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Constructive Characterization of 2-connected Digraphs

Master's thesis

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Chapter 1

Introduction

A digraph[1] is *k-connected* or *k-vertex-connected* if it has more than k vertices and removing any $k-1$ of them results in a strongly connected[2] digraph on at least two points. This is equivalent to saying that the digraph has k internally vertex-disjoint paths from any vertex to any other vertex.

The main goal of this thesis is to present a constructive characterization for 2-connected digraphs, based on a similar characterization for a family of bipartite graphs called braces.

A *constructive characterization* means building up a family of graphs, in this case the 2-connected digraphs, with relatively simple operations from a nicely describable starting set of graphs. Such a characterization can be useful in proving certain properties of the family: to prove a property, it is enough to show that it holds for the starting set and that it is preserved by the operations.

In order to see this problem more clearly, we will first take a look at known results about similar characterizations.

[1]Directed graph.

[2]There is a directed path from any node to any other.

1.1 The problem

1.1.1. Definition. Let $D = (V, A)$ be a directed graph.

By $D - v$ (for $v \in V$) we mean the following:

$$D - v := \big(V \setminus \{v\},\ \{a \in A:\ \forall u \in V\ a \neq uv \wedge a \neq vu\}\big).$$

Similarly by $D - a$ and $D + a$ (for $a \in A$) we mean:

$$D - a := \big(V,\ A \setminus \{a\}\big) \text{ and } D + a := \big(V,\ A \cup \{a\}\big)$$

Finally for a set H let $H - x := H \setminus \{x\}$ and $H + x := H \cup \{x\}$.

1.1.2. Definition. We say that a digraph $D = (V, A)$ is *strongly connected* if $\forall\, u, v \in V$ there is a directed path in D connecting u to v. In other words D is strongly connected if it contains no one-way cut.

1.1.3. Definition. A digraph $D = (V, A)$ is *k-connected* if $|V| > k$, and $D - v_1 - \ldots - v_l$ is strongly connected for any $l < k$ and $v_i \in V$.

1.1.4. Definition. Let $\mathcal{F}$ be a family of graphs. A *constructive characterization* of $\mathcal{F}$ is a well defined starting set $\mathcal{S} \subset \mathcal{F}$ and a well defined O set of operations for which:

$$\forall G \in \mathcal{F}, \forall o \in O:\ o(G) \in \mathcal{F}$$
$$\text{and}$$
$$\forall G \in \mathcal{F}:\ \exists\, o^1, \ldots, o^l \in O, \exists\, S \in \mathcal{S}:\ G = o^l \circ o^{l-1} \circ \ldots \circ o^1(S).$$

Our goal is to give a constructive characterization for the set of simple 2-connected digraphs. In this context, "simple" means that there are no loops and parallel arcs. However an arc uv and its reversed version vu are both allowed to be arcs of the same graph.

Chapter 2

Similar characterizations

In this chapter we take a look at some constructive characterization theorems for similar graph families. For more one can refer to [10].

2.1 k-edge-connectivity

First let us have a look at the constructive characterization of edge-connected graphs and digraphs. Unlike for vertex-connectivity, the characterization of edge-connectivity is known for both directed and undirected graphs. Results are similar for both cases, but interestingly the description is slightly simpler in the directed case.

Throughout this section we technically work with pseudographs. This means that multiple edges and loops are allowed. For the sake of simplicity the prefix "pseudo" will be omitted.

2.1.1 The directed case

Let us see the problem for digraphs first. The definition of k-arc-connectivity[1] is somewhat similar to the definition of k-connectivity.

2.1.1. Definition. A digraph $D = (V, A)$ is *k-arc-connected* if $D - a_1 - \ldots - a_l$ is strongly connected for any $l < k$ and $a_i \in A$, $(i = 1, \ldots, l)$.

[1] In order to emphasize that the graph in question is directed, we use the term k-arc-connectivity instead of k-edge-connectivity.

2.1.2. Theorem. (W. Mader [7]) *The set of k-arc-connected digraphs is constructively characterizable by $\mathcal{S} = \{K_1\}$ (a single point) and the following operations:*

o_1: Add an arc between two (not necessarily distinct) vertices.

o_2: Take k arcs, subdivide each of them by a new point, and merge these new vertices.

