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SOME PROBLEMS IN COMBINATORIAL GEOMETRY

Master’s thesis

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1 Introduction

Helly’s theorem is one of the fundamental theorems in discrete geometry. It states the following.

Let $C$ be a finite family of convex sets in $\mathbb{R}^d$ such that, for $k \leq d + 1$, any $k$ members have nonempty intersection. Then the whole family have nonempty intersection.

This theorem was discovered by Eduard Helly [17] in 1923. Since then it has played a central role in combinatorial convexity. It is used as a powerful tool in many results. One of the useful ways to think about it is the following. If a family of convex sets has empty intersection, then this fact is witnessed by $d + 1$ of the sets. Using this fact we can simplify many proofs and in lot of the cases we can get algorithmic results. For a survey on Helly’s theorem and its relatives the reader may refer to Danzer, Grünbaum, Klee [7] and Eckhoff [11].

The goal of this thesis is to present some new generalizations of Helly’s theorem. We also include the necessary background and context. Section 2 contains the basic definitions and results about convex sets and ellipsoids. We also discuss the general formulation of Helly-type theorems. Section 3 describes two kind of variations, the colorful and quantitative Helly-type theorems. In Section 4 we present our main result on a new version of Helly’s theorem concerning inscribed ellipsoids. It states the following

**Theorem.** Let $C_1, C_2, \ldots, C_{d(d+3)/2}$ be finite families of compact convex sets in $\mathbb{R}^d$, and suppose that for any choice $C_1 \in C_1, \ldots, C_{d(d+3)/2} \in C_{d(d+3)/2}$, the intersection $\bigcap_{i=1}^{d(d+3)/2} C_i$ contains an ellipsoid of volume 1. Then for some $j$, the intersection of the sets in the family $C_j$ also contains an ellipsoid of volume 1.

We discuss some open questions and conjectures in Section 5.
Notations and terminology

- The d-dimensional Euclidean space: \( \mathbb{R}^d \).
- The space of \( d \times d \) matrices: \( M^{d \times d} \).
- Trace of a matrix \( M \): \( Tr(M) \).
- Identity matrix: \( I_d \).
- Unit vectors in \( \mathbb{R}^d \): \( e_1, \ldots, e_d \).
- Convex set: \( C, K, C_1, C_2, \ldots \).
- Family of convex sets: \( C, \mathcal{C}_1, \ldots \).
- Closed halfspace: \( H \).
- Unit ball in \( \mathbb{R}^d \): \( B_2^d \).
- Ellipsoid: \( E \).
- For a family of sets \( \mathcal{C} \), let \( \cap \mathcal{C} \) be defined as \( \cap \mathcal{C} = \bigcap_{c \in \mathcal{C}} c \).
- If \( K \) is a convex body, \( vol(K) \) will mean the volume measure of the appropriate dimension.
- Boundary of a set \( K \): \( bd(K) \).
- For \( u, v \in \mathbb{R}^d \), \( \langle u, v \rangle \) denotes their scalar product.
- For \( u, v \in \mathbb{R}^d \), \( u \otimes v \) denotes the rank 1 matrix \( uv^T \).
- For \( a, b \in \mathbb{R}^d \), \( [a, b] \) denotes the segment connecting \( a \) and \( b \).
2 Background Material

The goal of this section is to introduce the definitions and results that are needed in later sections, and to present some examples. We introduce the basics of convex geometry, ellipsoids and Helly’s theorem. Since these results are well known, we present most of them without proofs. For more details see [2], [7], [13], and [16].

2.1 Convex sets

A set \( C \) in \( \mathbb{R}^d \) is called convex if, for all \( x, y \in C \) and for all \( t \in [0, 1] \), the point \( (1 - t)x + ty \) belongs to \( C \). In other words, for all \( x, y \in C \), the segment \( [x, y] \) is in \( C \). Another useful formulation uses convex combinations.

**Definition 2.1.** Let \( x_1, x_2, \ldots, x_k \) be points in \( \mathbb{R}^d \). Then \( y \) is called a **convex combination** of \( x_1, x_2, \ldots, x_k \) if there exists \( \lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}_+ \) such that \( \sum_{i=1}^{k} \lambda_i = 1 \) and \( y = \sum_{i=1}^{k} \lambda_i x_i \).

It is not hard to see that \( C \subset \mathbb{R}^d \) is convex if and only if it is closed under convex combination.

![Figure 1: Some convex sets in \( \mathbb{R}^2 \).](image)

For a set \( S \subset \mathbb{R}^d \), the **convex hull** of \( S \) is the containment minimal convex set that contains \( S \). We will denote this by \( \text{conv}(S) \). Since the intersection of convex sets is convex, this can also be defined as the intersection of all convex sets that contain \( S \).

We list a few properties of the convex hull operation. For any set \( S \subset \mathbb{R}^d \), \( \text{conv}(S) = \text{conv}(\text{conv}(S)) \). Also \( S_1 \subset S_2 \) implies \( \text{conv}(S_1) \subset \text{conv}(S_2) \). It is not
hard to see that we can obtain the convex hull by taking the convex combinations of all subsets of \( S \) with at most \( d + 1 \) points.

**Example 2.2.** The following sets are convex.

- The \( d \)-dimensional Euclidean space \( \mathbb{R}^d \).
- For any \( 0 \neq v \in \mathbb{R}^d \) and \( t \in \mathbb{R} \), the set \( \{ x | x^Tv \leq t \} \) is a halfspace and \( \{ x | x^Tv = t \} \) is a hyperplane.
- For some \( a \in \mathbb{R}^d \), \( r \in \mathbb{R}_+ \), the set \( B_r(a) = \{ x | \| x - a \| \leq r \} \) is called a ball.
- For some affine independent \( a_1, \ldots, a_{d+1} \in \mathbb{R}^d \) points, the set \( \text{conv}\{a_1, \ldots, a_{d+1}\} \) is called a \( d \)-dimensional simplex.
- \( \text{conv}\{e_1, -e_1, e_2, -e_2, \ldots, e_d, -e_d\} \) is called the \( d \)-dimensional cross-polytope.
- For a norm \( \| \cdot \| \), the set \( \{ x | \| x \| \leq 1 \} \) is called the unit ball of the norm.

We will denote the **unit ball** of the Euclidean norm in \( \mathbb{R}^d \) as \( B_2^d \). If we do not specify the norm, we always consider the Euclidean norm. We denote the volume of the unit ball as \( v_d \). It is well known that

\[
v_d = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)} \tag{2.1}\]

For details see Ball [2]. From Stirling’s formula we obtain that the this volume is roughly \( \left( \sqrt{\frac{2\pi e}{d}} \right)^d \).

Let \( S \) be a set in \( \mathbb{R}^d \). A hyperplane is called a **supporting hyperplane** of \( S \) if it has at least one common point with \( S \) and \( S \) is contained in one of the half-spaces determined by the hyperplane. If \( C \) is a convex set, then there is at least one supporting hyperplane for every boundary point of \( C \). For every supporting hyperplane, the half-space that contains the set is called a **supporting half-space**. If \( C \) is closed convex set and \( v \in \mathbb{R}^d \), then there is a unique supporting half-space with normal vector \( v \).

