# Covering questions in discrete geometry

MSc Thesis

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# Contents

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# 1. Introduction

According to Rogers' theorem [Rog57], the space $\mathbb{R}^d$ can be covered by translates of a given convex body of density at most $d \ln d + d \ln \ln d + 5d$. This was the first milestone in the theory of translative coverings, earlier only exponential bounds were obtained in this question. Rogers' theorem also implies some bounds in the problem of covering a convex body by a minimal number of translates of another convex body. The aim of this thesis is to discuss some versions of these questions.

Let $K$ and $L$ be convex bodies in $\mathbb{R}^d$ and $\mathcal{F} = \{\lambda_1 K, \lambda_2 K, \dots\}$ $(0 < \lambda_i < 1)$ a family of positive, smaller homothets of $K$. We look for bounds $t$, depending on $K$ and $L$, for which if the total volume of the family $\mathcal{F}$ is at least $t$, then $\mathcal{F}$ permits a translative covering of $L$, i.e. $L$ can be covered by translates of members of $\mathcal{F}$. We are also interested in the question of finding the most economical covering of the whole space by translates of members of $\mathcal{F}$ if $\sum_i \lambda_i^d = \infty$. Then, we will discuss the case of families with an unbounded set of ratios of homothety. Finally, returning to translates, we will consider multiple coverings of the space $\mathbb{R}^d$ by translates of a given convex body.

The structure of this thesis is the following.

In Chapter 2, we introduce some basic definitions and notations, related to covering questions.We mention some relevant problems and results.

In Chapter 3, we state and prove Rogers' theorem and a theorem of Rogers and Zong [RZ97].

In Chapter 4, we prove a number of new results.

First, let $K$ be a convex body in $\mathbb{R}^d$, and $\mathcal{F} = \{\lambda_1 K, \lambda_2 K, \dots\}$ a family of its positive smaller homothets. Let $f(K, K)$ denote the infimum of those $t$ for which the following holds: If

$$\sum_i \lambda_i^d \geq t$$

then $\mathcal{F}$ permits a translative covering of $K$. The known best upper bound for $f(K, K)$ was $3^d$ if $K$ is centrally symmetric, and $6^d$ in the general case [Nas10]. We improve this bound, and show that:

$$f(K,K) \leq \begin{cases} (d^3 \cdot \ln d \cdot \vartheta(K) + e) \cdot 2^d & \text{if } K = -K \\ d^3 \cdot \ln d \cdot \vartheta(K) \cdot \binom{2d}{d} + e \cdot 4^d & \text{in general,} \end{cases}$$

where $\vartheta(K)$ is the lowest density that can be attained by the coverings of $\mathbb{R}^d$ by translates of $K$.

Second, we will also prove that if $\sum_i \lambda_i^d = \infty$ then $\mathcal{F}$ permits a translative covering of $\mathbb{R}^d$ of density at most $d \ln d + d \ln \ln d + 5d$.

Finally, we will show that if $\mathcal{F}$ is a family of homothets of $K$ such that the homothet-ratios are not bounded, then we can bound the covering multiplicity. Namely, then $\mathcal{F}$ permits a translative covering of the whole space such that every point is covered at most $2d$ times if $K$ has smooth boundary, and at most $4d$ times in general. Note that, this implies that the covering density of $\mathcal{F}$ is at most $2d$ (resp. $4d$).

In Chapter 5, we consider $k$-fold covering of $\mathbb{R}^d$ of low density by translates of a given convex body $K$, that is a covering in which every point is covered at least $k$ times. The trivial bound on the density based on Rogers' result would be around $k \ln d$. We prove the bound $18\mathrm{e} \ln d$, as long as $k$ is at most $d \ln d$. This bound is independent of $k$ (provided that $k$ is in the given range).

# 2. Basic definitions and results

We will denote the origin by $o$, and the closed solid ball of radius $r$ centered at $a$ (in the relevant dimension) by $\mathbf{B}(a, r)$. A subset $K$ of $\mathbb{R}^d$ is a *convex body*, if it is compact, convex and has non-empty interior. We say that a convex body $K$ is *smooth* at a point $x \in \partial K$, if $K$ has a unique supporting hyperplane through $x$.

## 2.1 Covering by translates of a convex body

We say that the subsets $A_1, A_2, \dots$ of $\mathbb{R}^d$ *cover* the set $K \subseteq \mathbb{R}^d$ if $\bigcup_i A_i \supseteq K$.

**Definition 2.1.** For two convex bodies $K$ and $L$ in $\mathbb{R}^d$ let $N(L, K)$ denote the *translative covering number* of $L$ by $K$, that is the minimum number of translates of $K$ that cover $L$.

By considering volumes, we obtain an obvious lower bound:

$$N(L, K) \geq \frac{\text{vol}\,(K)}{\text{vol}\,(L)}.$$

There is also a well-known upper bound.

**Claim 2.2.** *Let $K$ and $L$ be convex bodies in $\mathbb{R}^d$ with $o \in \operatorname{int} K$. Then*

$$N(L, K) \leq 2^d \frac{\text{vol}\left(L + \frac{1}{2}(K \cap (-K))\right)}{\text{vol}\,((K \cap (-K))}.$$

We say that $K$ is *symmetric about the origin* if $K = -K$. Clearly, $K \cap (-K)$ is symmetric. We state without proof the following statement, according to which, the origin can be chosen such that the volume of $K \cap (-K)$ is not too small.

**Claim 2.3** ([Ste13])**.** *For any convex body $K$ in $\mathbb{R}^d$ there is a point $x \in K$ such that*

$$\text{vol}\,(K \cap (2x - K)) \geq \frac{1}{2^d} \text{vol}\,(K)\,. \tag{2.1}$$

In fact, in [MP00] it is shown that (2.1) holds if $x$ is the centroid of $K$.

*Proof of Claim 2.2:* Let $\Lambda \subset \mathbb{R}^d$ be a finite set and $\Lambda + \frac{1}{2}(K \cap (-K))$ a saturated packing of translates of $\frac{1}{2}(K \cap (-K))$ in $L + \frac{1}{2}(K \cap (-K))$, that is a maximal family of translates of $\frac{1}{2}(K \cap (-K))$ with pairwise disjoint interiors. Then

$$L \subseteq \Lambda + K,$$

since, if a point $x \in L$ was not covered by $\Lambda + K$, then $x + \frac{1}{2}(K \cap (-K))$ would be disjoint from any member of $\Lambda + \frac{1}{2}(K \cap (-K))$, contradicting the assumption that $\Lambda + \frac{1}{2}(K \cap (-K))$ is maximal. Now the statement clearly follows by considering volumes. □

**Remark 2.4.** If $K = -K$, then the statement above yields

$$N(L, K) \le 2^d \frac{\mathrm{vol}\left(L + \frac{1}{2}K\right)}{\mathrm{vol}\,(K)}.$$

The quantity $N(K, L)$ is related to the *illumination conjecture*. We say that a point $p \in \mathbb{R}^d \setminus K$ *illuminates* the boundary point $q$ of the convex body $K$, if the half line emenating from $p$ passing trough $q$ intersects the interior of $K$ at a point not between $p$ and $q$. Furthermore, the exterior points $p_1, p_2, \ldots p_n$ *illuminate* $K$ if each boundary point of $K$ is illuminated by at least one of the light sources $p_1, p_2, \ldots p_n$.

**Definition 2.5.** The smallest $n$ for which there exist $n$ exterior points of $K$ that illuminate $K$, is called the the *illumination number* of $K$. We denote the illumination number of $K$ by $i(K)$.

**Conjecture 2.6** (Gohberg-Markus-Levi-Boltyanski-Hadwiger)**.** *For a convex body $K \subseteq \mathbb{R}^d$ the illumination number $i(K)$ of $K$ is at most $2^d$ and $i(K) = 2^d$ if and only if $K$ is an affine $d$-dimensional cube.*

It is easy to see that $i(K)$ equals the infimum over $1 > \varepsilon > 0$ of those $n$ for which $K$ can be covered by $n$ translates of $(1-\varepsilon)K$. From this, $i(Q^d) = 2^d$, where $Q^d$ is a $d$-dimensional cube, clearly follows: Any translate of $(1-\varepsilon)Q$ can cover only one vertex of $Q^d$. Levi [Lev55] observed that $i(K) \ge d+1$ for any convex body $K$ in $\mathbb{R}^d$, and, if $K$ has smooth boundary, than $i(K) = d+1$.

In Chapter 3, we will recall the current best upper bound on $i(K)$ in general. For more background about the illumination number see [Bez10].

**Remark 2.7.** The quantities $N(L, K)$ and $i(K)$ are *affine invariant*, that is, if $T$ is an invertible affine transformation of $\mathbb{R}^d$, then $N(L, K) = N(T(L), T(K))$ and $i(K) = i(T(K))$.

## 2.2 Homothetic coverings

A *homothet* of a convex body $K$ is a set of the form $\lambda K$, where $\lambda \in \mathbb{R}^+$. (In general, $\lambda K$ with negative $\lambda$ is also called a homothet of $K$, but we will consider only positive homothets.) We say that a family $\mathcal{F} = \{\lambda_1 K, \lambda_2 K, \dots\}$ of positive, smaller homothets of the convex body $K$ (i.e. $0 < \lambda_i < 1$) permits a *translative covering* of the convex body $L$, if there exist translation vectors $x_1, x_2, \dots$ such that

$$L \subseteq \bigcup_i x_i + \lambda_i K.$$

**Definition 2.8.** For two given convex bodies $K$ and $L$ in $\mathbb{R}^d$ we define $f(L, K)$ as the minimum of those $t > 0$ such that, for any family $\mathcal{F}$ of positive, smaller homothets of $K$ with coefficients $0 < \lambda_1 \leq \lambda_2 \leq \cdots < 1$, the following holds:

$$\text{If } \sum_i \lambda_i^d \geq t \text{ then } \mathcal{F} \text{ permits a translative covering of } L.$$

Moreover, we set

$$f(d) := \sup\{f(K, K) : K \subset \mathbb{R}^d \text{ convex body}\}.$$

The problem of bounding $f(K, L)$, in the case when $K = L$ and the dimension is two, was originally posed by L. Fejes Tóth [BMP05]. An early upper bound, given by Januszewski [Jan98] on $f(K, K)$ was of order $d^d \operatorname{vol}(K)$. In Chapter 4, we will prove better bounds on $f(L, K)$ and $f(K, K)$.