This operation is commonly called pinching, see Figure 2.1.

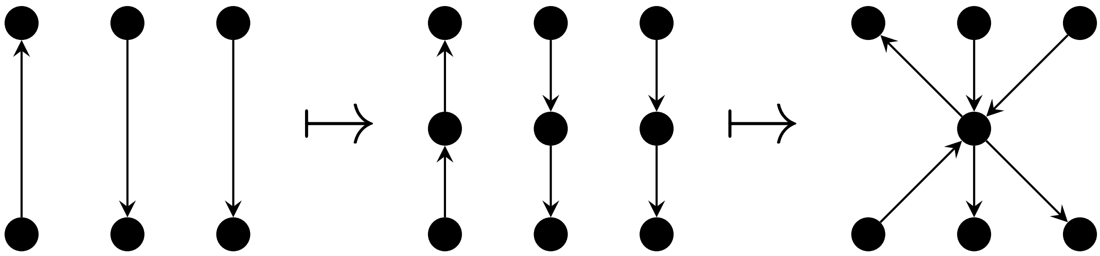

Figure 2.1: Example[2] for directed pinching for $k = 3$.

2.1.2 The undirected case

2.1.3. Definition. A graph $G = (V, E)$ is *k-edge-connected* if $G - e_1 - \ldots - e_l$[3] is connected for any $l < k$ and $e_i \in E$, $(i = 1, \ldots, l)$.

We have to formulate the characterization of k-edge-connected graphs for odd and even values of k separately. The characterization of $2k$-edge-connected graphs is very similar to the characterization of k-arc-connected digraphs.

2.1.4. Theorem. (L. Lovász [5, problem 6.52]) *The set of $2k$-edge-connected graphs is constructively characterizable by $\mathcal{S} = \{K_1\}$ and the following operations:*

[2] Unless stated otherwise, figures show only the relevant parts of the graphs.

[3] $G - e$ and $G - v$ for undirected graphs is analogous to Definition 1.1.1.

o_1: Add an edge between two (not necessarily distinct) vertices.

o_2: Pinch[4] k edges together.

A well known special case of the theorem above is the characterization of 2-edge-connected graphs. Here the pinching reduces to just subdividing an edge with a vertex. This means we can merge o_1 an o_2 into a single operation o_{12}.

o_{12}: Add a path between two (not necessarily distinct) vertices, with all of the internal points being newly created, see Figure 2.2.

These paths in o_{12} are commonly called ears. An ear is called an open ear if its endpoints are different, and closed if its endpoints are the same. $G = o_{12}^l \circ o_{12}^{l-1} \circ \ldots \circ o_{12}^1(K_1)$ is called an ear decomposition of G.

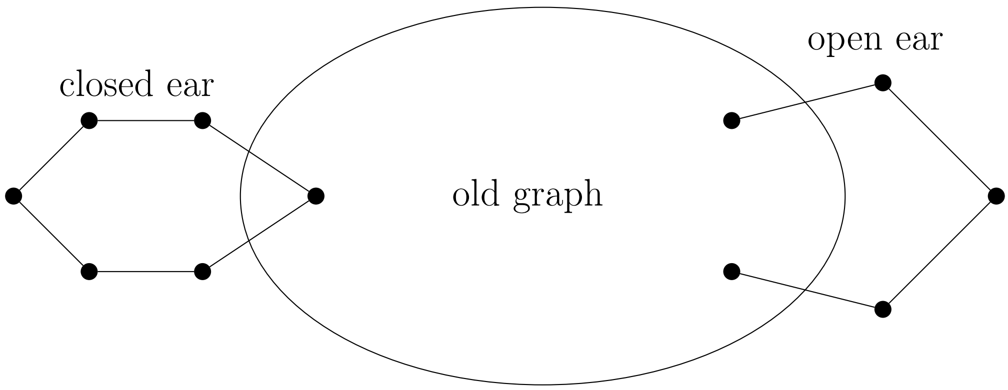

Figure 2.2: Examples for o_{12}.