Two sets, \( S \subset \mathbb{R}^d \) and \( Q \subset \mathbb{R}^d \) are **strictly separated**, if there exists a nonzero vector \( v \) and a real number \( c \) such that \( \langle x, v \rangle < c \) and \( \langle y, v \rangle > c \) for every \( x \in S, y \in Q \). In this case the hyperplane \( \{ x | x^Tv = c \} \) is called a strictly separating hyperplane. There are various separation theorems for convex sets, we will use the following version.
Theorem 2.3. If $S$ and $Q$ are closed convex sets in $\mathbb{R}^d$ and at least one of them is compact, then there is a hyperplane that strictly separates them.

In particular a closed convex set can be separated from a point outside of the set. This implies that a closed convex set is equal to the intersection of its supporting half-spaces, or it is the whole space.

2.2 Helly’s theorem

Helly’s theorem has many proofs. We present the proof of Radon [7] which was one of the earliest ones. The following theorem was proven by him as a lemma for Helly’s theorem. Later, it gained importance on its own right.

Theorem 2.4 (Radon). Given a set $S \subset \mathbb{R}^d$ such that $|S| \geq d + 2$. Then there is a nontrivial partition $S_1 \cup S_2 = S$, $S_1 \cap S_2 = \emptyset$ such that $\text{conv}(S_1) \cap \text{conv}(S_2) \neq \emptyset$.

Proof. It is enough to prove the case when $|S| = d + 2$. Let us denote the elements of $S$ as $s_1, s_2, \ldots, s_{d+2}$. Let $s^j_i$ denote the $j$-th coordinate of $s_i$. Consider the following homogeneous linear equations.

\begin{align}
\lambda_1 s^1_1 + \cdots + \lambda_{d+2} s^j_{d+2} &= 0, (i = 1, \ldots, d) \quad (2.2) \\
\lambda_1 + \cdots + \lambda_{d+2} &= 0 \quad (2.3)
\end{align}

The first line gives $d$ equations for the coordinates, hence we have $d + 1$ equations. These equations have a nontrivial solution, since we have more variables than equations. We fix a solution $\lambda_1, \ldots, \lambda_{d+2}$. Let $I_1$ be the set of all $i$ for which $\lambda_i > 0$, $I_2$ the set of all $i$ for which $0 < \lambda_i$, and $c := \sum_{i \in I_1} \lambda_i$. We have

\[
\sum_{i \in I_1} \lambda_i s_i = \sum_{i \in I_2} -\lambda_i s_i
\]

\[
\sum_{i \in I_1} \frac{\lambda_i}{c} s_i = \sum_{i \in I_2} -\frac{\lambda_i}{c} s_i
\]

By equations (2.2) and (2.3) both sides are convex combination. Hence, the point $\sum_{i \in I_1} \frac{\lambda_i}{c} s_i$ is the convex combination of $S_1 := \{s_i\}_{i \in I_1}$ and also of $S_2 := \{s_i\}_{i \in I_2}$. Since we had a nontrivial solution, none of these sets are empty. \qed

5
**Theorem 2.5** (Helly). Let $\mathcal{C}$ be a finite family of convex sets in $\mathbb{R}^d$, and suppose that for any choice $C_1 \in \mathcal{C}, \ldots, C_{d+1} \in \mathcal{C}$, the intersection $\bigcap_{i=1}^{d+1} C_i$ is nonempty. Then $\bigcap \mathcal{C}$ is also nonempty.

**Proof.** We will use induction on the number of the sets. For $|\mathcal{C}| \leq d + 1$, the statement is trivial. Suppose the statement is known for all families of $k - 1$ sets, for some $k \geq d + 2$. Let $\mathcal{C}$ be a family of $k$ sets, each $d + 1$ having nonempty intersection. By the induction hypothesis for each $K \in \mathcal{C}$ there is a point $p_K$ such that $p_K \in \bigcap_{C \in \mathcal{C}, C \neq K} C$. Using Theorem 2.4 on these at least $d + 2$ points, we obtain a partition $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, such that $\text{conv}(\{p_C\}_{C \in \mathcal{C}_1}) \cap \text{conv}(\{p_C\}_{C \in \mathcal{C}_2}) \neq \emptyset$. Clearly $\text{conv}(\{p_C\}_{C \in \mathcal{C}_1}) \subset \bigcap_{C \in \mathcal{C}_2} C$ and $\text{conv}(\{p_C\}_{C \in \mathcal{C}_2}) \subset \bigcap_{C \in \mathcal{C}_1} C$. Hence, any point $p$ in $\text{conv}(\{p_C\}_{C \in \mathcal{C}_1}) \cap \text{conv}(\{p_C\}_{C \in \mathcal{C}_2})$ lies in $(\bigcap \mathcal{C}_1) \cap (\bigcap \mathcal{C}_2) = \bigcap \mathcal{C}$. □

A two dimensional example is presented Figure 2.

![Figure 2: A case of Helly’s theorem in $\mathbb{R}^2$](image)

Helly’s own proof is based on the separation theorem (see Theorem 2.3), and uses induction on the dimension of the space. Other proofs use topological methods, or other theorems from combinatorial geometry, such as the following theorem of Carathéodory [6].

**Theorem 2.6** (Carathéodory). Given a set $S \subset \mathbb{R}^d$ and a point $p \in \text{conv}(S)$. Then there is a set $Q \subset S$ such that $|Q| \leq d + 1$ and $p \in \text{conv}(Q)$. 
An other closely related theorem is the following generalization of Theorem 2.4.

**Theorem 2.7** (Tverberg). Given a set $S \subseteq \mathbb{R}^d$ such that $|S| > (d + 1)(m - 1)$ for some $m$. Then $S$ can be partitioned into $m$ disjoint parts $S_1, \ldots, S_m$ in such a way that the $m$ convex hulls $\text{conv}(S_1), \ldots, \text{conv}(S_m)$ have a point in common.

### 2.3 On Helly-type theorems in a general setting

Helly’s theorem inspired many analogous results. They usually have the following form. Given a family of objects $\mathcal{A}$ such that any subfamily of size $k$ has property $P$, then $\mathcal{A}$ has property $Q$. Theorems of this type are called **Helly-type theorems**.

In these theorems $k$ is usually referred to as the **Helly number** of property $P$. In most of the cases we can find a simple construction to show that the Helly number is not smaller than a certain number.

Consider for example the Helly’s original theorem. We want to show that is $d + 1$ sharp. Hence, we need to show a family of convex sets such that every $d$ of them have a point in common but not all of them. Consider the facets of a $d$ dimensional simplex. These facets are convex sets and any $d$ of them have a point in common, namely one of the vertices. On the other hand, their intersection is empty. Based on this, we can easily create many examples, see Figure 3 for a 2-dimensional one.

![Figure 3](image)

Figure 3: Three pairwise intersecting sets, without a point in common.

Usually we also try to strengthen the property $Q$. The theorem is more interesting if the properties $P$ and $Q$ are the same or at least similar.

As an example for a Helly-type theorem, we give a different formulation of Carathéodory’s theorem.
Theorem 2.8 (Carathéodory). Given a family of points in $\mathbb{R}^d$ and a point $p$. If for any subfamily of size $d + 1$, the point $p$ is not in the convex hull of the subfamily, then $p$ is not in the convex hull of the family.