From the remarks about the illumination number, $f(d) \geq 2^d - 1$, and $f(K, K) \geq d$ ($K \subseteq \mathbb{R}^d$ convex body) easily follow.

Meir and Moser [MM68], and later Bezdek and Bezdek [BB84] proved that for the $d$-dimensional cube $Q^d$ we have $f(Q^d, Q^d) = 2^d - 1$.

**Conjecture 2.9** (L. Fejes Tóth)**.** *$f(2) = 3$, i.e. for any convex body $K$ in the plane, if for a family of its homothets $\mathcal{F} = \{\lambda_1, \lambda_2, \dots\}$ we have $\sum_i \lambda_i^2 \geq 3$, then $\mathcal{F}$ permits a translative covering of $K$.*

The best known result for the case of the plane is $f(3) \leq 6.5$ [Jan98]. For more background on this problem, see Chapter 3 of [BMP05]. Our conjecture for dimension $d$ is the following.

**Conjecture 2.10.** *$f(d) = 2^d - 1$.*

## 2.3 Covering of the whole space, density

Now, let us define the density of a family of convex bodies.

**Definition 2.11.** Let $\mathcal{F}$ be a collection of convex bodies in $\mathbb{R}^d$. Then the density of $\mathcal{F}$ with respect to $\mathbf{B}(o, r)$ is defined as

$$d(\mathcal{F}, r) = \frac{\sum_{F \in \mathcal{F}, F \cap \mathbf{B}(o,r) \neq \emptyset} \operatorname{vol}(F)}{\operatorname{vol}(\mathbf{B}(o, r))}.$$

The *lower* and *upper densities* of $\mathcal{F}$ (with respect to the whole space) are defined as follows:

$$\underline{d}(\mathcal{F}, \mathbb{R}^d) = \liminf_{r \to \infty} d(\mathcal{F}, r)$$

and

$$\overline{d}(\mathcal{F}, \mathbb{R}^d) = \limsup_{r \to \infty} d(\mathcal{F}, r),$$

respectively. If $\underline{d}(\mathcal{F}, \mathbb{R}^d)$ and $\overline{d}(\mathcal{F}, \mathbb{R}^d)$ coincide, then their common value is called the *density* of $\mathcal{F}$, and is denoted by $d(\mathcal{F}, \mathbb{R}^d)$.

A family $\mathcal{F}$ is *bounded*, is there exists a constant $C$ such that for any member $F \in \mathcal{F}$ we have $\operatorname{diam} F \leq C$.

**Claim 2.12.** *Let $\mathcal{F}$ be a bounded family of convex bodies in $\mathbb{R}^d$. Then*

$$\underline{d}(\mathcal{F}, \mathbb{R}^d) = \liminf_{r \to \infty} \frac{\sum_{F \in \mathcal{F}, F \cap \mathbf{B}(o,r) \neq \emptyset} \operatorname{vol}(F \cap \mathbf{B}(o, r))}{\operatorname{vol}(\mathbf{B}(o, r))} = \liminf_{r \to \infty} \frac{\sum_{F \in \mathcal{F}, F \subseteq \mathbf{B}(o,r)} \operatorname{vol}(F)}{\operatorname{vol}(\mathbf{B}(o, r))}$$

*and*

$$\overline{d}(\mathcal{F}, \mathbb{R}^d) = \limsup_{r \to \infty} \frac{\sum_{F \in \mathcal{F}, F \cap \mathbf{B}(o,r) \neq \emptyset} \operatorname{vol}(F \cap \mathbf{B}(o, r))}{\operatorname{vol}(\mathbf{B}(o, r))} = \limsup_{r \to \infty} \frac{\sum_{F \in \mathcal{F}, F \subseteq \mathbf{B}(o,r)} \operatorname{vol}(F)}{\operatorname{vol}(\mathbf{B}(o, r))},$$

*moreover, $\underline{d}(\mathcal{F}, \mathbb{R}^d)$ and $\overline{d}(\mathcal{F}, \mathbb{R}^d)$ are independent of the choice of the origin.*

*Proof:* Let $\ell$ denote the diameter of the member of $\mathcal{F}$ with largest diameter. Consider the following inequalities:

$$\frac{\sum\limits_{F\in\mathcal{F},F\cap\mathbf{B}(o,r)\neq\emptyset}\operatorname{vol}(F\cap\mathbf{B}(o,r))}{\operatorname{vol}(\mathbf{B}(o,r))}\leq\frac{\sum\limits_{F\in\mathcal{F},F\cap\mathbf{B}(o,r)\neq\emptyset}\operatorname{vol}(F)}{\operatorname{vol}(\mathbf{B}(o,r))}\leq$$

$$\frac{\sum\limits_{F\in\mathcal{F},F\cap\mathbf{B}(o,r+\ell)\neq\emptyset}\operatorname{vol}(F\cap\mathbf{B}(o,r+\ell))}{\operatorname{vol}(\mathbf{B}(o,r))}=$$

$$\frac{\sum\limits_{F\in\mathcal{F},F\cap\mathbf{B}(o,r+\ell)\neq\emptyset}\operatorname{vol}(F\cap\mathbf{B}(o,r+\ell))}{\operatorname{vol}(\mathbf{B}(o,r+\ell))}\cdot\frac{\operatorname{vol}(\mathbf{B}(o,r+\ell))}{\operatorname{vol}(\mathbf{B}(o,r))}$$

and

$$\frac{\sum\limits_{F\in\mathcal{F},F\subseteq\mathbf{B}(o,r)}\operatorname{vol}(F)}{\operatorname{vol}(\mathbf{B}(o,r))}\leq\frac{\sum\limits_{F\in\mathcal{F},F\cap\mathbf{B}(o,r)\neq\emptyset}\operatorname{vol}(F)}{\operatorname{vol}(\mathbf{B}(o,r))}\leq$$

$$\frac{\sum\limits_{F\in\mathcal{F},F\subseteq\mathbf{B}(o,r+\ell)}\operatorname{vol}(F)}{\operatorname{vol}(\mathbf{B}(o,r))}=\frac{\sum\limits_{F\in\mathcal{F},F\subseteq\mathbf{B}(o,r+l)}\operatorname{vol}(F)}{\operatorname{vol}(\mathbf{B}(o,r+\ell))}\cdot\frac{\operatorname{vol}(\mathbf{B}(o,r+\ell))}{\operatorname{vol}(\mathbf{B}(o,r))}.$$

Taking supremums and infimums, we obtain the first two statements. For the origin-independence, consider

$$\frac{\sum\limits_{F\in\mathcal{F},F\subseteq B(o_1,r)}\operatorname{vol}(F)}{\operatorname{vol}(B(o_1,r))}\leq\frac{\sum\limits_{F\in\mathcal{F},F\subseteq B(o_2,r+t)}\operatorname{vol}(F)}{\operatorname{vol}(B(o_1,r))}=$$

$$\frac{\sum\limits_{F\in\mathcal{F},F\subseteq B(o_2,r+t)}\operatorname{vol}(F)}{\operatorname{vol}(B(o_2,r+t))}\cdot\frac{\operatorname{vol}(B(o_2,r+t))}{\operatorname{vol}(B(o_1,r))}\leq\frac{\sum\limits_{F\in\mathcal{F},F\subseteq B(o_1,r+2t)}\operatorname{vol}(F)}{\operatorname{vol}(B(o_1,r))},$$

where $t$ is the distance of $o_1$ and $o_2$.

□

For a convex body $K$ we define its covering density $\vartheta(K)$ as the density of the most economical covering of the whole space by translates of $K$. More precisely:

**Definition 2.13.** Let $K$ be a convex body in $\mathbb{R}^d$. Then the *translative covering density* of $K$ is

$$\vartheta(K)=\inf_{\mathcal{F}_K} d(\mathcal{F}_K,\mathbb{R}^d)$$

where the infimum is taken over those coverings of $\mathbb{R}^d$ by translates of $K$, for which the density if defined.

**Remark 2.14.** Indeed, the infimum in Definition 2.13 is a minimum, i.e. it can be shown that $\vartheta(K)$ is attained by a suitable covering.

**Definition 2.15.** For a Borel measurable set $A \subseteq \mathbb{R}^d$, we define its *asymptotic lower* and *upper density* as

$$\underline{d}(A, \mathbb{R}^d) = \inf_{r \to \infty} \frac{\text{vol}\,(A \cap \mathbf{B}(o, r))}{\text{vol}\,(\mathbf{B}(o, r))}$$

and

$$\overline{d}(A, \mathbb{R}^d) = \sup_{r \to \infty} \frac{\text{vol}\,(A \cap \mathbf{B}(o, r))}{\text{vol}\,(\mathbf{B}(o, r))}$$

respectively. If $\underline{d}(A, \mathbb{R}^d) = \overline{d}(A, \mathbb{R}^d)$, then the asymptotic density of $A$ is $d(A, \mathbb{R}^d) = \underline{d}(A, \mathbb{R}^d) = \overline{d}(A, \mathbb{R}^d)$.

## 2.4 Lattices and periodic arrangements

We give two equivalent definitions of lattices in $\mathbb{R}^d$.

**Definition 2.16.** Let $v_1, \dots, v_d \in \mathbb{R}^d$ form a basis of $\mathbb{R}^d$. Then

$$\Lambda = \{\alpha_1 v_1 + \cdots + \alpha_d v_d : \alpha_i \in \mathbb{Z}\}$$

is a *lattice*, generated by the vectors $v_1, \dots, v_d$.

**Definition 2.17.** $\Lambda \subseteq \mathbb{R}^d$ is a *lattice* if

1. $\Lambda$ is topologically discrete
2. $\Lambda$ spans $\mathbb{R}^d$
3. $\Lambda$ is a subgroup of $(\mathbb{R}^d, +)$.

It is not hard to check that Definition 2.16 is equaivalent to Definition 2.17. $B = \{v_1, \dots v_d : v_i \in \mathbb{R}^d\}$ is a *basis* of the lattice $\Lambda$, if $\Lambda = \{\alpha_1 v_1 + \dots \alpha_d v_d : \alpha_i \in \mathbb{Z}\}$. Note that, the basis of a lattice is not unique.