2.1.5. Theorem. (Mader [6]) *The set of* $2k+1$*-edge-connected graphs is constructively characterizable by* $\mathcal{S} = \{K_1\}$ *and the following operations:*

o_1: Add an edge between two (not necessarily distinct) vertices.

o_2: Pinch k edges together, and add a new edge between an existing vertex and the pinched[5] vertex (Figure 2.3).

[4]Like above this means dividing each edge by a new point, and merging these new vertices.

[5]The new vertex created by the pinching.

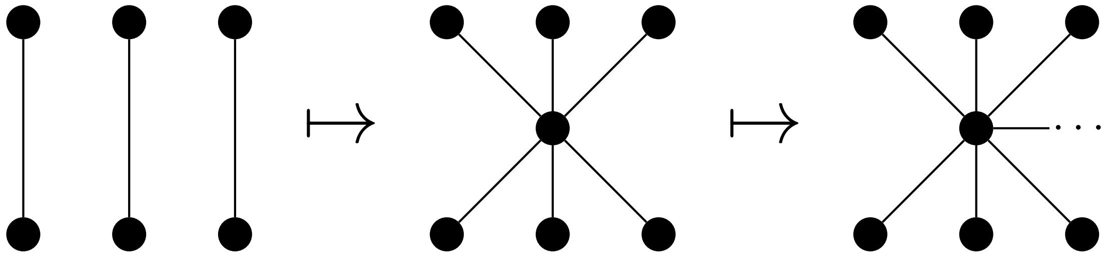

Figure 2.3: Example for o_2 for $k = 3$.

o_3: Pinch k edges together, in the resulting graph pinch together another k edges, and finally add a new edge between the two pinched vertices (Figure 2.4).

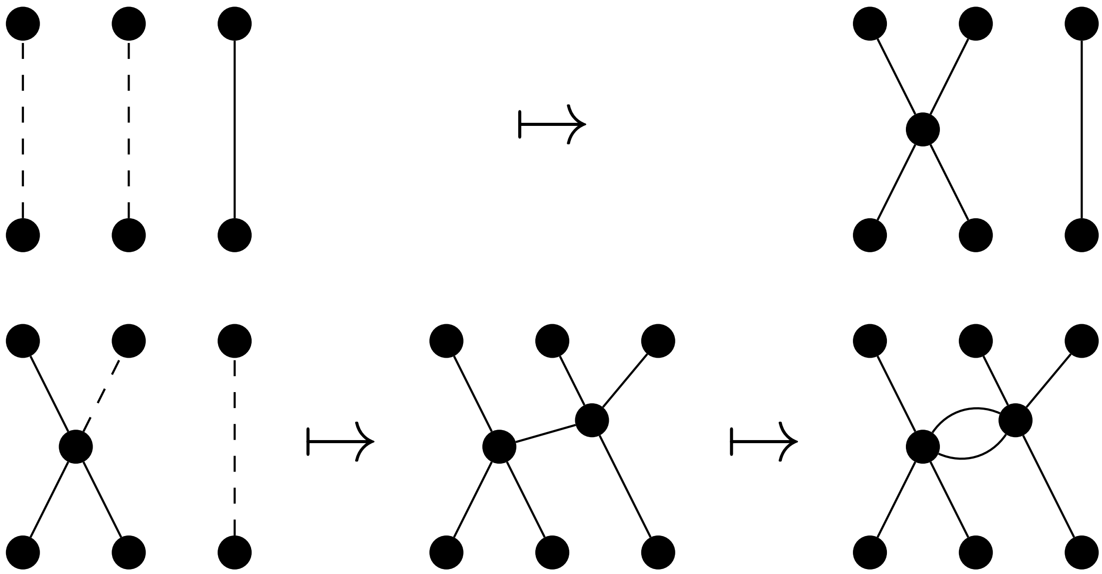

Figure 2.4: Example for o_3 for $k = 2$.

2.2 k-connectivity

2.2.1. Definition. A graph $G = (V, E)$ is *k-connected* if $|V| > k$ and $G - v_1 - \ldots - v_l$ is connected for any $l < k$ and $v_i \in V$ $(i = 1, \ldots, l)$.

This problem is the closest to the one that we would like to consider.

Unlike for k-edge-connectivity, there is no known useful characterization for k-connectivity when $k \geq 4$.