Helly-type theorems occur in many areas of mathematics, not just in geometry. Here is an example by Gavril [12] from graph theory.

Theorem 2.9 (Helly-type theorem for trees). Let $G$ be a tree graph and let $\mathcal{G}$ denote a family of subtrees of $G$. If any two members of $\mathcal{G}$ have a vertex in common, then there is vertex that belongs to every member of $\mathcal{G}$.

2.4 Geometric properties of ellipsoids

Let $A \in M^{d \times d}$ be a positive definite matrix and $a \in \mathbb{R}^d$, then they define an ellipsoid $E(A, a)$ by the following definition:

$$E(A, a) = \{x \in \mathbb{R}^d | (x - a)^T A^{-1} (x - a) \leq 1\}$$

Another way to define ellipsoid is as affine images of the unit ball. It is not hard to see that if $Q \in M^{d \times d}$ is a non-singular affine map, then $E(Q^2, a) = QB_d^d + a$. It also works the other way. For a positive definite matrix $A$, there is a unique positive definite matrix $A^{1/2}$ such that $A = (A^{1/2})^2$.

Figure 4: An ellipsoid in $\mathbb{R}^2$

There is a strong connection between the geometric properties of the ellipsoid and the algebraic properties of the corresponding matrix. Since $A$ is positive definite, there is an orthogonal basis of eigenvectors. The axes of the ellipsoid correspond to
the eigenvectors and their lengths correspond to the eigenvalues. An affine transformation scales the volume by the determinant of its matrix. Hence,

\[ \text{vol}(\mathcal{E}(A, a)) = \sqrt{\det(A)} v_d \quad (2.4) \]

Given a positive definite matrix \( A \in M^{n \times n} \), we can define a norm the following way.

\[ \|x\|_A = \sqrt{x^T A^{-1} x} \]

These types of norms are sometimes called ellipsoidal norms, since the unit ball in norm \( \|\cdot\|_A \) is \( \mathcal{E}(A, 0) \). From this we have \( \mathcal{E}(A, a) = \{x \in \mathbb{R}^d | \|x - a\|_A \leq 1\} \).

One of the great advantages of ellipsoids is the following idea: Suppose we have a problem that is invariant under affine transformation and we have an ellipsoid that is somehow related to the problem. Then we can use an affine transformation to transform our ellipsoid into a unit ball. After the transformation we have to deal with a ball instead of an arbitrary ellipsoid which is generally much easier. As an example we present a theorem, first published by Danzer et al. [8]. A convex compact set with non-empty interior is called a convex body.

**Theorem 2.10.** Let \( K \subset \mathbb{R}^d \) be a convex body. Then \( K \) contains a unique ellipsoid of maximal volume.

**Proof.** Since \( K \) is compact, it has finite volume. Hence, we have a bound on the volume of the ellipsoids inside \( K \). Since \( K \) is compact, this implies that there is an ellipsoid of maximal volume. We only have to show the uniqueness.

Suppose \( \mathcal{E}_1 = A B_2^d + a \) and \( \mathcal{E}_2 = B B_2^d + b \) are two ellipsoids of maximal volume. We will show that they are identical. Clearly this problem is affine invariant, since an affine transformation changes the volumes by a constant factor. Therefore, by an affine transformation, we can assume that \( \mathcal{E}_1 = B_2^d \). Then by a rotation we can assume that \( B \) is diagonal. Let \( b_{i,i} \) denote the \( i \)-th element of the diagonal. Then, by equation 2.4

\[ v_d = \text{vol}(B_2^d) = \text{vol}(\mathcal{E}(B^2, b)) = v_d \det(B) = v_d \prod_{i=1}^d b_{i,i} \]

Hence, \( \prod_{i=1}^d b_{i,i} = 1 \).
Consider now the ellipsoid \( \mathcal{E} = \frac{A + B}{2} B^d_2 + \frac{a + b}{2} \). Figure 5 shows a 2-dimensional example.

We will show that \( \mathcal{E} \subseteq \text{conv}(\mathcal{E}_1 \cup \mathcal{E}_2) \). For a point \( p \in \mathcal{E} \), there is a point \( q \in B^d_2 \) such that \( p = \frac{A + B}{2} q + \frac{a + b}{2} \). Clearly then \( p \) is the convex combination of \( Aq + a \) and \( Bq + b \) and those points are in \( \text{conv}(\mathcal{E}_1 \cup \mathcal{E}_2) \). Hence, \( \mathcal{E} \subseteq \text{conv}(\mathcal{E}_1 \cup \mathcal{E}_2) \subseteq \mathcal{K} \).

We can calculate the volume of \( \mathcal{E} \) using Equation 2.4.

\[
\text{vol}(\mathcal{E}) = v_d \det \left( \frac{A + B}{2} \right) = v_d \det \left( \frac{I_d + B}{2} \right) = v_d \prod_{i=1}^{d} \frac{1 + b_{i,i}}{2}
\]

Using the AM–GM inequality we conclude

\[
\text{vol}(\mathcal{E}) = v_d \prod_{i=1}^{d} \frac{1 + b_{i,i}}{2} \geq v_d \prod_{i=1}^{d} \sqrt{b_{i,i}} = v_d
\]

And equality holds if and only if \( b_{i,i} = 1 \) for every \( i = 1, \ldots, d \). Since \( \mathcal{E} \) cannot have larger volume than \( \mathcal{E}_1 \), equality must hold. Therefore \( B = I_d \), so both ellipsoids are balls. Figure 6 shows how we can find a larger ellipsoid again. By a rotation we can assume that \( b = c \cdot e_1 \) for some \( c \in \mathbb{R}_+ \). Consider the ellipsoid \( \mathcal{E}_3 = C B^d_2 + b \), where \( C \) is the diagonal matrix \( \text{diag}(\frac{c}{2} + 1, 1, 1, \ldots, 1) \). Then \( \mathcal{E}_3 \subseteq \text{conv}(\mathcal{E}(A, a) \cup \mathcal{E}(B, b)) \subseteq \mathcal{K} \). And its volume is \( v_d(\frac{c}{2} + 1) \geq v_d \). Since we chose
maximal ellipsoids, equality must hold, so \( c = 0 \). We conclude that \( A = B = I_d \) and \( a = b = 0 \), hence \( \mathcal{E}_1 = \mathcal{E}_2 \). So the maximal ellipsoid is unique.

![Figure 6: The convex hull of the union of two balls contains a larger ellipsoid.](image)

For a convex body, the inscribed ellipsoid of largest volume is called the **Löwner ellipsoid** of the set.

For \( u, v \in \mathbb{R}^d \), let \( u \otimes v \) denote the rank 1 matrix \( uv^T \). The following fundamental result of John [18] characterizes the Löwner ellipsoid.

