_+$ with one vertex at t.

Consider the $\mathcal{R}$-module maps $c_- : tGC^-(\mathbb{G}_+) \to tGC^-(\mathbb{G}_-)$ and $c_+ : tGC^-(\mathbb{G}_-) \to tGC^-(\mathbb{G}_+)$ defined on a grid state $\mathbf{x}_+ \in \mathbf{S}(\mathbb{G}_+)$ and $\mathbf{x}_- \in \mathbf{S}(\mathbb{G}_-)$ respectively in the following way:

$$c_-(\mathbf{x}_+) = \sum_{\mathbf{y}_- \in \mathbf{S}(\mathbb{G}_-)} \sum_{p \in \mathrm{Pent}^\circ{}_s(\mathbf{x}_+, \mathbf{y}_-)} U^{t \cdot |\mathbb{X} \cap p| + (2-t) \cdot |\mathbb{O} \cap p|} \cdot \mathbf{y}_-$$

$$c_+(\mathbf{x}_-) = \sum_{\mathbf{y}_+ \in \mathbf{S}(\mathbb{G}_+)} \sum_{p \in \mathrm{Pent}^\circ{}_t(\mathbf{x}_-, \mathbf{y}_+)} U^{t \cdot |\mathbb{X} \cap p| + (2-t) \cdot |\mathbb{O} \cap p|} \cdot \mathbf{y}_+.$$

Lemma 4.10. *The map c_- preserves the t-grading, while c_+ drops the t-grading by $2 - t$.*

Proof. The four marking between the circles β_i and γ_i divide β_i into four segments, call them **A**, **B**, **C** and **D** according to the followings: let **A** be the segment between the two O-markings, and from this part on, the order of the other segments to south is **B**, **C** and **D**. There is a natural one-to-one correspondence between $\mathbf{S}(\mathbb{G}_+)$ and $\mathbf{S}(\mathbb{G}_-)$, assigning to $\mathbf{x}_+ \in \mathbf{S}(\mathbb{G}_+)$ the grid state $\mathbf{x}_- \in \mathbf{S}(\mathbb{G}_-)$ that agrees with $\mathbf{x}_+$ in $n-1$ elements. It is not hard to compute the difference between the grading of $\mathbf{x}_-$ and $\mathbf{x}_+$:

$$\mathrm{gr}_t(\mathbf{x}_-) = \mathrm{gr}_t(\mathbf{x}_+) + \begin{cases} -1+t & \text{if} \quad \mathbf{x}_+ \cap \beta_i \in \mathbf{A} \\ 1 & \text{if} \quad \mathbf{x}_+ \cap \beta_i \in \mathbf{B} \\ 1-t & \text{if} \quad \mathbf{x}_+ \cap \beta_i \in \mathbf{C} \\ 1 & \text{if} \quad \mathbf{x}_+ \cap \beta_i \in \mathbf{D}. \end{cases}$$

We partition pentagons into left and right ones depending on whether they lie on the left or on the right side of the cross-commutation. Associate to each pentagon $p \in \mathrm{Pent}^\circ(\mathbf{x}_+, \mathbf{y}_-)$ the rectangle $r \in \mathrm{Rect}^\circ(\mathbf{x}_+, \mathbf{y}_+)$ with the same local multiplicities of the small cells as p, except from the four bigons between β_i and γ_i. From Proposition 3.24 we know that

$$\mathrm{gr}_t(\mathbf{x}_+) = \mathrm{gr}_t(\mathbf{y}_+) - t \cdot |\mathbb{X} \cap r| - (2-t) \cdot |\mathbb{O} \cap r| + 1.$$

Consider the left pentagons. We will do the computations by distinguishing four cases depending on the position of $\mathbf{y}_+ \cap \beta_i$:

- If $\mathbf{y}_+ \cap \beta_i \in \mathbf{A}$, then

$$\mathrm{gr}_t(\mathbf{x}_+) = \mathrm{gr}_t(\mathbf{y}_+) - t \cdot |\mathbb{X} \cap r| - (2-t) \cdot |\mathbb{O} \cap r| + 1 =$$

$$= \mathrm{gr}_t(\mathbf{y}_-) + 1 - t - t \cdot |\mathbb{X} \cap p| - (2-t) \cdot (|\mathbb{O} \cap p| + 1) + 1 = \mathrm{gr}_t(U^{t \cdot |\mathbb{X} \cap p| + (2-t) \cdot |\mathbb{O} \cap p|} \cdot \mathbf{y}_-).$$