**Definition 2.18.** Let $\Lambda$ be a lattice generated by the basis $v_1, \dots, v_d$. The *fundamental domain* (or the *base parallelotope*) of $\Lambda$ corresponding to the basis $v_1, \dots, v_d$ is the parallelotope

$$P_\Lambda = \{\alpha_1 v_1 + \cdots + \alpha_d v_d : \alpha_i \in [0, 1]\}.$$

We define the *determinant* of $\Lambda$ as

$$\det \Lambda = |\det(v_1, \dots, v_d)| = \text{vol}\,(P)\,.$$

**Definition 2.19.** Let $K$ be a convex body, $\Lambda$ a lattice and $T$ a finite set in $\mathbb{R}^d$. We call the family

$$\mathcal{F} = K + \Lambda + T = \{K + v + t : v \in \Lambda, t \in T\}$$

a *periodic arrangement* of translates of $K$. The density of $\mathcal{F}$ is defined as

$$\delta(\mathcal{F}) = \frac{|T| \operatorname{vol}(K)}{\det \Lambda}.$$

The *periodic translative covering density* $\vartheta_p(K)$ of $K$ is the infimum of the densities of periodic coverings of $\mathbb{R}^d$ by translates of $K$.

It can be shown that the definition of $\delta(\mathcal{F})$ agrees with the definition of $d(\mathcal{F}, \mathbb{R}^d)$. As it was true for the translative covering density, the periodic translative covering density $\vartheta_p(K)$ is also attained by a suitable periodic arrangement.

If $T = \{0\}$, then we call the arrangement a *lattice arrangement.* In this special case $\delta(\mathcal{F}) = \frac{\operatorname{vol}(K)}{\det \Lambda}$. The *lattice covering density* $\vartheta_L(K)$ of $K$ is the infimum of the densities of lattice coverings of $\mathbb{R}^d$ by translates of $K$.

Clearly $1 \le \vartheta(K) \le \vartheta_P(K) \le \vartheta_L(K)$. We note that all known upper bound on $\vartheta(K)$ obtained by a periodic arrangements. In Chapter 3, we will recall Rogers' theorem which gives the current best known upper bound on $\vartheta_P(K)$ which is approximately $d \ln d$. The current upper bounds on the lattice covering density are much weaker. Rogers showed that [Rog59] for any convex body $K$ there is a lattice $\Lambda$ such that $\Lambda + K$ is a covering of $\mathbb{R}^d$ of density at most $d^{\log_2 \ln d + O(1)}$.

A natural approach to finding a covering by translates of $K$ would be the following. Take a cube $Q \subseteq K$ and consider the tiling of the space by $Q$, with translation vectors $x_1, x_2, \ldots$. Then $\bigcup\limits_i x_i + K$ is a covering of $\mathbb{R}^d$ of density $\frac{\operatorname{vol}(K)}{\operatorname{vol}(Q)}$. However, this method gives weak bounds, since the ratio $\frac{\operatorname{vol}(K)}{\operatorname{vol}(Q)}$ may be very large for every cube $Q$ contained in $K$. From the opposite direction, in general, the periodic covering density can not be too small. For the $d$-dimensional ball, $\vartheta_P(\mathbf{B}(o,1)) \ge Cd$ with a universal constant $C > 0$ (cf. [CFR59]).

In the plane we have the following theorem.

**Theorem 2.20** (Fáry [Fár50])**.** *For any convex body $K \subseteq \mathbb{R}^2$ $\vartheta_L(K) \le \frac{2}{3}$, and equality holds if and only if $K$ is a triangle.*

## 2.5 John's ellipsoid

In the proof of some theorems in the next chapters, we will need some tools.

**Definition 2.21.** Let $A \in \mathbb{R}^{d\times d}$ be a positive definite, symmetric matrix. Then $\mathcal{E} = \{x \in \mathbb{R}^d : x^T Ax \leq 1\}$ is an *ellipsoid* centered at the origin.

It is a basic fact from linear algebra that $\varepsilon$ is an ellipsoid, if and only if it is an affine image of a ball. Note that any ellipsoid centered at $\underline{0}$ can be written as

$$\mathcal{E} = \left\{x \in \mathbb{R}^d : \sum_{i=1}^{d} \frac{< x, e_i >^2}{\alpha_i^2} \leq 1\right\}$$

for some orthonormal basis $e_1, \ldots, e_d$ and reals $\alpha_1, \ldots, \alpha_d > 0$.

**Theorem 2.22** (John [Joh48]). *Each convex body $K \subseteq \mathbb{R}^d$ contains a unique ellipsoid of maximal volume. This ellipsoid is $\mathbf{B}(o, 1)$ if and only if the following conditions are satisfied: $\mathbf{B}(o, 1) \subseteq K$ and for some integer $m$, there are unit vectors $u_1, \ldots, u_m$ on the boundary of $K$ and positive numbers $c_1, \ldots, c_m$ satisfying*

$$\sum_{i=1}^{m} c_i u_i = o \tag{2.2}$$

*and*

$$\sum_{i=1}^{m} c_i < x, u_i >^2 = |x|^2 \text{ for each } x \in \mathbb{R}^d \tag{2.3}$$

The condition 2.2 is equivalent to the statement that

$$x = \sum c_i < x_i, u_i > u_i \text{ for all } x \in \mathbb{R}^d, \tag{2.4}$$

or it can be written in matrix notation as

$$\sum c_i u_i \otimes u_i = I_d,$$

where $I_n$ is the identity map on $\mathbb{R}^d$, and for a unit vector $u$ $u \otimes u$ is the map $x \mapsto < x, u > u$ from $\mathbb{R}^d$ to itself.

If the maximal volume ellipsoid of a convex body $K$ is the ball $\mathbf{B}(o, 1)$, then we say that the body is in *John's position*. For any convex body $K$ there exists an affine transformation $T$ such that $T(K)$ is in John's position.

**Corollary 2.23.** *For any convex body $K$ in $\mathbb{R}^d$, after an affine transformation we can arrange that*

$$\mathbf{B}(o, 1) \subseteq K \subseteq d\mathbf{B}(o, 1).$$

*If $K$ is centrally symmetric, a stronger statement holds*

$$\mathbf{B}(o, 1) \subseteq K \subseteq \sqrt{d}\mathbf{B}(o, 1).$$

For details and proofs see Lecture 3. in [Bal97].

The main reason for recalling John's theorem, is the following lemma.

**Lemma 2.24.** *Let $K \subseteq \mathbb{R}^d$ be a convex body. Then there exist $j \leq 2d$ points $\{x_1, x_2, \ldots, x_j\}$ on the boundary of $K$, so that $K$ is smooth in each $x_i$ $(1 \leq i \leq j)$ and $\bigcap_i H_{i+} = L$ is a bounded set, where $H_{i+}$ is the halfspace that contains $K$, bounded by the tangent hyperlpane $H_i$ at $x_i$.*

Before proving it, we recall Steinitz's theorem.

**Theorem 2.25** (Steinitz [Ste13])**.** *Let $A \subset \mathbb{R}^d$ and $x \in \operatorname{int}\operatorname{conv} A$. then there exist at most $2d$ points $x_1, x_2, \ldots, x_j$ in $A$ such that $x \in \operatorname{int}\operatorname{conv}\{x_1, x_1, \ldots x_j\}$.*

*Proof of Lemma 2.24:* We may assume that $K$ is in John's position. Consider the unit vectors $u_i$ in Theorem 2.22. In each $u_i$, $K$ is clearly smooth (since $u_i$ is a contact point of an inscribed ellipsoid and $K$), and by condition (2.2), $o \in \operatorname{int}\operatorname{conv}\{u_1, u_2, \ldots, u_m\}$. By Theorem 2.25 there exist at most $2d$ point $x_1, \ldots, x_j$ in $\{u_1, \ldots, u_m\}$ such that $\underline{0} \in \operatorname{int}\operatorname{conv}\{x_1, \ldots, x_j\}$. Then for the tangent hyperplanes $H_i$ at $x_i$ $\bigcap_i H_{i+} = L$ is bounded. $\square$

# 3. Classical results

In this chapter, first we prove Rogers' theorem about the existence of an economical (low density) covering of space by translates of any given convex body. Then, we prove the theorem of Rogers and Zong, which gives a bound on the covering number. As a corollary of these, we obtain an upper bound on the illumination number.

**Theorem 3.1** (Rogers [Rog57]). *Let $K$ be a convex body in $\mathbb{R}^d$ ($d \geq 3$). Then the translative covering density of $K$ is at most:*

$$\vartheta(K) \leq d \ln d + d \ln \ln d + 5d.$$

We will follow the step of the proof in [Rog57]. For any $S \subseteq \mathbb{R}^d$ let $\chi_S$ denote the characteristic function of $S$.

*Proof:* We may assume that $\mathrm{vol}\,(K) = 1$, $o$ is the centroid of $K$ and $K$ is open. Let $R \in \mathbb{R}^+$ to be set later and

$$\Lambda = \{(\alpha_1 R, \alpha_2 R, \dots \alpha_d R) : \alpha_i \in \mathbb{Z}\}$$

be a lattice. If $R$ is sufficiently large, then

$$(g_1 + K) \bigcap (g_2 + K) = \emptyset,$$

for different elements $g_1, g_2$ of $\Lambda$.

Fix $N$, and choose $N$ points $x_1, x_2, \dots x_N$ independently of uniform random distribution from $C = [0, R)^d$.

Consider the system

$$\{K + x_i + g : 1 \leq i \leq N, g \in \Lambda\}. \tag{3.1}$$

We clearly have that $(K + x_i + g_1) \bigcap (K + x_i + g_2) = \emptyset$, if $g_1 \neq g_2$, therefore

$$\chi_{K+x_i+\Lambda}(x) = \sum_{g \in \Lambda} \chi_K(x - x_i - g).$$

This yields

$$\chi_E(x)=\prod_{i=1}^{N}\left(1-\sum_{g\in\Lambda}\chi_K\left(x-x_i-g\right)\right),$$

where $E$ is the set of the points, that are not covered by the system (3.1).