**Theorem 2.11 (John).** Let \( K \subset \mathbb{R}^d \) be a convex body. Then \( K \) contains a unique ellipsoid of maximal volume. This ellipsoid is \( \mathbf{B}_2^d \) if and only if \( \mathbf{B}_2^d \subset K \) and there exist contact points \( u_1, \ldots, u_m \in \text{bd}(K) \cap \text{bd}(\mathbf{B}_2^d) \) and positive numbers \( \lambda_1, \ldots, \lambda_m \) such that

\[
\sum_{i=1}^{m} \lambda_i u_i = 0 \quad (2.5)
\]

and

\[
I_d = \sum_{i=1}^{m} \lambda_i u_i \otimes u_i \quad (2.6)
\]

For us it will be important that there is a bound on the number of the contact points. Intuitively this means that only \( d(d+3)/2 \) contact points is needed, they determine the Löwner ellipsoid. One might wonder where the number \( d(d+3)/2 \) comes from. As we have seen an ellipsoid can be defined by a symmetric matrix and a vector. The dimension of the space of symmetric matrices is \( \frac{d(d+1)}{2} \), hence an ellipsoid
is defined by $\frac{d(d+1)}{2} + d = \frac{d(d+3)}{2}$ parameters. It has been shown by Gruber [14] that, for most of the convex bodies, the Löwner ellipsoid touches the boundary in at least $d(d+3)/2$ points.

On the contact points we also get an algebraic relation. Equation 2.5 means that the center of the Löwner ellipsoid is in the convex hull of the contact points.

Equation 2.6 expresses that the contact points should not be too close to a subspace because in that case we could squeeze our ellipsoid along that subspace so that it would grow in volume but it would remain in the set (see Figure 7).

![Figure 7: The dotted ellipse has larger volume.](image)

As an application of Theorem 2.11 we prove the following.

**Theorem 2.12.** Let $C$ be a symmetric convex body and let $\mathcal{E}$ be its Löwner ellipsoid. Then if we enlarge $\mathcal{E}$ from its center by a factor of $\sqrt{d}$, we get an ellipsoid that contains $C$.

**Proof.** Since the problem is affine invariant, we can assume that $\mathcal{E} = B^d_2$. Then we need to prove that $|x| \leq \sqrt{d}$ for each $x \in C$. Based on Theorem 2.11, we can pick $u_1, \ldots, u_m$ vectors and $\lambda_1, \ldots, \lambda_m$ positive reals such that $\sum_{i=1}^m \lambda_i u_i = 0$ and $I_d = \sum_{i=1}^m \lambda_i u_i \otimes u_i$. Since $B^d_2 \subset C$, the supporting hyperplane at $u_i$ is $\{x| \langle x, u_i \rangle = 1\}$. Hence, for every $x \in C$ and for every $j \in \{1, 2, \ldots, m\}$, we have $\langle x, u_j \rangle \leq 1$. Since $C$ is symmetric, this implies $|\langle x, u_j \rangle| \leq 1$. We also have the following relation.

\[
d = \text{Tr}(I_d) = \text{Tr}(\sum_{i=1}^m \lambda_i u_i \otimes u_i) = \sum_{i=1}^m \lambda_i \text{Tr}(u_i \otimes u_i) = \sum_{i=1}^m \lambda_i\]
Thus, we have
\[ |x|^2 = x^T x = x^T (x I_d) = x^T (x \sum_{i=1}^{m} \lambda_i u_i \otimes u_i) = \sum_{i=1}^{m} \lambda_i \langle x, u_i \rangle^2 \leq \sum_{i=1}^{m} \lambda_i = d \]

In the nonsymmetric case, a similar argument yields that enlarging the Löwner ellipsoid by \(d\) will result in a containing ellipsoid. Figure 8 show this theorem when \(C\) is a triangle in \(\mathbb{R}^2\). For the symmetric case, the cube proves that \(\sqrt{d}\) is the best possible, and for the nonsymmetric case, a simplex is extremal.

![Figure 8: A triangle and its Löwner ellipsoid.](image)

These theorems imply a bound on the size of the Löwner ellipsoid. The volume of the enlarged ellipsoid is larger than the volume of the set, so we get the following.

**Corollary 2.13.** Let \(C\) be a convex body in \(\mathbb{R}^d\) and let \(E\) be its Löwner ellipsoid. Then
\[
vol(E) \geq \frac{vol(C)}{d^d}
\]

This result yields a rough approximation for the volume of a convex body.

We note that there is a dual notion, for a convex body we can consider the ellipsoid of smallest volume that contains the set. Most of the theorems about inscribed ellipsoids have a dual theorem. For example in his original paper John [18] proves the following theorem:
Theorem 2.14. Let $C$ be a convex body in $\mathbb{R}^d$ and let $E$ be the ellipsoid of smallest volume that contains $C$. Then, if we shrink $E$ from its center by a factor of $d$, we get an ellipsoid that is contained in $C$.

Another famous application of the Löwner ellipsoids is the Ellipsoid method. It was the first polynomial algorithm for solving a system of linear inequalities. The main idea is to maintain an ellipsoid that contains all possible solutions, and to replace this ellipsoid with smaller ones until we either find a solution or we can conclude that there is no solution to the system. The replacement of the ellipsoid is done by calculating the dual Löwner ellipsoid of a section of the previous ellipsoid. A detailed explanation can be found in [13].
3 Generalizations of Helly’s theorem

In this section we will discuss two types of generalizations of Helly’s theorem. Later we provide a combination of these two generalizations.

3.1 Colorful theorems

Let $C$ be a family of convex sets and suppose that every set is painted with one of $d + 1$ colors. Assume that every subfamily that uses $d + 1$ distinct colors have nonempty intersection. What conclusion can we draw now? It is possible that the whole family has empty intersection. On the other hand there will be a color such that the sets of that color have a point in common.

If we partition $C$ according to the colors, we get $C_1, C_2, \ldots, C_{d+1}$ nonempty families of sets. We will call these the **color classes**. A selection of sets $C_1 \in C_1, \ldots, C_{d+1} \in C_{d+1}$ will be called a **colorful selection**.

The first colorful generalization of Helly’s theorem was proven by Lovász.

**Theorem 3.1** (Colorful Helly). *Let $C_1, C_2, \ldots, C_{d+1}$ be finite families of compact convex sets in $\mathbb{R}^d$, and suppose that for any colorful selection $C_1 \in C_1, \ldots, C_{d+1} \in C_{d+1}$, the intersection $\bigcap_{i=1}^{d+1} C_i$ is nonempty. Then for some $j$, the intersection $\bigcap C_j$ is also nonempty.*

This is a generalization of Helly’s theorem in a strong sense, since taking $C_1 = C_2 = \ldots = C_{d+1} = C$ immediately yields Helly’s original theorem.

Lovász did not publish this theorem but he presented it as a problem in a Hungarian math competition. Later Bárány [3] published it with another proof that uses a colorful version of Theorem 2.6. We will present Lovász’s proof here because we will use his ideas later.

First we prove a lemma that can be viewed as a different formulation of Helly’s theorem. We will use Helly’s theorem to prove it and we will see that it implies even the Colorful Helly’s theorem.

Let $x_1, x_2$ be points in $\mathbb{R}^d$. We say that $x_1$ is **lexicographically smaller** than $x_2$ if $x_1$ is smaller in the last coordinate where they differ. Usually they differ in the last coordinate, so we can think about it simply as $x_1$ is lower than $x_2$. 

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Lemma 3.2. Let $\mathcal{C}$ be a family of convex compact sets in $\mathbb{R}^d$ with at least $d$ elements such that $\cap \mathcal{C} \neq \emptyset$. Let $p$ be the lexicographical minimum of $\cap \mathcal{C}$. Then there exist a subfamily $\mathcal{C}' \subset \mathcal{C}$ with at most $d$ elements such that the lexicographical minimum of $\cap \mathcal{C}'$ is $p$.