- If $\mathbf{y}_+ \cap \beta_i \in \mathbf{B}$, then

$$\mathrm{gr}_t(\mathbf{x}_+) = \mathrm{gr}_t(\mathbf{y}_+) - t \cdot |\mathbb{X} \cap r| - (2-t) \cdot |\mathbb{O} \cap r| + 1 =$$

$$= \mathrm{gr}_t(\mathbf{y}_-) - 1 - t \cdot |\mathbb{X} \cap p| - (2-t) \cdot |\mathbb{O} \cap p| + 1 = \mathrm{gr}_t(U^{t \cdot |\mathbb{X} \cap p| + (2-t) \cdot |\mathbb{O} \cap p|} \cdot \mathbf{y}_-).$$

- If $\mathbf{y}_+ \cap \beta_i \in \mathbf{C}$, then

$$\mathrm{gr}_t(\mathbf{x}_+) = \mathrm{gr}_t(\mathbf{y}_+) - t \cdot |\mathbb{X} \cap r| - (2-t) \cdot |\mathbb{O} \cap r| + 1 =$$

$$= \mathrm{gr}_t(\mathbf{y}_-) - 1 + t - t \cdot (|\mathbb{X} \cap p| + 1) - (2-t) \cdot |\mathbb{O} \cap p| + 1 = \mathrm{gr}_t(U^{t \cdot |\mathbb{X} \cap p| + (2-t) \cdot |\mathbb{O} \cap p|} \cdot \mathbf{y}_-).$$

- If $\mathbf{y}_+ \cap \beta_i \in \mathbf{D}$, then

$$\mathrm{gr}_t(\mathbf{x}_+) = \mathrm{gr}_t(\mathbf{y}_+) - t \cdot |\mathbb{X} \cap r| - (2-t) \cdot |\mathbb{O} \cap r| + 1 =$$

$$= \mathrm{gr}_t(\mathbf{y}_-) - 1 - t \cdot |\mathbb{X} \cap p| - (2-t) \cdot |\mathbb{O} \cap p| + 1 = \mathrm{gr}_t(U^{t \cdot |\mathbb{X} \cap p| + (2-t) \cdot |\mathbb{O} \cap p|} \cdot \mathbf{y}_-).$$

For the right pentagons and for the map c_+ the computation goes the same way. □

Lemma 4.11. *The maps c_- and c_+ are chain maps.*

Proof. The proof is similar to the reasoning of Lemma 3.30. If we consider the expression $\partial_t^- \circ c_-(\mathbf{x}_+) + c_- \circ \partial_t^-(\mathbf{x}_+)$, then most of the domains contribute in pairs.

However, there might be two exceptional ones that admit a unique decomposition. These exceptional domains connect grid states $\mathbf{x}_+ \in \mathbb{G}_+$ and $\mathbf{x}_- \in \mathbb{G}_-$ that agree in all but one component. There are two thin annular regions A_1 and A_2 that have exactly three corners: one vertex is at s, and the other two are the components which distinguish $\mathbf{x}_+$ and $\mathbf{x}_-$ (see Figure 6.4 and 6.5 in [16]). Both A_1 and A_2 have a unique decomposition as either a juxtaposition of an empty pentagon with a vertex at s followed by an empty rectangle in $\mathbb{G}_-$ or as a juxtaposition of an empty rectangle in $\mathbb{G}_+$ followed by an empty pentagon with a vertex at s. Since A_1 and A_2 contains exactly the same X- and O-markings, their contributions to $\partial_t^- \circ c_-(\mathbf{x}_+) + c_- \circ \partial_t^-(\mathbf{x}_+)$ cancel. Therefore $\partial_t^- \circ c_-(\mathbf{x}_+) + c_- \circ \partial_t^-(\mathbf{x}_+) = 0$.