In the first few characterizations presented in this section, graphs with multiple edges are allowed. The term *simple graph* will be used in order to signal when multiple edges are prohibited.

2.2.2. Theorem. (H. Whitney [14]) *The set of* 2*-connected graphs is constructively characterizable by* $\mathcal{S} = \{K_3\}$ *(a complete graph on* 3 *points) and the operation of adding an open ear (see Figure 2.2).*

The decomposition which follows from this theorem is a special case of the ear decompositions mentioned in the previous section. Because only open ears are allowed, ear decompositions such as this are called *open ear decompositions.*

Interestingly there are several different characterizations for $k = 3$.

2.2.3. Theorem. (W. T. Tutte [13]) *The set of* 3*-connected graphs is constructively characterizable by* $\mathcal{S} = \{K_4\}$ *and the following operations:*

o_1: Duplicate an edge.

o_2: For a vertex v with degree at least 4 do the following:

- Split the edges incident to v into two groups such that each group contains at least two nonparallel edges.
- Substitute v with an edge $v'v''$, so that any edge uv from the first group is substituted by an edge uv', and any edge uv from the second group is substituted by an edge uv''.

This operation is often called vertex splitting, see Figure 2.5

This characterization can be used to prove Kuratowski's theorem, additionally it can also be used to prove the following stronger special case:

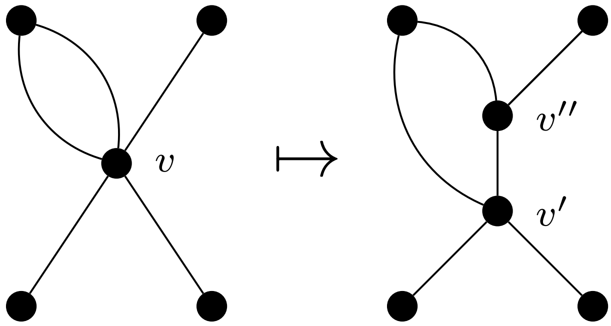

Figure 2.5: Splitting vertex v.

2.2.4. Theorem. (C. Thomassen [12]) *Let G be a simple 3-connected graph. If G does not contain a subdivided K_5 or $K_{3,3}$, then it can be embedded in the plane, so that edges are drawn with straight lines, and the faces are convex.*

The following characterization can be used to give a short proof of Steinitz's Theorem, which states that the graphs of polyhedra are 3-connected planar graphs.

2.2.5. Theorem. (D. W. Barnette and B. Grünbaum [2]) *The set of 3-connected graphs is constructively characterizable by $\mathcal{S} = \{K_4\}$ and the following operations:*

o_1: Add a new edge.

o_2: Subdivide an edge by a new vertex v, and connect v to an other vertex different from its two neighbours.

o_3: Subdivide two nonparallel edges by new vertices u and v, and add the edge uv.

The next characterization is the first presented here which uses a starting set that is infinite. It is similar to the above characterization by Tutte.

2.2.6. Definition. *Wheels* are the graphs of the set $\mathcal{W} = \{W_3, W_4, \ldots\}$ such that $W_n = (V_n, E_n)$ where $V_n := \{v_0, v_1, \ldots, v_n\}$ and:

$$E_k := \{v_i v_{i+1},\, v_{n-1}v_0,\, v_j v_n :\ i, j \in \{1, \ldots, n-1\},\ i \neq n-1\}.$$

It is easy to check that wheels are 3-connected. Also one can see that W_k is the graph of a k-gon based pyramid.

2.2.7. Theorem. (W. T. Tutte [13]) *The set of* 3*-connected simple graphs is constructively characterizable by* $\mathcal{S} = \mathcal{W}$ *and the following operations:*

o_1: Add a new edge.

o_2: Split a vertex.

The last characterization is by Barnette. It is a characterization for 3-connected simple graphs where the starting set is finite.

2.2.8. Definition. Let G be a graph. Let $v \in G$ be a vertex of degree 2. Let u and w be the two neighbours of v. We say that we get G' by *coalescing* the edges uv and vw, if $G' = G - v + uw$.