Figure 9: The lexicographical minimum is already determined by the circle and the triangle.

Proof. Let $K$ denote the points in $\mathbb{R}^d$ that are lexicographically smaller than $p$. We leave it as an exercise to see that $K$ is convex. Hence, we can use Helly’s theorem on $\mathcal{C} \cup \{K\}$. By the choice of $K$ we have $(\cap \mathcal{C}) \cap K = \emptyset$. Therefore there are $d + 1$ sets in $\mathcal{C} \cup \{K\}$ such that their intersection is empty. Since $\cap \mathcal{C} \neq \emptyset$, $K$ must be one of these sets. $K$ contains every point that are lexicographically smaller than $p$, therefore the other $d$ selected set do not contain a smaller point than $p$ in their intersection. Since they contain $p$, their lexicographical minimum must be $p$. Figure 9 depicts an example in $\mathbb{R}^2$. \[ \square \]

Now we can turn to the proof of Theorem 3.1.

Proof. For each colorful selection $C_1 \in \mathcal{C}_1, \ldots, C_{d+1} \in \mathcal{C}_{d+1}$, we can choose a lexicographically minimal point in their intersection. Among these minimal points there is a lexicographically maximal one, since there are only finitely many of them. Let us call it $p$ and let the defining sets be $C_1 \in \mathcal{C}_1, \ldots, C_{d+1} \in \mathcal{C}_{d+1}$. By the Lemma 3.2 this $p$ is already determined by $d$ sets. So there is a $j \in \{1, \ldots, d + 1\}$ such that $p$ is the lexicographical minimum in $\bigcap_{i=1, i \neq j}^{d+1} C_i$. We will show that $p \in \cap \mathcal{C}_j$. 

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Let $K$ be an arbitrary element of $C_j$ and suppose that $K$ does not contain $p$. Since \( \{C_1, \ldots, C_{j-1}, K, C_{j+1}, \ldots C_{d+1}\} \) is a colorful selection, by the assumption of the theorem \( (\bigcap_{i=1, i \neq j}^{d+1} C_i) \cap K \) is non-empty. Since the lexicographical minimum of \( \bigcap_{i=1, i \neq j}^{d+1} C_i \) is $p$, and $p \notin K$, the lexicographical minimum of \( \bigcap_{i=1, i \neq j}^{d+1} C_i \cap K \) is lexicographically larger than $p$. This contradicts the maximality of $p$. \( \square \)

In the case of Helly’s theorem, the Helly number was best possible. Similarly, in the colorful version the number of the color classes can not be decreased. See Figure 10 for a counterexample in $\mathbb{R}^2$ with two colors. Every blue set intersects every red one, but the two red and the two blue sets are disjoint.

![Figure 10: In $\mathbb{R}^2$ two color classes is not enough.](image)

In $\mathbb{R}^d$ we can create a similar counterexample using the facets of a cube. In each color class we put two opposite facets of the cube. Since the cube has $2d$ facets, we get $d$ classes, each containing two disjoint sets.

Recall that a general Helly-type theorem has the following form. Given a family of objects $\mathcal{A}$ such that any subfamily of size $k$ has property $\mathcal{P}$, then $\mathcal{A}$ has property $\mathcal{Q}$.

For every Helly-type theorem we can try to create a corresponding colorful version. Given a family of objects $\mathcal{A}$, that are colored with $k$ colors. Suppose that any subfamily of size $k$ that uses all $k$ colors has property $\mathcal{P}$. Then there is a color $i$ such that the subfamily containing the sets of color $i$ have property $\mathcal{Q}$.

It is important that a Helly-type theorem does not immediately implies its colorful version. In some cases the colorful version does not hold!
As examples we state the colorful versions of Theorem 2.6 and Theorem 2.7. For more details see [1], [3], [20] and [24].

**Theorem 3.3** (Colorful Carathéodory). Given sets \( S_1, S_2, \ldots, S_{n+1} \subset \mathbb{R}^d \) and \( p \in \mathbb{R}^d \) such that \( p \in \text{conv}(V_i) \) for \( i = 1, \ldots, n + 1 \). Then there exists a colorful selection of vectors \( s_i \in S_i \), such that \( p \in \text{conv}(s_1, \ldots, s_{n+1}) \).

**Theorem 3.4** (Colorful Tverberg Theorem). For every \( d \geq 1 \) and \( r \geq 2 \) there exists \( t \) such that whenever \( S \subset \mathbb{R}^d \) is a set of \( (d+1)t \) points partitioned into \( t \)-point subsets \( S_1, \ldots, S_{d+1} \), then there is a partition of \( S \) into sets \( (R_1, \ldots, R_r) \) such that \(|R_i \cap S_j| \leq 1\) for each \( i \) and \( j \), and \( \bigcap_{i=1}^{r} \text{conv}(R_i) \neq \emptyset \).

It has been observed by Minki [19], that Theorem 2.9 admits a colorful generalization.

**Theorem 3.5** (Colorful Helly-type theorem for trees). Let \( G = (V, E) \) be a tree graph and let \( G_1 \) and \( G_2 \) denote families of subtrees of \( G \). Suppose that for any choice \( T_1 \in G_1 \) and \( T_2 \in G_2 \), the two graph have a vertex in common, then there is a vertex that either belongs to every member of \( G_1 \) or to every member of \( G_2 \).

**Proof.** First we collect the essential steps from the proof of Theorem 3.1, and then we show that they can be applied in this context.

1. Create an appropriate ordering on the points.

2. For any family of sets, we can select the unique point in their intersection that is minimal by the ordering. We will call it the minimal point of the family.

3. From any colorful selection, one of the sets can be left out, such that the minimal point does’nt change. (Use the monochromatic version to prove this).

4. There is a colorful selection \( C_1, C_2, \ldots \), where the minimal point \( p \) is the maximal among all colorful selections. By the previous step there is an \( i \) such that the set \( C_i \) is not required to determine \( p \).

5. If \( K \) is an element of the \( i \)-th colorclass, then it must contain \( p \). Otherwise \( C_1, \ldots C_{i-1}, K, C_{i+1}, \ldots \) would be a colorful selection with higher minimal point than \( p \). This follows from the fact that \( v \) is minimal in \( C_1, \ldots C_{i-1}, C_{i+1}, \ldots \), but it is not in \( K \). Hence, \( p \) is in the intersection of the \( i \)-th colorclass.
The first step is easy, see Figure 11. We choose an arbitrary vertex as root. Then
the order of the vertices is determined by a breadth first search.

![Figure 11: One of the possible orderings on a tree.](image)

Step 2 is trivial, since we have a total ordering.

Step 3 is also not hard. Let $T_1, T_2$ be two subtrees, and $p$ be the minimal point in
their intersection. If both trees contained a smaller vertex, then both would contain
the parent of $p$. But this contradicts the minimality of $p$.

Step 4 and 5 are straightforward. A tree has finitely many vertices, hence we
can pick a maximal one in step 4, even if $G_1$ or $G_2$ is not finite.