The same argument shows that c_+ is also a chain map. □

The above chain maps c_- and c_+ induce the desired maps C_- and C_+ on the homologies. In order to verify Proposition 4.9, we have to show that $C_- \circ C_+$ and $C_+ \circ C_-$ are both the multiplication by U^{2-t}. For this aim, we construct chain homotopies between the composites $c_- \circ c_+$ respectively $c_+ \circ c_-$ and multiplication by U^{2-t}.

For $\mathbf{x}_-, \mathbf{y}_- \in \mathbf{S}(\mathbb{G}_-)$, let $\mathrm{Hex}^\circ{}_{s,t}(\mathbf{x}_-, \mathbf{y}_-)$ denote the set of empty hexagons with two consecutive corners at s and at t in the order consistent with the orientation of the hexagon. The set $\mathrm{Hex}^\circ{}_{t,s}$ for $\mathbf{x}_+, \mathbf{y}_+ \in \mathbf{S}(\mathbb{G}_+)$ is defined analogously.

Let $H_- : tGC^-(\mathbb{G}_-) \to tGC^-(\mathbb{G}_-)$ be the $\mathcal{R}$-module map whose value on any $\mathbf{x}_- \in \mathbf{S}(\mathbb{G}_-)$ is

$$H_-(\mathbf{x}_-) = \sum_{\mathbf{y}_- \in \mathbf{S}(\mathbb{G}_-)} \sum_{h \in \mathrm{Hex}^\circ{}_{s,t}(\mathbf{x}_-, \mathbf{y}_-)} U^{t \cdot |\mathbb{X} \cap h| + (2-t) \cdot |\mathbb{O} \cap h|} \cdot \mathbf{y}_-.$$

The analogous map $H_+ : tGC^-(\mathbb{G}_+) \to tGC^-(\mathbb{G}_+)$ is defined in the same way using $\mathrm{Hex}^\circ{}_{t,s}(\mathbf{x}_+, \mathbf{y}_+)$.

It can be shown that H_- and H_+ drops the t-grading by $1-t$: if a hexagon h from $\mathbf{x}_+$ to $\mathbf{y}_+$ is counted in H_+, then there exists a corresponding empty rectangle r from $\mathbf{x}_+$ to $\mathbf{y}_+$ that contains one more X-marking, and the same number of O-markings as h.

Following the lines of the proof of Lemma 3.33, we can easily verify that H_+ is a chain homotopy between $c_+ \circ c_-$ and the multiplication by U^{2-t}, and that H_- is a chain homotopy between $c_- \circ c_+$ and U^{2-t}, i.e.:

$$\partial_t^- \circ H_+ + H_+ \circ \partial_t^- = c_+ \circ c_- + U^{2-t},$$

$$\partial_t^- \circ H_- + H_- \circ \partial_t^- = c_- \circ c_+ + U^{2-t}.$$

Therefore, we have that $C_- \circ C_+$ and $C_+ \circ C_-$ are both the multiplication by U^{2-t}, which completes the proof of Proposition 4.9. $\square$

As a corollary of this proposition, we can determine the rank of t-modified grid homology (recall that the rank of an $\mathcal{R}$-module M is the number of the free summands in ${}^{M}\!/_{\mathrm{Tors}(M)}$).

Lemma 4.12. *Let M and N be two modules over $\mathcal{R}$. If $\varphi : M \to N$ and $\psi : N \to M$ are two module maps with the property that $\psi \circ \varphi$ is the multiplication by U^α for some $\alpha \in \mathbb{R}_{\geq 0}$, then φ induces an injective map from ${}^{M}\!/_{\mathrm{Tors}(M)}$ into ${}^{N}\!/_{Tors(N)}$.*

Proof. Let $\mathbf{x} \in \mathrm{Tors}(M)$ be a torsion element, i.e. there exist an $r \in \mathcal{R}$ such that $r \cdot \mathbf{x} = 0$. Then $\varphi(\mathbf{x})$ is also a torsion element, since $r \cdot \varphi(\mathbf{x}) = \varphi(r \cdot \mathbf{x}) = 0$. Thus φ maps torsion elements to torsion elements and the same holds for ψ. So φ and ψ indeed induce well-defined homomorphisms between ${}^{M}\!/_{\mathrm{Tors}(M)}$ and ${}^{N}\!/_{Tors(N)}$.