2.2.9. Definition. A graph G' is obtained from G by *extending* G with vertex v, if we can get G from G' by removing v, and then coalescing any pairs of edges meeting at resulting vertices of degree 2.

In other words we add v, then we may connect it with some existing vertices, and we might subdivide some edges and connect v with the subdividing vertices.

2.2.10. Definition. A *T-triangle* is a triangle whereby at least one of its vertices has exactly 3 neighbours.

2.2.11. Definition. We say that a graph G' is obtained from G by *adding a T-edge*, if G is obtained from G' by removing an edge of a T-triangle opposite to a vertex of degree 3 and then coalescing edges at vertices of degree 2.

2.2.12. Theorem. (D. W. Barnette [1]) *The set of* 3*-connected simple graphs is constructively characterizable by* $\mathcal{S} = \{K_4, K_{3,3}\}$ *and the following operations:*

- o_1*: Extend the graph* G *with a vertex, so that the newly created edges don't all meet the same edge of* G *(Figure 2.6).*
- o_2*: Add a* T*-edge (Figure 2.7).*

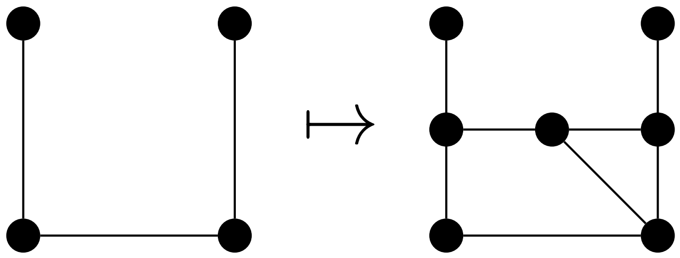

Figure 2.6: Example for o_1.

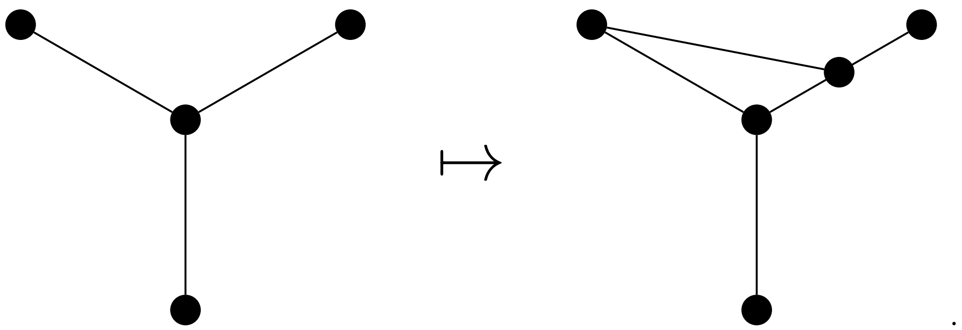

Figure 2.7: Example for adding a T-edge.

Chapter 3

Directed 2-connectivity

In this chapter we present the constructive characterization of braces given by William McCuaig in [9]. This will be translated into a characterization of 2-connected digraphs. From now on we use the word *2-digraph* instead of *2-connected digraph*.

3.1 Braces and 2-digraphs

3.1.1. Definition. A connected (undirected) bipartite graph G on at least 6 vertices is called a *brace* if it has a perfect matching and any matching in G of size 2 can be extended to a perfect matching.

For example, $K_{3,3}$ is a brace on 6 vertices. Braces are also called 2-extendible or 2-elementary bipartite graphs. They are helpful, because they correspond to 2-digraphs.

3.1.1 The correspondence

3.1.2. Lemma. *Let $D = (V, A)$ be a digraph. Let us construct the bipartite graph $G = (V', V'', E)$ in the following way (Figure 3.1):*

$$V' := \{v' : v \in V\},\ V'' := \{v'' : v \in V\}$$
$$E := \{u'u'', v'w'' : u \in V, vw \in A\}.$$

If D is 2-connected then G is a brace.