### 3.2 Quantitative theorems

One of the most natural way to create new Helly-type theorems is to replace the
property from non-empty intersection to something else. Of course, we have to
change the conclusion and the Helly number accordingly. Bárány, Katchalski and
Pach [4] considered the variation when the property is changed to having an inter-
section of large volume. More precisely they showed the following theorem.

**Theorem 3.6** (Quantitative Helly). *There exists a constant $q(d)$ depending only on
the dimension such that the following holds. Let $\mathcal{C}$ be a family of compact convex
sets in $\mathbb{R}^d$, and suppose that for any choice $C_1 \in \mathcal{C}, \ldots, C_{2d} \in \mathcal{C}$, the intersection
$\bigcap_{i=1}^{2d} C_i$ has volume greater or equal 1. Then $\cap \mathcal{C}$ has volume at least $q(d)$.*

They also showed that $q(d) \geq d^{-2d^2}$ for the best possible value of $q(d)$ and
conjectured that $q(d) \sim d^{-cd}$ can be achieved for some constant $c$. This conjecture was proved by Naszódi [21], he showed that $q(d) \geq e^{-d-1}d^{-2d-\frac{1}{2}}$.

As we can see, this theorem is somewhat different from the Helly-type theorems we have seen so far. Usually, we can conclude the same property for the whole family that we assumed for any selection. Here we get something weaker, the volume of $\cap C$ is just a fraction of what we assumed for the intersection of any $2d$. There is a natural question, can we improve the Helly number or the bound on $q(d)$?

First consider the Helly number. As we can see it is $2d$ instead of the usual $d+1$. An easy example shows that we can not do better than that. A cube in $\mathbb{R}^d$ is determined by $2d$ half-spaces. The intersection of any $2d - 1$ of these half-spaces is unbounded. So no matter how small $q(d)$ we choose, we can create a counterexample for Helly number $2d - 1$. See Figure 12 for the 2-dimensional case. (A half-space is not compact, but we can just intersect everything with a large enough ball).

![Figure 12](image)

Figure 12: The shaded region is unbounded.

Clearly $q(d) \leq 1$ and it would be nice to have $q(d) = 1$. Unfortunately, it is not possible even if we consider a larger Helly number. If we take the halfspaces that define a regular $k + 1$-gon, then the intersection of any $k$ of them has larger volume than the the $k + 1$-gon. Hence, $q(d) < 1$ for any Helly number.

If we do not change the Helly number form $2d$, the situation is even worse, $q(d)$ is less than $d^{-cd}$ for some absolute constant $c$. This can be seen from an important example that was shown by Naszódi [21]. To see this, consider the unit sphere and its supporting half-spaces. The volume of the intersection of any $2d$ of these half-spaces is at least $c_1^d$ for some absolute constant $c_1$. But the intersection of the whole family is the unit sphere, which has volume $v_d$. Hence, $q(d) \leq \frac{v_d}{c_1^d} \sim d^{-cd}$ for some constant
Figure 13 depicts the 2-dimensional case. In this case the intersection of any 4 of the half-spaces has volume at least 4. Hence \( q(2) \leq \frac{\pi}{4} \).

A natural question arises. Can we improve \( q(d) \) if we allow a larger Helly-number? It turns out that we can improve our fraction to be arbitrarily close to 1 if we take enough color classes. These type of questions have been investigated in detail recently. For a detailed discussion of quantitative theorems see the work of De Lorea et al. [10].

**Theorem 3.7** (Quantitative Continuous Helly). For every \( d \) and \( \varepsilon > 0 \) there exists a number \( n^*(d, \varepsilon) \) such that following holds. Let \( \mathcal{C} \) be a finite family of convex set such that for any subfamily \( \mathcal{C}' \) of size at most \( n^*(d, \varepsilon)d \), we have \( \text{vol}(\cap \mathcal{C}') \geq 1 \). Then, \( \text{vol}(\cap \mathcal{C}) \geq \frac{1}{1+\varepsilon} \).

They also give a bound, \( n^*(d, \varepsilon) \leq \left( \frac{cd^2}{\varepsilon} \right)^{(d-1)/2} \) for some absolute constant \( c \). This bound is quite big, but they showed that we can not do better asymptotically. They also created a colorful version of this theorem, where the number of color classes is even worse than \( n^*(d, \varepsilon) \).

There is a missing version here, namely the quantitative colorful theorem with few color classes. In other words, a colorful version of Theorem 3.6 with as few color classes as possible and we do not require \( q(d) \) to be close to 1. At the end of the next section we show the following new result to fill this hole.

**Theorem 3.8** (Quantitative Colorful Helly). There exists a constant \( qc(d) \) depending only on the dimension such that the following holds. Let \( \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{d(d+3)/2} \) be nonempty families of compact convex sets in \( \mathbb{R}^d \), and suppose that for any choice
$C_1 \in \mathcal{C}_1, \ldots, C_{d(d+3)/2} \in \mathcal{C}_{d(d+3)/2}$ the intersection $\bigcap_{i=1}^{d(d+3)/2} C_i$ has volume greater or equal 1. Then for some $i$, the intersection of all the sets in the family $\mathcal{C}_i$ has a volume at least $qc(d)$.

We will also see that $qc(d)$ can be chosen such that $qc(d) > d^{-d}$.

We have omitted the proofs of Theorem 3.6 and Theorem 3.7. The proof of Theorem 3.6 heavily relies on the use of Löwner ellipsoids.
4 Helly-type theorem for ellipsoids

In this section we will show a new generalization of Helly’s theorem concerning inscribed ellipsoids. Then we will use this result to prove Theorem 3.8. First we discuss the monochromatic version in detail, then we create a colorful version.

4.1 Monochromatic version

Danzer, Grünbaum and Klee wrote an survey on the generalizations of Helly’s theorem [7]. In this article statement 6.17 states the following

**Theorem 4.1** (Behrend [5]). Let \( C \) be a convex body in \( \mathbb{R}^2 \) such that for every 5 halfplanes, each containing \( C \), the intersection’s Löwner ellipsoid has volume at least 1. Then the Löwner ellipsoid of \( C \) has volume at least 1.

We will show a more general version of this theorem in arbitrary dimension. This theorem is an easy consequence of Theorem 2.11.

**Theorem 4.2** (Helly-type theorem for ellipsoids). Let \( \mathcal{C} \) be a finite family of compact convex sets in \( \mathbb{R}^d \), and suppose that for any choice \( C_1 \in \mathcal{C}, \ldots, C_{d(d+3)/2} \in \mathcal{C} \), the intersection \( \bigcap_{i=1}^{d(d+3)/2} C_i \) contains an ellipsoid of volume 1. Then \( \bigcap\mathcal{C} \) also contains an ellipsoid of volume 1.

**Proof.** Suppose that \( \bigcap\mathcal{C} \) does not contain an ellipsoid of volume 1. We will show \( d(d+3)/2 \) sets from \( \mathcal{C} \) such that the Löwner ellipsoid of their intersection has volume smaller than 1.

The problem is clearly affine invariant. The only thing that changes, is the constant 1. Hence, by an affine transformation we can assume that the Löwner ellipsoid of \( \bigcap\mathcal{C} \) is the unit ball \( B_d^2 \). This means that the intersection of any \( d(d+3)/2 \) set from \( \mathcal{C} \) contains an ellipsoid of volume strictly greater than \( v_d \).