For the injectivity it is enough to show that φ and ψ maps non-torsion elements to non-torsion elements. Suppose that for $\mathbf{x} \in M$ we have that $\varphi(\mathbf{x}) \in \mathrm{Tors}(N)$, that is, there exists an $s \in \mathcal{R}$ such that $s \cdot \varphi(\mathbf{x}) = 0$. Then $\psi(s \cdot \varphi(\mathbf{x})) = s \cdot \psi(\varphi(\mathbf{x})) = s \cdot U^t \cdot \mathbf{x} = 0$, so $\mathbf{x} \in \mathrm{Tors}(M)$, which means that φ (and ψ as well) maps non-torsion elements to non-torsion elements. $\square$

Lemma 4.13. *For any grid diagram $\mathbb{G}$ with grid number $n \geq 2$,*

$$ {}^{tGH^-(\mathbb{G})}\!/_{Tors(tGH^-(\mathbb{G}))} \cong \mathcal{R}^{2^{n-1}}. $$

Proof. For any $n \geq 2$ we can draw an $n \times n$ grid diagram $\mathbb{G}_n$ for the unknot, such that the X-markings are in the main diagonal of $\mathbb{G}_n$, and the O-markings are the eastern neighbours of the X-markings. With a straightforward calculation it can be shown that $tGH^-(\mathbb{G}_2) \cong \mathcal{R}^2$. Note that the diagram $\mathbb{G}_{n+1}$ can be obtained from $\mathbb{G}_n$ by a stabilization. Therefore the rank of $tGH^-(\mathbb{G}_{n+1})$ is twice the rank of $tGH^-(\mathbb{G}_n)$. Hence ${}^{tGH^-(\mathbb{G}_n)}\!/_{\mathrm{Tors}(tGH^-(\mathbb{G}_n))} \cong \mathcal{R}^{2^{n-1}}$.

Since any grid diagram $\mathbb{G}$ with grid number n can be connected to $\mathbb{G}_n$ by a sequence of crossing changes, from Proposition 4.9 and Lemma 4.12 we get an injective module map from ${}^{tGH^-(\mathbb{G})}\!/_{\mathrm{Tors}(tGH^-(\mathbb{G}))}$ to $\mathcal{R}^{2^{n-1}}$. As every submodule of $\mathcal{R}^m$ is of the form $\mathcal{R}^k$, where $k \leq m$, we have that ${}^{tGH^-(\mathbb{G})}\!/_{\mathrm{Tors}(tGH^-(\mathbb{G})} \cong \mathcal{R}^r$, where $r \leq 2^{n-1}$. Proposition 4.9 and Lemma 4.12 also give an inclusion of $\mathcal{R}^{2^{n-1}}$ into ${}^{tGH^-(\mathbb{G})}\!/_{\mathrm{Tors}(tGH^-(\mathbb{G}))}$, from which $r = 2^{n-1}$ follows. $\square$

Another corollary of Proposition 4.9 is that we can give a bound for the change of Υ under crossing changes:

Theorem 4.14. *If the knots K_+ and K_- differ in a crossing change, then for $t \in [1,2]$*

$$\Upsilon_{K_+}(t) \leq \Upsilon_{K_-}(t) \leq \Upsilon_{K_+}(t) + (2-t),$$

and from the symmetry of Υ, for $t \in [0,1]$:

$$\Upsilon_{K_+}(t) \leq \Upsilon_{K_-}(t) \leq \Upsilon_{K_+}(t) + t.$$

Proof. Consider a non-torsion element $\xi \in tGH^-(K_-)$ that has grading $\Upsilon_{K_-}(t)$.
As we have seen in the proof of Lemma 4.12, according to Proposition 4.9, $C_+(\xi)$ is non-torsion, and its grading is $\Upsilon_{K_-}(t) - 2 + t$. Thus $\Upsilon_{K_-}(t) \leq \Upsilon_{K_+}(t) + 2 - t$.
Similarly, if $\eta \in tGH^-(K_+)$ is a non-torsion element with grading $\Upsilon_{K_+}(t)$, then its image $C_-(\eta)$ has grading $\Upsilon_{K_+}(t)$ too. Since $C_-(\eta)$ is non-torsion, $\Upsilon_{K_+}(t) \leq \Upsilon_{K_-}(t)$. □

4.3 Saddle moves

Now we will examine, how Υ changes under saddle moves. Recall that a saddle move on a link L is performed by taking an embedded rectangle in S^3 that has two opposite sides incident to two arbitrary arcs of L, then cutting out these two arcs from the link and replace them by the other two sides of the rectangle. Note, that a saddle move changes the number of components by one. We distinguish two types of saddle moves by the property that they increase or decrease the number of components of the link. We call the former type *split move*, while the latter *merge move*.