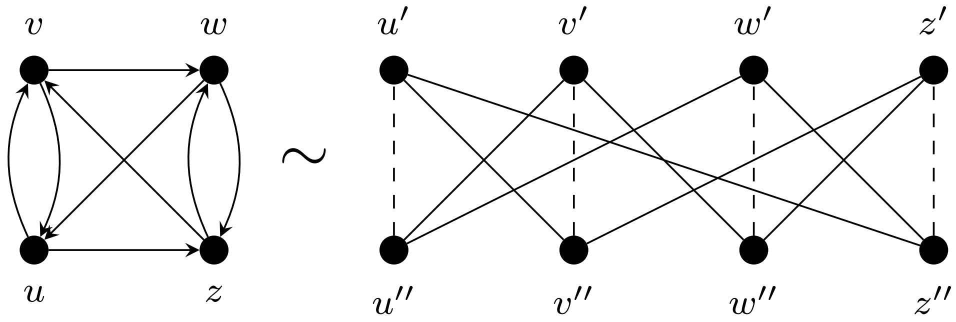

Figure 3.1: The correspondence.

Proof. The connectedness of G follows from the fact that D is strongly connected. Moreover $\{u'u'' : u \in V\}$ is a perfect matching on G. Finally G has at least 6 vertices, since D must have at least 3 vertices to be 2-connected. The only remaining property to check is the extendibility of every matching of size 2.

Assume indirectly that there are 2 independent edges x_1y_1 and x_2y_2 in G, such that there is no perfect matching in $G_{1,2} = (V'_{1,2}, V''_{1,2}, E_{1,2}) := G - x_1 - y_1 - x_2 - y_2$. According to Hall's theorem this means that there is a deficient set $X' \subset V'_{1,2}$ for which the neighbourhood $|N_{G_{1,2}}(X')| < |X'|$.

In G this means that $|N_G(X')| < |X'| + 2$ because $N_G(X')$ may contain y_1 or y_2. Let $X'' := \{v'' \in V'' : v' \in X'\}$. Observe that $X'' \subseteq N_G(X')$, which means that $N_G(X')$ contains at most one vertex w'' outside of X''.

Returning to D, X' corresponds to a set $X \subset V$ for which $|X| \leq \leq |V| - 2$, and all of the arcs leaving X[1] meet in the same vertex w. But this means that $(V \setminus X - w, X)$ is a one-way cut[2] in $D - w$, in other words $D - w$ is not strongly connected, which is a contradiction. □

3.1.3. Lemma. *Let $G = (S, T, E)$ be a bipartite graph with a perfect matching M. Let us construct the digraph $D = (V, A)$ as follows. $V := = M$ and $A := \{(s_1t_1, s_2t_2) : s_1t_2 \in E \setminus M\}$. In other words direct the edges of G from S to T, and then contract the arcs we got from M.*

If G is a brace then D is 2-connected.

[1] If there were no arcs leaving X, that would mean that D is not even strongly connected.

[2] There is no arc going from X to $V \setminus X - w$.

Proof. Assume indirectly that for a brace $G = (S, T, E)$ and a perfect matching M on G, the resulting digraph D is not 2-connected. This means that there is a vertex $v \in V$ for which $D - v$ is not strongly connected. In other words there is a one-way cut (V_1, V_2) in $D - v$.

The vertex v corresponds to an edge $x_1y_1 \in M$. For $i = 1, 2 : V_i$ corresponds to $S_i \subset S$ and $T_i \subset T$ where $|S_i| = |T_i|$. Since (V_1, V_2) is a one-way cut, there is no edge between S_2 and T_1.

Suppose first that there is an edge x_2y_2 between S_1 and T_2 (Figure 3.2). G is a brace thus x_1y_1 and x_2y_2 should be extendable to a perfect matching on G. But in the graph $G_{1,2} := G - x_1 - y_1 - x_2 - y_2$ we have that $N_{G_{1,2}}(S_2) = T_2 - y_2$ therefore $|N_{G_{1,2}}(S_2)| < |S_2|$ which means that there cannot be a perfect matching in $G_{1,2}$. This contradicts our assumption that G is a brace.