Since $\chi_E$ is periodic in each coordinate with period $R$, $d(E,\mathbb{R}^d)=\frac{\mathrm{vol}(C\cap E)}{\mathrm{vol}(C)}$. The volume of $C\cap E$ is

$$\int_C\prod_{i=1}^{N}\left(1-\sum_{g\in\Lambda}\chi_K(x-x_i-g)\right)dx.$$

We compute now the expected value of $\mathrm{vol}\,(C\cap E)$:

$$\begin{aligned}&\mathbb{E}(\mathrm{vol}\,(C\cap E))=\\&R^{-dN}\int_C\int_C\cdots\int_C\left(\int_C\prod_{i=1}^{N}\left(1-\sum_{g\in\Lambda}\chi_K\left(x-x_i-g\right)\right)dx\right)dx_1dx_2\dots dx_N=\\&\qquad R^{-dN}\int_C\left(\prod_{i=1}^{N}\int_C\left(1-\sum_{g\in\Lambda}\chi_K(x-x_i-g)\right)dx_i\right)dx=\\&\qquad R^{-dN}\int_C\left(\prod_{i=1}^{N}\left(R^d-\sum_{g\in\Lambda}\int_{C-x+g}\chi_K(-y)dy\right)\right)dx.\end{aligned}$$

Since the sets $C-x+g$ $(g\in\Lambda)$ are disjoint and cover the space $\mathbb{R}^d$, the expected value of $\mathrm{vol}\,(C\cap E)$ is

$$R^{-dN}R^d\big(R^d-\int_{\mathbb{R}^d}\chi_K(-y)dy\big)^N=R^d(1-R^{-d})^N.$$

Therefore, we can choose vectors $x_1,x_2,\dots x_N$ of $C$ such that

$$\mathrm{vol}\,(C\cap E)\le R^d(1-R^{-d})^N,$$

and hence,

$$d(E,\mathbb{R}^d)\le(1-R^{-d})^N.$$

Note that $(1-R^{-d})^N$ is "small" if $R^{-d}N$ is "large". Now we only need to cover the set $E$.

Let $\eta\in\mathbb{R}$ so that $0<\eta<\frac{1}{d}$, and let $M$ be the largest integer for which there exist translation vectors $x_1,y_2,\dots y_M$ such that the system

$$\{-\eta K+y_i+g:1\le i\le M,g\in\Lambda\}\tag{3.2}$$

is a packing and

$$\left(\bigcup_{g\in\Lambda}\bigcup_{i=1}^{N}(K+x_i+g)\right)\bigcap\left(\bigcup_{h\in\Lambda}\bigcup_{j=1}^{M}(-\eta K+y_i+h)\right)=\emptyset.$$

The volume of the intersection of the system (3.2) with $C$ is clearly $M\eta^d$.

The bodies in the system (3.2) lie in $E$, hence $M\eta^d$ does not exceed the volume of $C\cap E$, that is,

$$M\eta^d \le R^d(1-R^-d)^N,$$

or, equivalently,

$$M \le \eta^{-d}R^d(1-R^-d)^N. \tag{3.3}$$

Since $0 < \eta < 1$, therefore, by the choice of $R$, it follows that $(-\eta K + y + g_1)\bigcap(-\eta K + y + g_2) = \emptyset$ if $g_1$ and $g_2$ are different elements of $\Lambda$. By the maximality of $M$, for every $y \in \mathbb{R}^d$ there is a $g_1 \in \Lambda$, for which $-\eta K + y + g_1$ intersects at least one member of the systems (3.1) or (3.2).

First assume that there is an element $K+x_i+g_2$ of the system (3.1), which intersects $-\eta K + y + g_1$, i.e. there exist $k_1, k_2 \in K$ such that

$$-\eta k_1 + y + g_1 = k_2 + x_i + g_2,$$

or, equivalently,

$$y = k_2 + \eta k_1 + x_i + (g_2 - g_1).$$

Hence, by the convexity of $K$, the point $y$ belongs to the set

$$(1+\eta)K + x_i + (g_2 - g_1).$$

Now consider the case when there is an element $-\eta K + y_j + g_2$ of the system (3.2), which intersects $-\eta K + y + g_1$, i.e. there exist $k_1, k_2 \in K$ such that

$$-\eta k_1 + y + g_1 = -\eta k_2 + y_j + g_2,$$

or, equivalently,

$$y = -\eta k_2 + \eta k_1 + y_j + (g_2 - g_1).$$

Since $o$ is the centroid of $K$, hence $-d^{-1}K \subseteq K$, and $0 < \eta < \frac{1}{d}$, thus, there is a point $k_3 \in K$, for which $-\eta k_2 = k_3$. Again, by the convexity of $K$, $y = k_3+\eta k_1+y_j+(g_2-g_1)$ belongs to

$$(1+\eta)K + y_j + (g_2 - g_1).$$

This shows that the whole space $\mathbb{R}^d$ is covered by the union of the systems

$$\{(1+\eta)K + x_i + g : i = 1, \dots N, g \in \Lambda\}$$

and

$$\{(1+\eta)K + y_i + g : i = 1, \dots N, g \in .\Lambda\}$$

The density of this covering is

$$(1+\eta)^d (N+M) R^{-d}.$$

From homogenity considerations, it follows that there is a covering of the whole of space by translates of $K$ with this density. By (3.3) now we have that

$$\vartheta_P(K) \leq (1+\eta)^d N R^{-d} + (1+\eta)^d \eta^{-d} (1 - R^{-d})^N.$$

For fixed $\eta$ let

$$R := \left( \frac{N}{d \ln \left( \frac{1}{\eta} \right)} \right)^{\frac{1}{d}}.$$

Note that we can choose $N$ large enough so that $R$ is sufficiently large. Since

$$(1 - R^{-d})^N \leq e^{-NR^{-d}},$$

we have that

$$\vartheta_P(K) \leq (1+\eta)^d d \ln \left( \frac{1}{\eta} \right) + \left( \frac{1}{\eta} \right)^d \eta^{-d} e^{d \ln \left( \frac{1}{\eta} \right)} = (1+\eta)^d \left( 1 + d \ln \left( \frac{1}{\eta} \right) \right).$$

Now set

$$\eta := \frac{1}{d \ln d},$$

which yields

$$\vartheta_P(K) \leq (1+\eta)^d \left( 1 + d \ln \left( \frac{1}{\eta} \right) \right) \leq e^{d\eta} \left( 1 + d \ln \left( \frac{1}{\eta} \right) \right) < \left( 1 + \frac{2}{\ln d} \right) (d \ln d + d \ln \ln d + 1) < d \ln d + d \ln \ln d + 5d.$$

□

**Theorem 3.2** (Rogers and Zong [RZ97]). *Let $K$ and $L$ be convex bodies in $\mathbb{R}$. Then*

$$N(K,L) \le \frac{\mathrm{vol}\,(K-L)}{\mathrm{vol}\,(L)}\vartheta(L).$$

*Proof:* Consider a discrete set $G$ of translation vectors, for which

$$\bigcup_{g\in G} g+L$$

is a covering of $\mathbb{R}^d$ of density $\vartheta(L)$. Fix $\varepsilon > 0$. If $C$ is a sufficiently large cube centered at the origin, then we have

$$\sum_{g\in G\cap C} \mathrm{vol}\,(L+g) \le \mathrm{vol}\,(C)\,(\vartheta(L)+\varepsilon),$$

which yields that

$$|G\cap C| \le \frac{\mathrm{vol}\,(C)}{\mathrm{vol}\,(L)}(\vartheta(L)+\varepsilon).$$

Note that for $g \in G$

$$(K+t)\bigcap(L+g) \ne \emptyset$$

holds if and only if $g \in K-L+t$. Since $\bigcup_{g\in G} g+L = \mathbb{R}^d \supset K+t$, the sets $L+g$ with $g \in K-L+t$ cover $K+t$. It is easy to see that the average value of $|G\cap(K-L+t)|$ for $t \in C$ is at most

$$\frac{\mathrm{vol}\,(K-L)}{\mathrm{vol}\,(L)}(\vartheta(L)+2\varepsilon),$$

hence, there is a $t$ such that

$$|G\cap(K-L+t)| \le \frac{\mathrm{vol}\,(K-L)}{\mathrm{vol}\,(L)}(\vartheta(L)+2\varepsilon),$$

which means that there is a covering of $K$ by at most

$$\frac{\mathrm{vol}\,(K-L)}{\mathrm{vol}\,(L)}(\vartheta(L)+2\varepsilon)$$

translates of $L$. Now $\varepsilon$ can be chosen arbitrarily small, and $N(K,L)$ is an integer, hence we obtained that

$$N(K,L) \le \frac{\mathrm{vol}\,(K-L)}{\mathrm{vol}\,(L)}\vartheta(L).$$

□

Theorems 3.1 and 3.2 combined with the *Rogers-Shephard inequality* [RS57], according to which

$$\mathrm{vol}\,(K-K) \le \binom{2d}{d} \tag{3.4}$$

for any convex body $K$ in $\mathbb{R}^d$, yield the following.

**Corollary 3.3.** *Let $K$ be a convex body in $\mathbb{R}^d$ $(d \ge 3)$. Then, for the illumination number $i(K)$, we have*

$$i(K) \le \begin{cases} 2^d(d\ln d + d\ln\ln d + 5d) & \text{if } K = -K \\ \binom{2d}{d}(d\ln d + d\ln\ln d + 5d) & \text{in general.} \end{cases}$$

# 4. Homothetic coverings

In this chapter, we discuss our new results about translative homothetic coverings. The main theorems are Theorem 4.1, 4.2, 4.5, 4.11 and 4.14.

## 4.1 Covering a convex body by a family of its homothets

The main result of this section is the following upper bound on $f(K,K)$ (see Definition 2.8).

**Theorem 4.1.** *Let $K \subseteq \mathbb{R}^d$ $(d \geq 3)$ be a convex body. Then*

$$f(K,K) \leq \begin{cases} (d^3 \cdot \ln d \cdot \vartheta(K) + e)2^d & \text{if } K = -K \\ d^3 \cdot \ln d \cdot \vartheta(K) \cdot \binom{2d}{d} + e \cdot 4^d & \text{in general.} \end{cases}$$

We will obtain this theorem as a corollary to the following.

**Theorem 4.2.** *For any dimension $d$ and any convex body $K$ in $\mathbb{R}^d$, we have*

$$f(K,K) \leq \inf_{\varepsilon>0} \left\{ \left\lceil \frac{-\ln \varepsilon}{\ln(1+\varepsilon)} \right\rceil (1+\varepsilon) d\vartheta(K) \frac{\operatorname{vol}(K-K)}{\operatorname{vol}(K)} + \left(1+\frac{\varepsilon}{2}\right)^d 2^d \frac{\operatorname{vol}(K)}{\operatorname{vol}(K \cap (-K))} \right\}.$$

Before we prove Theorem 4.2, we restate Theorem 1.3 of [Nas10] in a slightly more general form than the original.

**Theorem 4.3.** *Let $K$ and $L$ be convex bodies in $\mathbb{R}^d$ with $o \in \operatorname{int} K$, and $\mathcal{F} = \{\lambda_1 K, \lambda_2 K, \dots\}$ a family of its homothets with $0 < \lambda_i \leq \lambda_1 < 1$. Assume that*

$$\sum_{i=1} \lambda_i^d > 2^d \frac{\operatorname{vol}\left(L + \lambda_1 \frac{K \cap (-K)}{2}\right)}{\operatorname{vol}(K \cap (-K))}.$$

*Then $\mathcal{F}$ permits a translative covering of $L$.*

*Proof:* The proof is the same as the proof of Theorem 1.3 in [Nas10].