Remark 4.15. *From this point on, we sometimes work with homologies of (oriented) links. The theory can be extended to links the same way as we did for knots, and many of the former statements hold.*

Proposition 4.16. *If $\vec{L}'$ is obtained from $\vec{L}$ by a split move, then there are $\mathcal{R}$-module maps*

$$\sigma : tGH^-(\vec{L}) \rightarrow tGH^-(\vec{L}')$$
$$\mu : tGH^-(\vec{L}') \rightarrow tGH^-(\vec{L})$$

with the following properties:

- *σ is homogeneous of degree t,*
- *μ is homogeneous of degree 0,*
- *$\mu \circ \sigma$ is multiplication by U^t,*
- *$\sigma \circ \mu$ is multiplication by U^t.*

Proof. Consider two grid diagrams $\mathbb{G}$ and $\mathbb{G}'$ that differ only in the placement of their X-markings in two consecutive columns. In those two columns, fix the circular ordering of the O- and the X-markings for $\mathbb{G}$ and $\mathbb{G}'$ according to Figure 4.2. Label the distinguished X-markings X_1 and X_2 for the diagram $\mathbb{G}$ and X_1' and X_2' for $\mathbb{G}'$ as in the figure. The vertical circle β_i separating the two markings is divided into two disjoint arcs **A** and **B**, in a way that **B** is adjacent to the O-markings, but A is not; X_1 is at the lower left end of **A**, while X_2 is at the upper right; X_1' is at the lower right and X_2' is at the upper left.

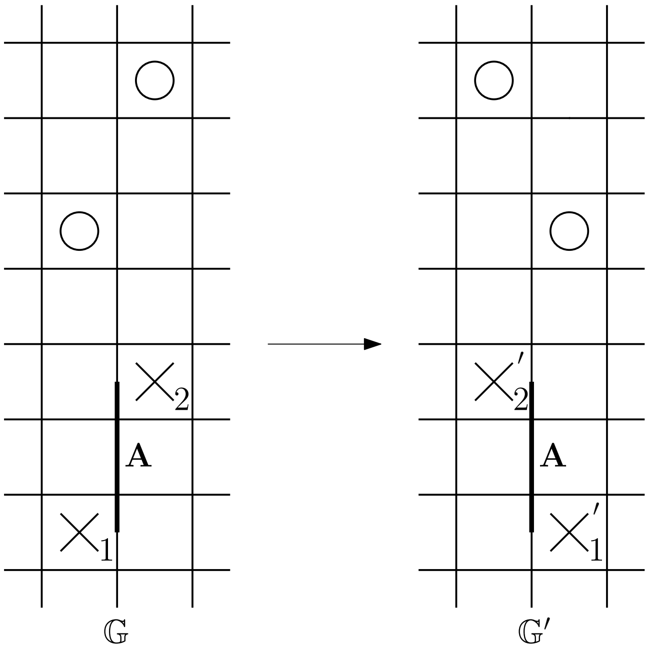


Figure 4.2:

If $\vec{L}$ and $\vec{L}'$ differ by a saddle move, then such grid diagrams $\mathbb{G}$ and $\mathbb{G}'$ for $\vec{L}$ and $\vec{L}'$ can be found.

Identify the grid states of $\mathbb{G}$ and $\mathbb{G}'$. We can classify them in two types: let $\mathcal{A}$ be the set of grid states whose element in the distinguished vertical circle β_i lies in the arc **A** between X_1 and X_2; and denote by $\mathcal{B}$ the set of grid states whose β_i-component is on the other arc **B**.