It is easy to check that each vertex of a brace has degree 3 or more. Using this and the fact that braces are connected we can show that, if there is no edge between S_1 and T_2, there are two independent edges x_1y_3 and y_1x_3 in G, where $\{x_3, y_3\} \cap (S_1 \cup T_1) = 1$. We can make a similar argument as above with edges x_1y_3 and y_1x_3. $\square$

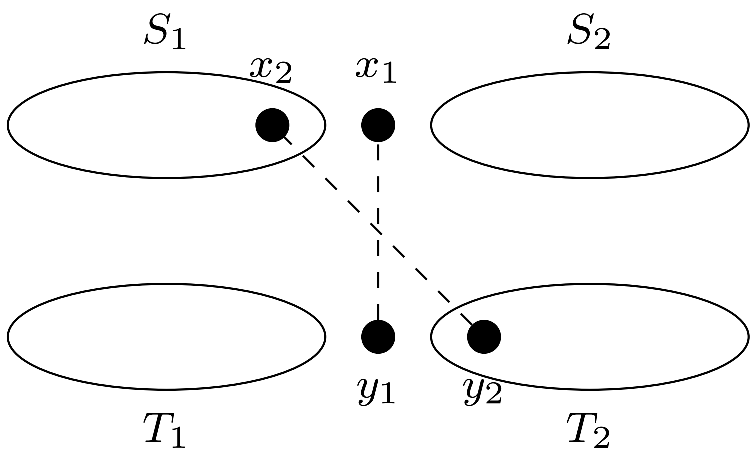

Figure 3.2: The proof when there is an edge between S_1 and T_2.

3.1.4. Definition. Let D be a 2-digraph. Let us denote the corresponding brace as in Lemma 3.1.2 by D^*.

3.1.5. Definition. Let G be a brace. Let us denote the set of corresponding 2-digraphs as in Lemma 3.1.3 by G^*. Let us denote a specific 2-digraph in G^* by $G^*(M, S)$, where M is the matching used in the translation, and S is the first color class of G.

3.1.2 Brace-classes

Therefore there is a correspondence between braces and 2-digraphs. Yet what kind of correspondence is it?

As we have seen, the two directions are not symmetric. From a brace we can get, in theory, many different[3] 2-digraphs (later we will see that this is indeed the case), but if we take a 2-digraph there is only one brace corresponding to it. This means that although there can be more 2-digraphs we can get from a brace, we cannot get the same digraph from two different braces. In other words braces induce a partition of 2-digraphs into equivalence classes in a natural way.

And what is the transformation that, when applied to a 2-digraph, results in an other 2-digraph in the same equivalence class?

If we take a 2-digraph D, get $D^* = (S, T, E)$, change the perfect matching to some M, and then get $D^{**}(M, S)$, we might as well end up with a different 2-digraph in the same equivalence class. We now need to formulate this directly.

3.1.6. Lemma. *Let $D = (V, A)$ be a 2-digraph. Take a cycle C in D.* Blow up *the vertices of C, that is, replace each vertex v with an arc $v'v''$ as follows. For each uv arc entering v replace uv with uv''. For each vw arc leaving v replace vw with $v'w$.*

We get an alternating cycle C' (see Figure 3.3) where the original arcs of C point forward and the newly created arcs point backwards. Contract each arc in C' which originally belonged to C. We get a new graph D'. Let us call this whole process rotating *the cycle C.*

The graph D' belongs to the same brace equivalence class as D.

[3] Non-isomorphic.

Proof. Let the brace $G = (S, T, E)$, and matching M be such that $D = G^*(M, S)$. The cycle C corresponds to a matching M_C on G. Furthermore $M \cup M_C$ contains an alternating cycle $M^C \cup M_C$, with $M^C \subseteq M$. This means that $M' := M_C \cup (M \setminus M^C)$ is a perfect matching on G. Now $D' = G^*(M', S)$, which proves the lemma. □