By Theorem 2.11, there exists some contact points \( u_1, \ldots, u_m \in bd(\bigcap\mathcal{C}) \cap bd(B_d^2) \) and positive numbers \( \lambda_1, \ldots, \lambda_m \) such that \( d+1 \leq m \leq \frac{d(d+3)}{2} \), \( \sum_{i=1}^{m} \lambda_i u_i = 0 \) and \( I_d = \sum_{i=1}^{m} \lambda_i u_i \otimes u_i \). Since \( \mathcal{C} \) is finite ant its elements are compact, \( bd(\bigcap\mathcal{C}) \subset \bigcup_{C \in \mathcal{C}} bd(C) \). Therefore, we can choose some \( C_1, C_2, \ldots C_m \in \mathcal{C} \) such that \( u_i \in bd(C_i) \) for \( i = 1, \ldots, m \).
Let us now consider the set \( K = \bigcap_{i=1}^{m} C_i \). We want to use the other direction of Theorem 2.11 on this set. We can see that \( B_2^d \subset C \subset K \). Also the same \( u_i \) contact points and \( \lambda_i \) values will be good. They satisfy the algebraic conditions trivially. The only thing we have to check is that \( u_i \in \text{bd}(K) \cap \text{bd}(B_2^d) \) for \( i = 1, \ldots, m \). In other words, we have to show that the \( u_i \)-s are contact points between \( B_2^d \) and \( K \).

Each \( u_i \) was chosen such that \( u_i \in \text{bd}(B_2^d) \). Since \( u_i \in K \) and \( C_i \) was chosen such that \( u_i \in \text{bd}(C_i) \), we also have \( u_i \in \text{bd}(K) \). Therefore the \( u_i \)-s are indeed contact points. By Theorem 2.11 we have that \( B_2^d \) is the Löwner ellipsoid of \( K \). This is a contradiction since we have seen that \( K \) contains an ellipsoid of volume greater than \( v_d \).

The Helly number in Theorem 4.2 is \( d(d + 3)/2 \), and this is the best possible that can be achieved. For every dimension \( d \), there exist \( d(d + 3)/2 \) half-spaces \( H_1, \ldots, H_{d(d+3)/2} \), such that the Löwner ellipsoid of \( \bigcap_{i=1}^{d(d+3)/2} H_i \) is smaller than the Löwner ellipsoid of \( \bigcap_{i=1, i \neq j}^{d(d+3)/2} H_i \) for any \( j \). In Figure 14 we can see that the five halfspaces that determine a regular pentagon give an example in 2-dimension.

Figure 14: The Löwner ellipsoid of the pentagon grows if we delete one of the sides.

Since an affine transformation does not change the structure of the sets, in Theorem 4.2 we could consider ellipsoids of volume \( c \) for any positive constant \( c \).

Using Theorem 4.2 we can give a quick proof for the quantitative helly theorem with helly number \( d(d+3)/2 \). Since the intersection of any \( d(d+3)/2 \) set has volume at least 1, by Corollary 2.13 each intersection contains an ellipsoid of volume at least
By Theorem 4.2 this implies that the intersection of all sets contains an ellipsoid of volume $d^{-d}$. Hence, the volume of the intersection is at least $d^{-d}$.

### 4.2 Colorful version

As we can see, Theorem 4.2 is a Helly-type result, so we can formulate a colorful version in the usual way.

**Theorem 4.3.** Let $C_1, C_2, \ldots, C_{d(d+3)/2} \subset \mathbb{R}^d$ be finite families of compact convex sets in $\mathbb{R}^d$, and suppose that for any colorful selection $C_1 \in C_1, \ldots, C_{d(d+3)/2} \in C_{d(d+3)/2}$ the intersection $\bigcap_{i=1}^{d(d+3)/2} C_i$ contains an ellipsoid of volume 1. Then for some $j$ the intersection of the sets in the family $C_j$ also contains an ellipsoid of volume 1.

**Proof.** We will follow the same steps that we presented in the proof of Theorem 3.5.

The first step is to create an ordering. This time we are working with ellipsoids instead of points, therefore we need an ordering on the ellipsoids.

**Definition 4.4.** For an an ellipsoid $E$ let $m_E$ denote the highest point inside $E$. Since an ellipsoid is strictly convex, this is unique. The **height** of an ellipsoid is just the last coordinate of $m_E$.

This height notion will provide our ordering. An important thing to note is that this is not a total ordering. Hence, the next step is not trivial in this case, we need to show that the minimal ellipsoid inside a set is unique.

**Lemma 4.5.** Let $C$ be a convex compact set, such that it contains an ellipsoid of volume 1. Then there is a unique ellipsoid of volume 1 such that every other ellipsoid of volume 1 inside $C$ has larger height. We will call this ellipsoid the **lowest** ellipsoid inside $C$.

**Proof.** For $t \in \mathbb{R}$, let $H_t$ denote the half-space $\{x \mid x^Te_d \leq t\}$, where $e_d = \{0,0,\ldots,0,1\}$.

Clearly the height of an ellipsoid $E$ is the smallest $t \in \mathbb{R}$ such that $E \subset H_t$. Hence, we can take the smallest $t \in \mathbb{R}$ such that $H_t \cap C$ contains an ellipsoid of volume 1. See Figure 6 for an example. Since $C$ is a compact set and it contains an ellipsoid of volume 1, this $t$ value is well defined. We will show that the Löwner ellipsoid of $H_t \cap C$ is the unique lowest ellipsoid of $C$. Let us call this ellipsoid $E$. 

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We will show that $\text{vol}(E) = 1$. By the choice of $t$ we have $\text{vol}(E) \geq 1$. Suppose that $\text{vol}(E) > 1$. Then we could shrink $E$ from its center to get a new ellipsoid $E'$ with $\text{vol}(E') = 1$. Since $E' \subset \text{int}(E)$, we would have $E' \subset \text{int}(H_t \cap C) \subset H_t - \varepsilon \cap C$ for some small $\varepsilon > 0$. This contradicts the minimality of $t$. Hence, $\text{vol}(E) = 1$.

By the choice of $t$, no ellipsoid of volume 1 inside $C$ has smaller height than $t$. On the other hand the height of $E$ is $t$. By Theorem 2.10 there is no other ellipsoid of volume 1 inside $H_t \cap C$. Hence, $E$ is indeed the unique lowest ellipsoid inside $C$. \hfill \Box

The next step is the following lemma. As usual we want to show that the lowest ellipsoid in the intersection of some sets is determined by one less sets than the number of the color classes.

**Lemma 4.6.** Let $C_1, C_2, \ldots, C_{d(d+3)/2}$ be convex compact sets in $\mathbb{R}^d$. Let $K := \bigcap_{i=1}^{d(d+3)/2} C_i$, and $K_j := \bigcap_{i=1,i\neq j}^{d(d+3)/2} C_i$, and let $E$ denote the lowest ellipsoid in $K$. Then there exists a $j$ such that the $E$ is also the lowest ellipsoid of $K_j$.