We may assume that the homothety-ratios are ordered: $1 > \lambda_1 \geq \lambda_2 \ldots$. Assume that

$$\sum_{i=1}^{N} \lambda_i^d > 2^d \frac{\operatorname{vol}\left(L + \lambda_1 \frac{K\cap(-K)}{2}\right)}{\operatorname{vol}(K \cap (-K))}.$$

First, we define a sequence of translation vectors $x_1, x_2, \ldots$ by an inductive procedure as follows. Let $x_1$ be an arbitrary point in $L$. Once we have chosen $x_1, x_2, \ldots, x_k$, the next element $x_{k+1}$ is chosen such that $x_{k+1} \in L$ and the sets

$$x_1 + \frac{\lambda_1}{2}(K \cap (-K)), x_2 + \frac{\lambda_2}{2}(K \cap (-K)), \ldots, x_{k+1} + \frac{\lambda_{k+1}}{2}(K \cap (-K))$$

form a packing, that is, they have pairwise disjoint interiors. If $k = N$ or $x_{k+1}$ cannot be chosen, we stop. There are two cases.

Case 1: When the algorithm stops because $k = N$. We show that this case is impossible. First note that

$$x_1 + \frac{\lambda_1}{2}(K \cap (-K)), x_2 + \frac{\lambda_2}{2}(K \cap (-K)), \ldots x_k + \frac{\lambda_k}{2}(K \cap (-K))$$

is a packing in $L + \frac{\lambda_1}{2}(K \cap (-K))$, hence the total volume

$$\sum_{i=1}^{k} \operatorname{vol}\left(\frac{\lambda_i}{2}(K \cap (-K))\right) \leq \operatorname{vol}\left(L + \frac{\lambda_1}{2}(K \cap (-K))\right).$$

On the other hand by 4.3 we have that

$$\sum_{i=1}^{k} \operatorname{vol}\left(\frac{\lambda_i}{2}(K \cap (-K))\right) > \operatorname{vol}\left(L + \frac{\lambda_1}{2}(K \cap (-K))\right).$$

This is a contradiction.

Case 2: The algorithm terminates because there is no $x_{k+1} \in L$ such that 4.1 is a packing. In this case, it is easy to see that

$$L \subseteq \bigcup_{i=1}^{k} x_i + \lambda_i(K \cap (-K)) \subseteq \bigcup_{i=1}^{k} x_i + \lambda_i K$$

is a covering of $L$, which finishes the proof.

□

**Remark 4.4.** We obtain the bound

$$f(K,K) \leq \begin{cases} 3^d & \text{if } K=-K \\ 6^d & \text{in general} \end{cases}$$

that was mentioned in the introduction, as a corollary of Theorem 4.3. For the non-symmetric case we use (3.4) .

The proof of Theorem 4.2 is based on considering two cases. The first is the case of a family, whose members are roughly of the same size, hence we can use Rogers' theorem. The second is the case of a family which has only small members, therefore Theorem 4.3 gives a good bound.

*Proof of Theorem 4.2:* We fix $\varepsilon > 0$. Now, we are given $\mathcal{F} = \{\lambda_1 K, \lambda_2 K, \dots\}$ with $0 < \lambda_i < 1$ for all $i$, such that

$$\sum_i \lambda_i^d > \left\lceil \frac{-\ln\varepsilon}{\ln(1+\varepsilon)} \right\rceil (1+\varepsilon) d\vartheta(K) \frac{\mathrm{vol}\,(K-K)}{\mathrm{vol}\,(K)} + \left(1+\frac{\varepsilon}{2}\right)^d 2^d \frac{\mathrm{vol}\,(K)}{\mathrm{vol}\,(K\cap(-K))}$$

holds.

We consider two cases:

Case 1: There exists a subfamily $\mathcal{F}' = \{\mu_1 K, \mu_2 K, \dots\}$ of $\mathcal{F}$ in which

$$(1+\varepsilon)^{-1} \leq \frac{\mu_i^d}{\mu_j^d} \leq (1+\varepsilon)$$

for all $i$ and $j$ and

$$\sum_i \mu_i^d > \vartheta(K)(1+\varepsilon)\frac{\mathrm{vol}\,(K-K)}{\mathrm{vol}\,(K)}.$$

In this case $\mathcal{F}'$ has at least $\frac{1}{\mu_1^d(1+\varepsilon)}\vartheta(K)(1+\varepsilon)\frac{\mathrm{vol}(K-K)}{\mathrm{vol}(K)} = \frac{1}{\mu_1^d}\vartheta(K)\frac{\mathrm{vol}(K-K)}{\mathrm{vol}(K)}$ members.

We may assume that $\mu_1$ is the smallest homothety ratio in $\mathcal{F}'$. By Theorem 3.2 we can cover $K$ by at most $\frac{\mathrm{vol}(K-\mu_1 K)\cdot\vartheta(K)}{\mathrm{vol}(\mu_1 K)}$ translates of $\mu_1 K$. Since $\mu_i K \subseteq \mu_1 K$ and $|\mathcal{F}'| \geq \frac{1}{\mu_1^d}\vartheta(K)\frac{\mathrm{vol}(K-K)}{\mathrm{vol}(K)} \geq \frac{\mathrm{vol}(K-\mu_1 K)\cdot\vartheta(K)}{\mathrm{vol}(\mu_1 K)}$ we obtain that $\mathcal{F}'$ permits a translative covering of $K$.

Suppose that now Case 1 does not hold, and consider the intervals

$$(\varepsilon^d, \varepsilon^d(1+\varepsilon)], (\varepsilon^d(1+\varepsilon), \varepsilon^d(1+\varepsilon)^2], \dots, (\varepsilon^d(1+\varepsilon)^{c(\varepsilon)-1}, \varepsilon^d(1+\varepsilon)^{c(\varepsilon)}],$$

where $c(\varepsilon) = d\left\lceil\frac{-\ln\varepsilon}{\ln 1+\varepsilon}\right\rceil$. Since $\varepsilon^d(1+\varepsilon)^{c(\varepsilon)} \geq 1$, we have

$$\sum_{\lambda_i K\in\mathcal{F}, \lambda_i^d > \varepsilon^d} \lambda_i^d < c(\varepsilon) d(1+\varepsilon)\vartheta(K)\frac{\mathrm{vol}\,(K-K)}{\mathrm{vol}\,(K)},$$

therefore Case 2 holds.

Case 2: There exists a subfamily $\mathcal{F}' = \{\mu_1 K, \mu_2 K, \dots\}$, in which

$$\mu_i^d \le \varepsilon^d$$

and

$$\sum_i \mu_i^d > \left(1 + \frac{\varepsilon}{2}\right)^d 2^d \frac{\mathrm{vol}\,(K)}{\mathrm{vol}\,(K \cap (-K))}.$$

Then $\mathcal{F}'$ permits a translative covering of $K$, since by, Theorem 4.3, $\mathcal{F}'$ permits a translative covering of $K$ if

$$\sum_{i=1} \mu_i^d > 2^d \frac{\mathrm{vol}\left(K + \mu_1 \frac{K \cap (-K)}{2}\right)}{\mathrm{vol}\,(K \cap (-K))}$$

which clearly follows from the condition

$$\sum_{i=1}^{\infty} \mu_i^d > (1 + \frac{\varepsilon}{2})^d 2^d \frac{\mathrm{vol}\,(K)}{\mathrm{vol}\,(K \cap (-K))}.$$

□

To obtain Theorem 4.1 chose $\varepsilon = \frac{1}{d}$. For the non-symmetric case use Claim 2.3 and (3.4).

## 4.2 A generalisation of Rogers' theorem

In this section we show that if instead of translates of a convex body we use given homothtes of a convex body (with divergent volume sum) to cover the space $\mathbb{R}^d$, then the same density can be reached.

**Theorem 4.5.** *Let $K$ be a convex body in $\mathbb{R}^d$, and let $\mathcal{F} = \{\lambda_1 K, \lambda_2 K, \dots\}$ $(0 < \lambda_i \le 1)$ be a family of its homothets with*

$$\sum_{i=1}^{\infty} \lambda_i^d = \infty.$$

*Then if the set $\{\lambda_1, \lambda_2, \dots\}$ has a limit point other than zero, then $\mathcal{F}$ permits a covering of space of density $\vartheta(K)$. Otherwise, $\mathcal{F}$ permits a covering of space of density one.*

Recall that $\vartheta(K)$ is the covering density of $K$. For definition on estimates see Section 2.3.

**Remark 4.6.** Regarding the covering density that we can reach, it does not matter if we require that we use every element of $\mathcal{F}$, or only a subfamily. Indeed, assume that a subfamily of $\mathcal{F}$ permits a translative covering of $\mathbb{R}^d$ of density $\vartheta$, and let $\mathcal{G}$ denote the subfamily of the unused homothets. Then we can take a zero-density arrangement of $\mathcal{G}$, and add it to the covering. This way, the combined density will remain $\vartheta$.

We make some preparations for the proof of Theorem 4.5.

**Definition 4.7.** A collection of $\mathcal{V}$ of Lebesgue-measurable subsets of $\mathbb{R}^n$ is a *regular family* if there exists a constant $C$ for which

$$\operatorname{diam}(V)^d \leq C \operatorname{vol}(V)$$

holds for every $V \in \mathcal{V}$.

**Definition 4.8.** A collection $\mathcal{V}$ of subsets of $\mathbb{R}^d$ is a *Vitali-covering* of $E \subseteq \mathbb{R}^d$, if for every $x \in E$ and $\delta > 0$, there exists an element $U$ of $\mathcal{V}$ such that $x \in U$ and $0 < \operatorname{diam}(U) < \delta$.

We recall Vitali's covering theorem (Page 425 of [93].)