Define

$$\sigma : \ tGH^-(\vec{L}) \ \rightarrow \ tGH^-(\vec{L}') \quad \text{and} \quad \mu : \ tGH^-(\vec{L}') \ \rightarrow \ tGH^-(\vec{L})$$

by

$$\sigma(\mathbf{x}) = \begin{cases} U^t \cdot \mathbf{x} & \text{if } \mathbf{x} \in \mathcal{A} \\ \mathbf{x} & \text{if } \mathbf{x} \in \mathcal{B} \end{cases} \quad \text{and} \quad \mu(\mathbf{x}) = \begin{cases} \mathbf{x} & \text{if } \mathbf{x} \in \mathcal{A} \\ U^t \cdot \mathbf{x} & \text{if } \mathbf{x} \in \mathcal{B} \end{cases}$$

Obviously, both composite maps $\sigma \circ \mu$ and $\mu \circ \sigma$ are multiplications by U^t.

To prove that σ is a chain map, consider a rectangle $r \in \mathrm{Rect}^\circ(\mathbf{x}, \mathbf{y})$ connecting the grid states $\mathbf{x}$ and $\mathbf{y}$. Since the O-markings are at the same positions in both diagrams, r contains as many O in $\mathbb{G}$ as in $\mathbb{G}'$. It is easy to see that if both grid states $\mathbf{x}$ and $\mathbf{y}$ are in $\mathcal{A}$ or both are in $\mathcal{B}$, then the rectangle r intersects $\{X_1, X_2\}$ with the same multiplicity as it intersects $\{X_1', X_2'\}$ viewed as a rectangle in either $\mathbb{G}$ or $\mathbb{G}'$. If $\mathbf{x} \in \mathcal{A}$ and $\mathbf{y} \in \mathcal{B}$, then r, thought of as a rectangle in $\mathbb{G}$, contains exactly one of X_1 and X_2, but it does not contain either of X_1' or X_2'. Similarly, if $\mathbf{x} \in \mathcal{B}$ and $\mathbf{y} \in \mathcal{A}$, r contains neither X_1 or X_2, but it contains exactly one of X_1' and X_2'. It follows from these observations that σ is a chain map. The map μ is a chain map by the same logic.

Comparing the gradings for $\mathbb{G}$ and $\mathbb{G}'$, note that for an element $\mathbf{x} \in \mathcal{A}$ we have $M_{\mathbb{X}'}(\mathbf{x}) = M_{\mathbb{X}}(\mathbf{x}) + 1$, while for $\mathbf{x} \in \mathcal{B}$, $M_{\mathbb{X}'}(\mathbf{x}) = M_{\mathbb{X}}(\mathbf{x}) - 1$. Since for $\mathbf{x} \in \mathbf{S}(\mathbb{G})$ $M_{\mathbb{O}'}(\mathbf{x}) = M_{\mathbb{O}}(\mathbf{x})$, the Alexander gradings are given by

$$A(\mathbf{x}) = \frac{1}{2}(M_{\mathbb{O}}(\mathbf{x}) - M_{\mathbb{X}}(\mathbf{x})) - \frac{n-l}{2} \quad \text{and}$$

$$A'(\mathbf{x}) = \frac{1}{2}(M_{\mathbb{O}}(\mathbf{x}) - M_{\mathbb{X}'}(\mathbf{x})) - \frac{n-l-1}{2},$$

where l denotes the number of components of $\vec{L}$.
Hence, for any $\mathbf{x} \in \mathcal{A}$

$$\mathrm{gr}_t(\mathbf{x}) - \mathrm{gr}_t{}'(\mathbf{x}) = -t \cdot (A(\mathbf{x}) - A'(\mathbf{x})) =$$

$$= -\frac{t}{2} \cdot (-M_{\mathbb{X}}(\mathbf{x}) + M_{\mathbb{X}'}(\mathbf{x}) - 1) = 0,$$

and for any $\mathbf{x} \in \mathcal{B}$

$$\mathrm{gr}_t(\mathbf{x}) - \mathrm{gr}_t{}'(\mathbf{x}) = -t \cdot (A(\mathbf{x}) - A'(\mathbf{x})) =$$

$$= -\frac{t}{2} \cdot (-M_{\mathbb{X}}(\mathbf{x}) + M_{\mathbb{X}'}(\mathbf{x}) - 1) = t.$$