**Proof.** Let $H_t$ denote the same half-space as before, so $E$ is the Löwner ellipsoid of $K \cap H_t$.

Suppose that $E$ is not the lowest ellipsoid in $K_j$ for every $j \in \{1, \ldots, d(d+3)/2\}$. Since $E \subset K \subset K_j$, this means that each $K_j$ contains a lower ellipsoid than $E$. Therefore we can choose a small $\varepsilon > 0$ such that $K_j \cap H_{t-\varepsilon}$ contains an ellipsoid of volume 1 for each $j$. 

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Figure 15: The lowest ellipsoid of volume 1 inside the intersection of some half-spaces.
Let us consider now the following $\frac{d(d+3)}{2} + 1$ sets: $K_1, K_2, \ldots, K_{d(d+3)/2}, H_{t-\epsilon}$. We can use Theorem 4.2 on these sets. If we choose $\frac{d(d+3)}{2}$ of these sets and take intersection, we obtain either $K$, or $K_j \cap H_{t-\epsilon}$ for some $j$. By the assumption of Theorem 4.3, $K$ contains an ellipsoid of volume 1. We have also seen that $K_j \cap H_{t-\epsilon}$ contains an ellipsoid of volume 1. Hence, $C_1 \cap \cdots \cap C_{d(d+3)/2} \cap H_{t-\epsilon} = K \cap H_{t-\epsilon}$ also contains an ellipsoid of volume 1. This contradicts the fact that $E$ is the lowest ellipsoid in $K$.

Using Lemma 4.5 and Lemma 4.6 we can finish the proof of Theorem 4.3.

For every colorful selection $C_1 \in C_1, \ldots, C_{d(d+3)/2} \in C_{d(d+3)/2}$, there is an ellipsoid of volume 1 inside $\bigcap_{i=1}^{d(d+3)/2} C_i$. By Lemma 4.5, we can choose the lowest ellipsoid in each of these intersections. Let us denote the set of these ellipsoid as $B$. Since we have finitely many intersections, there is a highest one among these ellipsoids. Let us denote this ellipsoid by $E_{\text{max}}$.

Figure 16 depicts a very special and trivial case of the theorem. Three of the color classes only contain a single ellipsoid, and the other two contains two parallelograms. In this case $B$ only contains the four dark ellipsoids.

Figure 16: A very special case of Theorem 4.3.

$E_{\text{max}}$ is defined by some $C_1 \in C_1, \ldots, C_{d(d+3)/2} \in C_{d(d+3)/2}$. Once again let $K_j = \bigcap_{i=1,i\neq j}^{d(d+3)/2} C_i$ and $K = \bigcap_{i=1}^{d(d+3)/2} C_i$. By Lemma 4.6, there is a $j$ such that $E_{\text{max}}$ is the lowest ellipsoid in $K_j$. We will show that $E_{\text{max}}$ lies in every element of $C_j$ for this $j$. 

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Let $C \in \mathcal{C}_j$ be a fixed set from $\mathcal{C}_j$. Suppose that $\mathcal{E}_{\text{max}} \not\subset C$. Then $\mathcal{E}_{\text{max}} \not\subset C \cap K_j$.

By the assumption of Theorem 4.3, $C \cap K_j$ contains an ellipsoid of volume 1, since it is the intersection of a colorful selection of sets. Since $C \cap K_j \subset K_j$, the lowest ellipsoid of $C \cap K_j$ is at least as high as the lowest ellipsoid of $K_j$. But the unique lowest ellipsoid of $K_j$ is $\mathcal{E}_{\text{max}}$, and $\mathcal{E}_{\text{max}} \not\subset C \cap K_j$. So the lowest ellipsoid of $C \cap K_j$ lies higher than $\mathcal{E}_{\text{max}}$. This contradicts that $\mathcal{E}_{\text{max}}$ was chosen to be the highest among the ellipsoids in $B$. So $\mathcal{E}_{\text{max}} \subset C$, which gives us $\mathcal{E}_{\text{max}} \subset \cap C_j$.

The authors learned that this type of proof for a colorful theorem works in a more general setting (De Lorea et al. [9]). If a property $\mathcal{P}$ satisfies certain conditions, then the Helly-type theorem for $\mathcal{P}$ immediately implies a colorful version for this property.

If we take $C_1 = C_2 = \cdots = C_{d(d+3)/2}$ in Theorem 4.3, we obtain the monochromatic version of the theorem. Since the Helly number was optimal in that case, the number of the color classes is optimal in the colorful case.

4.3 Quantitative Colorful Helly

Now we are ready to prove Theorem 3.8. As in the monochromatic case, the proof comes from the combination of Theorem 4.3 and Corollary 2.13.

Proof. Since the intersection of any colorful selection has volume at least 1, by Corollary 2.13 each intersection contains an ellipsoid of volume at least $d^{-d}$. By Theorem 4.3, this implies that there is a color class such that the intersection of all sets in that color class contains an ellipsoid of volume $d^{-d}$. Hence, the volume of the intersection of the sets in the color class is at least $d^{-d}$. \qed
5 Open questions and further work

In this section we collect some open questions related to the topic.

In Theorem 3.6 the highest possible value of $q(d)$ is not known. As we have seen that we obtain a bound if we consider the sphere and its supporting half-spaces. We believe that this is the worst case.

Conjecture 5.1. $q(d) = \frac{\nu_d}{2}$.  

We are quite far from this conjecture. Every proof of Theorem 3.6 uses ellipsoids to approximate the volume, which is usually a very rough approximation.

Our knowledge in the colorful case is even farther from the conjectured optimum. In Theorem 3.8 the number of the color classes is $d(d+3)/2$, which is larger than $2d$, the Helly number of the monochromatic version.

Conjecture 5.2. In Theorem 3.8, the number of the color classes can be decreased to $2d$.

The author is currently working on a method to achieve $cd$ color classes for some absolute constant $c$. As we have seen the number $d(d+3)/2$ came from the fact that the Löwner ellipsoid is determined by $d(d+3)/2$ contact points. Rudelson [22] and Sirastava [23] showed that for any convex body $C$, we can find an approximating body where the Löwner ellipsoid is determined by $cd$ contact points for some constant $c$. The constant only depends on the level of the approximation. The author is trying to combine these results and the methods presented in Section 4, to decrease the number of the color classes.

In [15] a purely arithmetic proof of Theorem 2.11 can be found. They consider the $d(d+3)/2$ dimensional space of ellipsoids, where the theorem follows from Carathéodory theorem. We believe that in the same manner, Theorem 4.3 can be achieved, using the colorful Carathéodory’s theorem.

Helly’s theorem also has a fractional version.

Theorem 5.3 (Fractional Helly). For every dimension $d > 1$ and every $\alpha > 0$ there exists a $\beta = \beta(d, \alpha)$ with the following property. Let $C_1, \ldots, C_n$ be convex sets in $\mathbb{R}^d$, $n > d + 1$, and suppose that for at least $\alpha \binom{n}{d+1}$ of the $(d+1)$-point index sets $I \subset \{1, 2, \ldots, n\}$, we have $\cap_{i \in I} C_i \neq \emptyset$. Then there exists a point contained in at least $\beta n$ sets among the $C_i$. 

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Most of the time when we can create a colorful version of a Helly-type theorem, we can also create a fractional version. It would be interesting to consider a fractional version of Theorem 4.2.
References