**Theorem 4.9.** *Let $E \subset \mathbb{R}^d$ be a measurable with finite Lebesgue-measure, and let $\mathcal{V}$ be a regular family of closed subsets of $\mathbb{R}^d$ that is a Vitali covering for $E$. Then there exist a finite or countably infinite disjoint subcollection $\{U_j\} \subseteq \mathcal{V}$ such that*

$$\operatorname{vol}\left(E \setminus \bigcup_j U_j\right) = 0.$$

We will also use the following simple lemma:

**Lemma 4.10.** *Let $A = \{a_1 \geq a_2 \geq a_3, \dots\}$ be a set of positive real numbers with $\sum_i a_i = \infty$. Then there is a partition $A = \bigsqcup_{i=0}^{\infty} B_i$ of $A$ into a countably infinite many parts, such that the sum of the elements in each $B_i$ is divergent, i.e. $\sum_{b \in B_i} b = \infty$.*

*Proof:* Let

$$B_i := \{a_j : 2^i | j, 2^{i+1} \nmid j\}.$$

We prove by induction on $i$ that $\sum_{b \in B_i} b = \infty$ and $\sum_{a \in A \setminus \bigcup_{j \leq i} B_j} a = \infty$. First we have to show that $\sum_{b \in B_0} b = \infty$. $\sum_{b \in B_0} b = \sum_{j=0}^{\infty} a_{2j+1}$. If this sum was finite, then $\sum_{i=1}^{\infty} a_{2i} = \infty$, but

$\sum_{j=0}^{\infty} a_{2j+1} \geq \sum_{i=1}^{\infty} a_{2i}$ which would be a contradiction. Assume now that $\sum_{b \in B_i} b = \infty$ and $\sum_{a \in A \setminus \bigcup_{j \leq i} B_j} a = \infty$. We have to show that $\sum_{b \in B_{i+1}} b = \infty$ also holds. By the construction of the sets $B_i$, we have $A \setminus \bigcup_{j \leq i} B_i = \{a_{2^{i+1} \cdot 1}, a_{2^{i+1} \cdot 2}, a_{2^{i+1} \cdot 3}, \dots\}$, and $B_{i+1} = \{a_j : 2^{i+1} | j, 2^{i+2} \nmid j\}$, hence the proof of this step is the same as the proof of the initial case. □

*Proof of Theorem 4.5:* We consider the two cases.

Case 1: The set $\{\lambda_1, \lambda_2, \dots\}$ has a limit point other than zero. Then for every $0 < \varepsilon$ there exists a subfamily $\mathcal{F}' = \{\mu_1 K, \mu_2 K, \dots\}$ of $\mathcal{F}$ in which

$$(1+\varepsilon)^{-1} \leq \frac{\mu_i^d}{\mu_j^d} \leq 1 + \varepsilon \text{ (for every } i \text{ and } j)$$

and

$$\sum_i \mu_i^d = \infty.$$

(That is, $\mathcal{F}'$ has infinitely many members.)

Let $\mu := \inf\{\mu_i\}$, and $x_1, x_2, \dots$ be translation vectors in $\mathbb{R}^d$ for which

$$\bigcup_i x_i + \mu K$$

is a covering of density at most $\vartheta(K)$. Then, clearly, we have

$$\mathbb{R}^d = \bigcup_i x_i + \mu_i K$$

and the density of this covering is at most $(1+\varepsilon)\vartheta(K)$, since $\frac{\mu_i^d}{mu^d} \leq 1 + \varepsilon$. Thus for every $0 < \varepsilon$ there is subcollection $\mathcal{F}'$, which permits a translative covering of $\mathbb{R}^d$ with density at most $(1+\varepsilon)\vartheta(K)$, hence by Remark 2.14 there is a translative covering by the elements of $\mathcal{F}$ of density at most $\vartheta(K)$.

Assume now that Case 1 does not hold. Then we have Case 2.

Case 2: For every $\varepsilon_0 > 0$ there exists a subcollection $\mathcal{F}' = \{\mu_1 K, \mu_2 K, \dots\}$ of $\mathcal{F}$ in which

$$\mu_i^d \leq \varepsilon_0$$

for each $i$, and

$$\sum_i \mu_i^d = \infty.$$

We may assume that $\mu_1 \geq \mu_2 \geq \dots$, and $\mathrm{vol}\,(K) = 1$. By Lemma 4.10, we can partition

$$\mathcal{F}' = (\bigsqcup_i \mathcal{A}_i) \bigsqcup (\bigsqcup_j \mathcal{B}_|)$$

into a countably infinite number of subcollections, so that

$$\sum_{\mu_l K \in \mathcal{A}_i} \mu_l^d = \sum_{\mu_l K \in \mathcal{B}_j} \mu_l^d = \infty$$

holds for every $i$ and $j$. Note that, each $\mathcal{A}_i$ and $\mathcal{B}_j$ has the same property as $\mathcal{F}'$, i.e. for every $\varepsilon_0$ the total volume of the members that have volume at most $\varepsilon_0$ is infinite. (Else Case 1 would hold.)

Let $\mathrm{vol}\,(K) = C^d(\mathrm{diam}(K))^d$, and fix $\varepsilon$. First we cover the cube $[0,1]^d$ by a subfamily of $\mathcal{A}_1$, in which the total volume is at most $(1+2\varepsilon C^{-1})^d + \varepsilon$.

Again, by Lemma 4.10 we can partition

$$\mathcal{A}_1 = \bigsqcup_j \mathcal{C}_j$$

into countably infinitely many subcollections, so that $\sum_{\mu_l K \in \mathcal{C}_j} \mu_l^d = \infty$. (And in each $\mathcal{C}_j$ the total volume of members whose volume does not exceed $\varepsilon_0$ is infinite.) Let $\mathcal{C}_{j,\varepsilon}$ denote the subset of those elements $\mu_l K$ of $\mathcal{C}_j$, for which $\mu_l^d \leq \varepsilon$. For every $j \in \mathbb{N}$ cover the cube $[0,1]^d$ by translates of the elements $\mathcal{C}_{j,\left(\frac{\varepsilon}{j}\right)^d}$. The union of these coverings is obviously regular, since $\frac{\mathrm{vol}(\mu_l K)}{\mathrm{diam}(\mu_l K)^d} = \frac{\mathrm{vol}(K)}{\mathrm{diam}(K)^d} = C^d$. Also, it is a Vitali-covering, hence for every $0 < \delta$ there is a $j \in \mathbb{N}$ for which $\frac{\varepsilon}{jC} < \delta$, that is the elements of $\mathcal{C}_{j,\left(\frac{\varepsilon}{j}\right)^d}$ have diameter at most $\delta$, and every point of $[0,1]^d$ is covered by a translate of an element of $\mathcal{C}_{j,\left(\frac{\varepsilon}{j}\right)^d}$.

Therefore, we can apply Vitali's covering theorem for the union of these coverings. There is a finite or countably infinite subfamily $\{\alpha_1 K, \alpha_2 K, \dots\}$ of disjoint members, of this union for which

$$\mathrm{vol}\left(\left([0,1]^d \setminus \bigcup_l \alpha_l K\right) = E\right) = 0.$$

As a next step, we cover $E$ by a translates of members of a subcollection of $B_1$, in which the total volume is at most $\varepsilon$. Partition $B_1 = \bigsqcup_k \mathcal{D}_k$ into countably many subcollection so that

$$\sum_{\mu_l K \in \mathcal{D}_k} \mu_l^d = \infty$$

for every $k$. (Each subfamily $\mathcal{D}_k$ has the same properties as $\mathcal{C}_j$-s.) Since $\mathrm{vol}\,(E) = 0$, for every $\varepsilon' > 0$, there is collection $\mathcal{I} = \{I_1, I_2, \dots\}$ of homothets if $K$ so that

$$E \subseteq \bigcup_l I_l$$

and

$$\sum_l \mathrm{vol}\,(I_l) \leq \varepsilon'.$$

By Theorem 4.1, there is a constant $f(d)$ (depending only on the dimension) such that if for a subcollection $\{\mu_1 K, \mu_2 K, \dots\}$ of $\mathcal{D}_l$ we have $\sum_k \mu_k^d \geq f(d)\,\mathrm{vol}\,(I_l)$, then this subcollection permits a translative covering of $I_l$. We show that using the members of $\mathcal{D}_{l,\frac{\varepsilon'}{2^l}}$ there is a covering of $I_l$ in which the total volume is at most

$$f(d)\,\mathrm{vol}\,(I_l) + \frac{\varepsilon'}{2^k}.$$

We may assume that in $\mathcal{D}_{k,\frac{\varepsilon'}{2^k}} = \{\mu_1 K, \mu_2 K, \dots\}$ the indices are ordered: $\mu_1 \geq \mu_2 \geq \dots$. If $m$ denotes the smallest index, for which

$$\sum_{l=1}^{m} \mu_l^d \geq f(d)\,\mathrm{vol}\,(I_l)\,,$$

then

$$f(d)\,\mathrm{vol}\,(I_l) \leq \sum_{l=1}^{m+1} \mu_l^d \leq f(d)\,\mathrm{vol}\,(I_l) + \frac{\varepsilon'}{2^k},$$

from which the statement follows. With this, we obtain a covering of $\bigcup_l I_l$, in which the sum of the volumes is at most

$$\sum_{l=1}^{\infty} f(d)\,\mathrm{vol}\,(I_l) + \frac{\varepsilon'}{2^l} \leq f(d)\varepsilon' + \varepsilon'.$$

Setting $\varepsilon' := \frac{\varepsilon}{1+f(d)}$, we get the desired covering of $E$.

Since we covered $[0,1]^d \setminus E$ by disjoint homothets of $K$ of diameter at most $C^{-1}\varepsilon$, (hence all of the translates are contained in the cube $(1+2C^{-1}\varepsilon)^d[0,1]^d$) and we covered $E$ by a subcollection of total volume at most $\varepsilon$, we obtained a covering of $[0,1]^d$ by translates of the elements of $\mathcal{A}_1$, such that the sum of the volumes of the translates used is at most $(1+2C^{-1}\varepsilon)^d + \varepsilon$.

Finally, let $a_1, a_2, \dots$ be an enumeration of the elements of $\mathbb{Z}^d$. Similarly we can cover $[0,1]^d + a_i$ by translates of $\mathcal{A}_i$ such that the sum of the translates were used is at most $(1+2C^{-1}\varepsilon)^d+\varepsilon$. Hence we obtain a translative covering of $\mathbb{R}^d$ by the elements of $\mathcal{F}$ of density at most $(1+2C^{-1}\varepsilon)^d+\varepsilon$. This is true for every $\varepsilon>0$, hence, by Remark 2.14, in this case, there is a covering of the space of density one. □

## 4.3 Covering with an unbounded family of homothets

Consider now unbounded families. In this case, we can bound not only the density, but also the covering multiplicity. Moreover, the bound we obtain for the density, is better by a $d\ln d$ factor than Rogers' result for translates.