It follows that σ and μ have the stated behaviour on the bigradings. □

Theorem 4.17. *Let $\vec{L}$ and $\vec{L'}$ be two oriented links that differ by a saddle move, and suppose that $\vec{L'}$ has one more component than $\vec{L}$. Then,*

$$\Upsilon_{\vec{L}}(t) - t \leq \Upsilon_{\vec{L'}}(t) \leq \Upsilon_{\vec{L}}(t). \tag{4.1}$$

Proof. Consider a non-torsion element $\xi \in tGH^-(\vec{L})$ that has grading $\Upsilon_{\vec{L}}(t)$.
According to Proposition 4.16, $\sigma(\xi)$ is non-torsion and its grading is $\Upsilon_{\vec{L}}(t) - t$.
Thus $\Upsilon_{\vec{L}}(t) - t \leq \Upsilon_{\vec{L'}}(t)$.
Similarly, if $\eta \in tGH^-(\vec{L'})$ is a non-torsion element with grading $\Upsilon_{\vec{L'}}(t)$, then its

image $\mu(\eta)$ has grading $\Upsilon_{\vec{L}'}(t)$ too. Since $\mu(\eta)$ is non-torsion, $\Upsilon_{\vec{L}'}(t) \leq \Upsilon_{\vec{L}}(t)$. We get Inequality (4.1) by rearranging these two inequalities. □

4.4 The genus bound

The aim of this section is to show that the Υ invariant gives a lower bound for the genus of a cobordism connecting two knots. To this, we will need the following proposition about the normal form of cobordisms. For the proof and details see Appendix B.5 in [16].

Proposition 4.18. *Suppose that two knots K_1 and K_2 can be connected by a smooth, oriented, genus g cobordism. Then, there are knots K_1' and K_2' and integers b and d with the following properties:*

- *$\mathcal{U}_b(K_1)$ can be obtained from K_1' by b simultaneous saddle moves.*
- *K_1' and K_2' can be connected by a sequence of $2g$ saddle moves.*
- *$\mathcal{U}_d(K_2)$ can be obtained from K_2' by d simultaneous saddle moves.*

Lemma 4.19. *Let K_1 and K_2 be two knots so that K_2 can be derived from K_1 by a sequence of $2g$ saddle moves. Then $|\Upsilon_{K_1}(t) - \Upsilon_{K_2}(t)| \leq g \cdot t$.*

Proof. Consider $\vec{L}_1$ and $\vec{L}_2$ to be two links that are connected by a sequence of saddle moves. Denote by m the number of merge moves and by s the number of split moves in this sequence. By iterating Inequality (4.1) from Theorem 4.17, we get

$$\Upsilon_{\vec{L}_1}(t) \leq \Upsilon_{\vec{L}_2}(t) + m \cdot t \quad \text{and} \quad \Upsilon_{\vec{L}_2}(t) \leq \Upsilon_{\vec{L}_1}(t) + s \cdot t.$$

In case of the starting and ending links are knots, $m = s = g$. Therefore, the claim follows from the previous inequalities. □

Lemma 4.20. *If K_1 and K_2 are two knots such that K_2 can be obtained from $\mathcal{U}_d(K_1)$ by exactly d saddle moves, then $\Upsilon_{K_1}(t) = \Upsilon_{K_2}(t)$.*

Using these, we can verify the following bound:

Theorem 4.21. *Suppose that K_1 and K_2 are two knots that can be connected by a genus g cobordism in $[0,1] \times S^3$. Then*

$$|\Upsilon_{K_1}(t) - \Upsilon_{K_2}(t)| \leq t \cdot g.$$

Proof. Fix a genus g cobordism from K_1 to K_2, and let K_1' and K_2' be two knots as in the statement of Proposition 4.18. From Lemma 4.20 it follows that $\Upsilon_{K_1}(t) = \Upsilon_{K_1'}(t)$ and $\Upsilon_{K_2}(t) = \Upsilon_{K_2'}(t)$. By Lemma 4.19 $|\Upsilon_{K_1'}(t) - \Upsilon_{K_2'}(t)| \leq t \cdot g$, from which $|\Upsilon_{K_1}(t) - \Upsilon_{K_2}(t)| \leq t \cdot g$ follows. □

To sum up, we proved some important properties and applications of the Υ invariant. However, the topic discussed in this thesis is surrounded by many unanswered questions, hence it may be rewarding to make further researches.

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