**Theorem 4.11.** *Let $K$ be a convex body with smooth boundary and $\mathcal{F}=\{\lambda_1K,\lambda_2K,\dots\}$ a family of its homothets so that the $\lambda_i$-s are not bounded. Then $\mathcal{F}$ permits a translative covering of $\mathbb{R}^d$ so that every point is covered at most $2d$ times.*

*Proof:* Let $p\in\partial K$, and let $H$ be the tangent hyperplane of $K$ through $p$. Then the ratio $\frac{t}{H_t}$ can be arbitrarily large, where $H_t$ is the volume of the intersection of $K$ and the translation of $H$ by $tv$, for its normalvector $v$.

We fix $\varepsilon>0$. We may assume that $\mathcal{F}$ has an element $\mu_0K$ so that $Q_0=[-\varepsilon,\varepsilon]^d\subseteq\mu_0K\subseteq[-1,1]^d$. We present an algorithm to produce the desired covering. We will define inductively a sequence of cubes $Q_i$ $(i\in\mathbb{N})$, which are centered at the origin and have side length at least $i$, a sequence of translation vectors $x_1^1,x_1^2,\dots x_1^{2d},x_2^1,\dots,x_2^{2d},x_3^1,\dots,$ and a sequence of elements of $\mathcal{F}$ $\mu_1^1K,\mu_1^2K,\dots\mu_1^{2d}K,\mu_2^1K,\dots,\mu_2^{2d}K,\mu_3^1K,\dots$ so that the following hold with the convention $x_0^j=0$ and $\mu_0^j=\mu_0$:

1. $Q_k\subseteq\bigcup_{i=0}^k\bigcup_{j=1}^{2d}x_i^j+\mu_i^jK$ for $k\in\mathbb{N}$
2. $\left(\bigcup_{i=0}^{k}\bigcup_{j=1}^{2d}x_i^j+\mu_i^jK\right)+\mathbf{B}(o,\varepsilon)\subseteq Q_{k+1}$ for $k\in\mathbb{N}$
3. $(x_i^j+\mu_i^jK)\bigcap\left(x_i^{j+d}+\mu_i^{j+d}K\right)=\emptyset$ for $1\leq j\leq d$
4. $(x_i^j+\mu_i^jK)\bigcap(x_l^m+\mu_l^mK)=\emptyset$ if $|i-l|\geq 2$.

Indeed, assume that we found $x_i^j$-s, $\mu_i^j$-s and $Q_i$-s for $i\leq k$. Choose $Q_{k+1}$ so that

$$\left(\bigcup_{i=0}^{k}\bigcup_{j=1}^{2d}x_i^j+\mu_i^jK\right)+\mathbf{B}(o,\varepsilon)\subseteq Q_{k+1}.$$

Let $H_i$ denote support hyperplane of the $i$-th facet of $Q_k$, and $H_{i,+}$ the half-space bounded by $H_i$ and does not contain $Q_k$. The condition on the $\lambda_i$-s, the remark at the beginning of the proof and property 2. together implies that we can choose a member $\mu_{k+1}^i K$ of $\mathcal{F}$ (from the unused member), and a translation vector $x_{k+1}^i$ such that

$$Q_{k+1} \bigcap H_{i,+} \subseteq x_{k+1}^i + \mu_{k+1}^i K$$

and

$$\left(x_{k+1}^i + \mu_{k+1}^i K\right) \bigcap \left(x_{k-1}^j + \mu_{k-1}^j K\right) = \emptyset$$

for all $j$.

Note that we only used the fact that $K$ is smooth at the points of intersection of $K$ with supporting hyperplanes that are parallel to one of the $d$ coordinate hyperplanes.

Also we have that if $H_i$ ($i \leq d$) and $H_{i+d}$ supports opposite sides of $Q_{k+1}$ then

$$\left(x_{k+1}^i + \mu_{k+1}^i K\right) \bigcap \left(x_{k+1}^{i+d} + \mu_{k+1}^{i+d} K\right) = \emptyset,$$

and

$$Q_{k+1} \setminus Q_k \subseteq \bigcup_{i=1}^{2d} x_{k+1}^i + \mu_{k+1}^i K$$

hence we can find the desired $Q_i$-s and translates.

Since $Q_i$ has side length at least $i$,

$$\mathbb{R}^d = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{2d} x_i^j + \mu_i^j K.$$

Property 3. ensures that, the subfamily $\bigcup_{i=1}^{2d} x_k^i + \mu_k^i K$ covers every point at most $d$ times, and property 4. yiedls that every point of $\mathbb{R}^n$ is covered at most two subfamilies $\bigcup_{i=1}^{2d} x_k^i + \mu_k^i K$, which finishes the proof.

□

**Remark 4.12.** At first, one may believe that, by some approximation argument, the condition of smoothness can be dropped in Theorem 4.11 Unfortunately, this is not the case, the standard argument does not work.

Let $K$ be a convex body in $\mathbb{R}^d$ and $\mathcal{F} = \{\lambda_1 K, \lambda_2 K, \dots\}$ a family of its homothets, such that the coefficients $\lambda_i$-s are not bounded. Let $L$ be a convex body with smooth boundary such that $L \subseteq K \subseteq (1+\varepsilon)L$. Consider the family $\mathcal{F}' = \{\lambda_1 L, \lambda_2 L, \dots\}$, and follow the steps of the proof of the smooth case in Theorem 4.11 for $\mathcal{F}'$.

We obtain a covering $x_i^j + \lambda_i^j L$ of $\mathbb{R}^d$, where every point is covered at most $2d$ times. Then $x_i^j + (1+\varepsilon)\lambda_i^j L$ is also a covering. However, it may happen that $x_i^j + (1+\varepsilon)\lambda_i^j L$ covers every point infinitely many times: If $\lambda_i^k$ is sufficiently large, then $x_i^j + (1+\varepsilon)\lambda_i^j K$ may contain $\mathbf{B}(o, i)$ for all $i$.

**Remark 4.13.** The same proof works, if $K$ is a cube, and the proof needs only a little modification if $K$ is a polytope.

Figure4.1 shows the first two generations, for cubes.

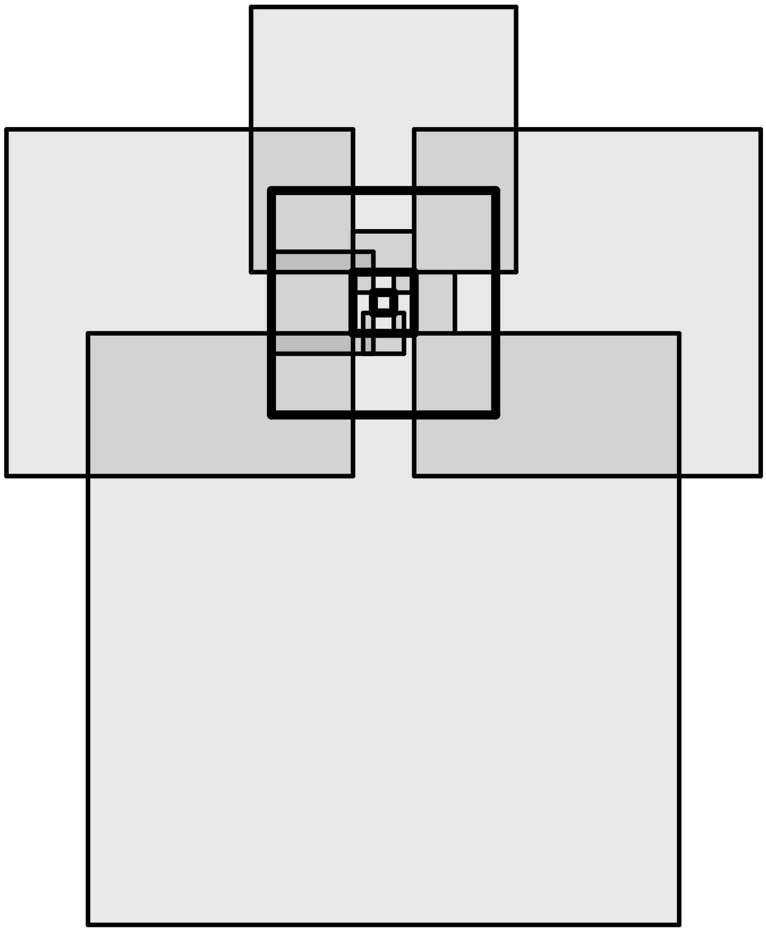

Figure 4.1: First two step, if $K$ is a cube (square)

For the general case, we prove a slightly weaker bound.

**Theorem 4.14.** *Let $K \subseteq \mathbb{R}^d$ be a convex body and let $\mathcal{F} = \{\lambda_1 K, \lambda_2 K, \dots\}$ a family of its homothets so that the $\lambda_i$-s are not bounded. Then $\mathcal{F}$ permits a translative covering of $\mathbb{R}^d$ so that every point is covered at most $4d$ times.*

*Proof:* Let $L$ be the polytope obtained in Lemma 2.24. We may assume that $L$ contains the origin, and let $\varepsilon > 0$ be fixed. We may assume that $\mathcal{F}$ has an element $\mu_0 K$

so that $-L_0 = \varepsilon - L =\subseteq \mu_0 K \subseteq -L$. Similarly to the proof of Theorem 4.11 we inductively define a sequence $-\alpha_1 L, -\alpha_2 L, \dots$ of homothets of $-L$, a sequence of translation vectors $x_1^1, x_1^2, \dots x_1^{2d}, x_2^1, \dots, x_2^{2d}, x_3^1, \dots$ and a sequence of members of $\mathcal{F}$ $\mu_1^1 K, \mu_1^2 K, \dots \mu_1^{2d} K, \mu_2^1 K, \dots, \mu_2^{2d} K, \mu_3^1 K, \dots$, so that $\alpha_i \geq i$ and the following hold:

1. $-L_k \subseteq \bigcup_{i=0}^{k} \bigcup_{j=1}^{2d} x_i^j + \mu_i^j K$ for $k \in \mathbb{N}$
2. $\left( \bigcup_{i=0}^{k} \bigcup_{j=1}^{2d} x_i^j + \mu_i^j K \right) + \mathbf{B}(o, \varepsilon) \subseteq -L_{k+1}$ for $k \in \mathbb{N}$
3. $\left(x_i^j + \mu_i^j K\right) \bigcap \left(x_i^{j+d} + \mu_i^{j+d} K\right) = \emptyset$ for $1 \leq j \leq d$

The proof of existence of such sequences is very similar to the proof of Theorem 4.11, and provides the desired covering.

□

**Remark 4.15.** In the previous theorem, we obtained covering multiplicity $4d$ instead of $2d$ because$-L$ may not have parallel pairs of facets.

# 5. Multiple covering

We consider now $k$-fold coverings of $\mathbb{R}^d$. As a new result, we will show that one can find a $k$-fold covering (that is every point is covered at least $k$ times) of the space by translates of a given convex body $K$ of density around $18ed \ln d$, provided $k$ is at most $d \ln d$.

**Definition 5.1.** Let $\mathcal{F}$ be a family of subsets of a finite set $X$, and $k \in \mathbb{Z}^+$. The *$k$-th covering number* of $\mathcal{F}$, denoted by $\tau_k(\mathcal{F})$, is the minimum cardinality of a multi-subfmaily of $\mathcal{F}$ such that each ponint of $X$ is contained it at least $k$ (with multiplicity) members of the subfamily.

**Definition 5.2.** A *fractional covering* of $X$ by $\mathcal{F}$ is a mapping $w$ from $\mathcal{F}$ to $\mathbb{R}^+$ with $\sum\limits_{x \in F \in \mathcal{F}} w(F) \geq 1$ for all $x \in X$. The total weight of a fractional covering is denoted by $w(\mathcal{F}) := \sum_{F \in \mathcal{F}} w(F)$, and its infimum is the *fractional covering number* of $\mathcal{F}$:

$$\tau^*(X, \mathcal{F}) := \inf\{w(\mathcal{F}) \ : \ w : \mathcal{F} \to \mathbb{R}^+ \text{ is a fractional-covering of } X\}.$$

We will use the following simple combinatorial statement.

**Lemma 5.3.** *Let $\mathcal{F}$ be a family of subsets of a base set $X$ of fractional covering number $\tau^* := \tau^*(\mathcal{F})$, and $k \in \mathbb{Z}^+$. Then*

$$\tau_k \leq \left\lceil \tau^* \left( k + \frac{3}{2} \ln |X| + \frac{3}{2} \sqrt{(4k + \ln |X|) \ln |X|} \right) \right\rceil \leq \lceil 6\tau^* \max\{\ln |X|, k\} \rceil.$$

The proof is a standard probabilistic argument.

*Proof:* Let $w$ be a fractional covering of $X$ by $\mathcal{F}$ of total weight $\tau^* := \tau^*(\mathcal{F})$, and let

$$m = \left\lceil \tau^* \left( k + \frac{3}{2} \ln |X| + \frac{3}{2} \sqrt{(4k + \ln |X|) \ln |X|} \right) \right\rceil.$$

We pick $m$ members of $\mathcal{F}$ randomly, independently with the same distribution: at each draw, each member $F$ of $\mathcal{F}$ is picked with probability $w(F)/w(\mathcal{F})$. For a fixed $x \in X$, the probability that $x$ is not covered at least $k$ times by the selected family is at most $\mathbb{P}(\xi < k)$, where $\xi = \xi_1 + \ldots + \xi_m$, with independent random 0-1-valued variables $\xi_1, \ldots, \xi_m$, each of expectation $1/\tau^*$. By Chernoff's inequality,

$$\mathbb{P}(\xi < k) \leq \exp\left(-\frac{(m - k\tau^*)^2}{3m\tau^*}\right).$$

Thus,

$$\mathbb{P}(\text{there is an } x \in X \text{ which is not covered}) \leq |X| \exp\left(-\frac{(m - k\tau^*)^2}{3m\tau^*}\right).$$

The lemma now clearly follows. □

**Definition 5.4.** Let $K$ and $L$ be two sets in $\mathbb{R}^d$. We define $N_k(L, K)$ the $k$-fold covering number of $L$ by $K$ as the minimum number of translates of $K$ that cover $K$ $k$-fold.

Note that $N_k(L, K) = \tau_k(\mathcal{F})$, where $\mathcal{F} = \{(x + K) \cap L : x \in \mathbb{R}^d\}$.

**Definition 5.5.** Let $K$ and $L$ be bounded Borel measurable sets in $\mathbb{R}^d$. A fractional covering of $L$ by translates of $K$ is a Borel measure $\mu$ on $\mathbb{R}^d$ with $\mu(x - K) \leq 1$ for all $x \in L$. The *fractional covering number* of $L$ by translates of $K$ is

$$N^*(L, K) = \inf\{\mu(\mathbb{R}^d) : \mu \text{ is a fractional covering of } L \text{ by translates of } K\}.$$

Clearly, in Definition 5.5 we may assume that the fractional cover $\mu$ is supported on $\mathrm{cl}(L - K)$. According to the Theorem 1.7 of [AAS15] we have

$$\max\left\{\frac{\mathrm{vol}\,(L)}{\mathrm{vol}\,(K)}, 1\right\} \leq N^*(L, K) \leq \frac{\mathrm{vol}\,(L - K)}{\mathrm{vol}\,(K)}. \tag{5.1}$$

The following straightforward corollary to Lemma 5.3 is the key element of the proof of the main theorem.

**Observation 5.6.** *Let $Y$ be a set, $\mathcal{F}$ a family of subsets of $Y$, and $X \subseteq Y$. Let $\Lambda$ be a finite subset of $Y$ and $\Lambda \subseteq U \subseteq Y$. Assume that for another family $\mathcal{F}'$ of subsets of $Y$ we have $\tau_k(X, \mathcal{F}) \leq \tau_k(\Lambda, \mathcal{F}')$. Then*

$$\tau_k(X, \mathcal{F}) \leq \tau_k(\Lambda, \mathcal{F}') \leq \lceil 6\tau^*(U, \mathcal{F}') \max\{\ln|\Lambda|, k\}\rceil.$$

For two sets, $K$ and $T$ we denote their *Minkowski difference* by $K \sim T = \{x \in \mathbb{R}^d : T + x \subseteq K\}$. The same proof as proof of Theorem 1.2 of [Nas16] yields the following.

**Theorem 5.7.** *Let $K$, $L$ and $T$ be bounded Borel measurable sets in $\mathbb{R}^d$ and let $\Lambda \subset \mathbb{R}^d$ be a finite set with $L \subseteq \Lambda + T$. Then*

$$N_k(L, K) \leq \lceil 6N^*(L - T, K \sim T) \max\{\ln |\Lambda|, k\} \rceil.$$

*If $\Lambda \subset K$, then we have*

$$N_k(L, K) \leq \lceil 6N^*(L, K \sim T) \max\{\ln |\Lambda|, k\} \rceil.$$

**Theorem 5.8.** *Let $K \subseteq \mathbb{R}^d$ be a convex body and $k \leq d(\ln d + \ln lnd)$ then we can cover the space $\mathbb{R}^d$ by translates of $K$ with density at most $6\mathrm{e}d(3\ln d + \ln\ln d + 15)$ so that every point is covered at least $k$ times.*

In the proof we will follow the steps of the proof of Theorem 2.1 of [Nas16].

*Proof:* By the Corollary 2.23 theorem we may assume that

$$\left[-\frac{1}{2}, \frac{1}{2}\right]^d \subseteq K \subseteq [-d, d]^d.$$

Let $C = \left[-\frac{a}{2}, \frac{a}{2}\right]^d$ be a cube of edge length $a$. (We will set $a$ later.)

Let $\delta > 0$ be fixed and let $\Lambda \subseteq \mathbb{R}^d$ be a finite set such that $\lambda + \frac{\delta}{2}(K \cap (-K))$ is a saturated (ie. maximal) packing of $\frac{\delta}{2}(K \cap (-K))$ in $C - \frac{\delta}{2}(K \cap (-K))$. Thus $C \subseteq \Lambda + \delta K \subseteq \lambda + \delta(K \cap (-K)) \subseteq \Lambda + \delta K$. By considering volume, we have that

$$|\Lambda| \leq \frac{\mathrm{vol}\left(C - \frac{\delta}{2}(K \cap (-K))\right)}{\mathrm{vol}\left(\frac{\delta}{2}(K \cap (-K))\right)} \leq \frac{(a + \frac{\delta d}{2})^d 2^d}{\mathrm{vol}\left(\mathbf{B}(o, 1)\right) \left(\frac{\delta}{2}\right)^d}$$

(5.1) yields that

$$N^*(C - \delta(K \cap (-K)), K \sim \delta(K \cap (-K))) \leq$$
$$N^*(C - \delta K, (1 - \delta)K) \leq \frac{\mathrm{vol}\,(C - K)}{\mathrm{vol}\,((1 - \delta)K)} \leq \frac{(a + d)^d}{(1 - \delta)^d\,\mathrm{vol}\,(K)}.$$

From Theorem 5.7 we have now

$$N_k(C, K) \leq 6\frac{(a + d)^d}{(1 - \delta)^d\,\mathrm{vol}\,(K)} \ln\left(\frac{(a + \frac{\delta d}{2})^d 2^d}{\mathrm{vol}\,(\mathbf{B}(o, 10))\,(\frac{\delta}{2})^d}\right).$$

On the other hand

$$\vartheta_k(K) \leq N_k(C, K)\frac{\mathrm{vol}\,(K)}{\mathrm{vol}\,(C)} \leq 6\frac{(a + d)^d}{(1 - \delta)^d} \ln\left(\frac{(a + \frac{\delta d}{2})^d 2^d}{\mathrm{vol}\,(\mathbf{B}(o, 1))\,(\frac{\delta}{2})^d}\right) (\mathrm{vol}\,(C))^{-1}.$$

Choose now $\delta = \frac{1}{2d \ln d}$, $a = d^2$, and estimate $\mathrm{vol}\,(\mathbf{B}(o,1))$ by the volume of the cube of side length $\frac{1}{2\sqrt{d}}$, which is contained in $\mathbf{B}(o,1)$.

$$\vartheta_k(K) \leq 6\left(\frac{d^2+d}{d^2}\right)^d \left(1 - \frac{1}{2d\ln d}\right)^{-1} \left(d \ln\left(4d^3 \ln d + d + 2 + 2d^{\frac{1}{2}}\right)\right) \leq$$
$$6d\left(1+\frac{1}{d}\right)^d \exp\left(\frac{1}{\ln d}\right) \ln(8d^3 \ln d) \leq$$
$$6d\left(1+\frac{2}{\ln d}\right)(3\ln d + \ln\ln d + \ln 8) \leq$$
$$6\mathrm{e}d(3\ln d + \ln\ln d + 15)$$

yields the desired bound.

□

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