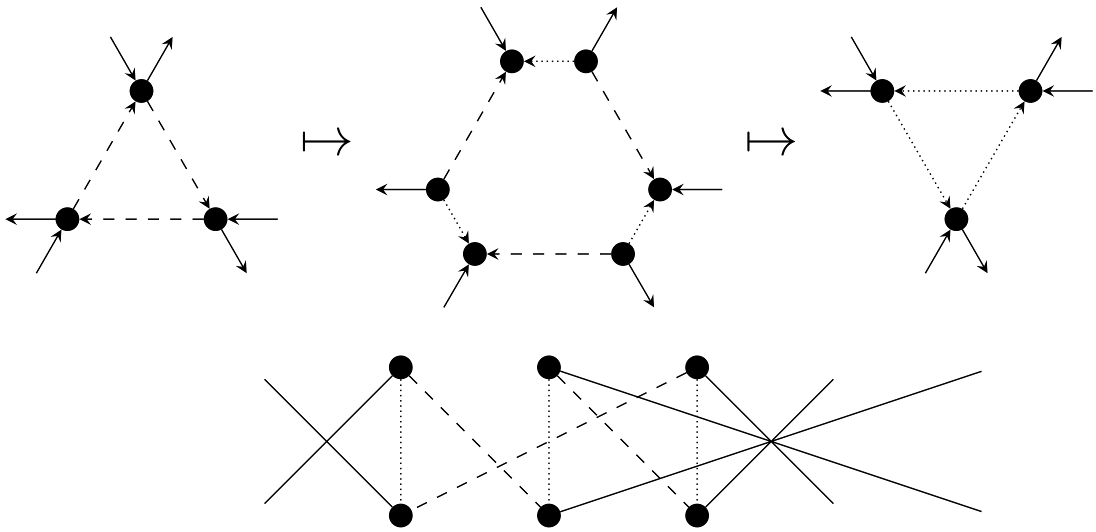

Figure 3.3: Rotating a cycle.

Why must the new arcs during the cycle rotation operation point backwards? This is because when we replace v with $v'v''$, v' corresponds to a vertex in S and v'' corresponds to a vertex in T for some brace $G = (S, T, E)$. This means that v' has to be a source and v'' has to be a sink.

Let $G = (S, T, E)$ be a brace. G and $G' = (T, S, E)$ are isomorphic[4] but even if we fix a perfect matching M, the 2-digraphs $G^*(M, S)$ and $G^*(M, T)$ can be different. This is because flipping the two color classes of G corresponds to reversing D, that is, reversing each arc of D. This means that if D' is the 2-digraph we get by reversing a 2-digraph D, then D and D' belong to the same brace equivalence class.

[4]If we forget about color classes and just consider them as normal graphs $(S \cup T, E)$.

3.1.7. Lemma. *Let D be a 2-digraph. We can get every D' 2-digraph belonging to the same brace-equivalence class as D by reversing D or rotating one or more cycles.*

Proof. Let D and D' be two different brace-equivalent 2-digraphs. Let $G := D^* = D'^*$. We have $D = G^*(M, S)$ and $D = G^*(M', S')$.

If $S \neq S'$ it is clear that the reversal of D yields $G^*(M, S')$.

Suppose now, that $S = S'$ and $M \neq M'$. It is well known that $M \Delta M'$ is the union of disjoint alternating cycles. These cycles tell us which directed cycles to rotate in D. Let C be a cycle in $M \Delta M'$. The edges of $C \cap M'$ correspond to a directed cycle C^* in D. Also $M_C := (M \setminus C) \cup (C \cap M')$ is a perfect matching. If we rotate C^* we get $G^*(M_c, S)$, where $M_C \Delta M'$ contains one less alternating cycle than $M \Delta M'$. After repeating this process enough times, we eventually reach $G^*(M', S)$. □

3.2 Generating braces

Now let us see the constructive characterization of braces given by William McCuaig in [9]. First we need to define some sets of graphs: ladders, Möbius-ladders, and biwheels.

3.2.1. Definition. Define the bipartite graph $LM_{2n} := (S, T, E)$, s.t.:

$$S := \{s_0, s_1, \dots, s_{n-1}\},$$
$$T := \{t_0, t_1, \dots, t_{n-1}\},$$
$$E := \big\{s_i t_i,\ s_i t_{i+1},\ s_i t_{i-1} : i \in \{0, \dots, n-1\}\big\}$$

3.2.2. Definition. *Ladders* are the graphs of the set $\mathcal{L} := \big\{LM_{4k} : k \in \{2, 3, \dots\}\big\}$. Also LM_{4k} is the graph of a $2k$-gon based prism.

The definition of Möbius-ladders is almost the same, but here $n = 2k + 1$.

3.2.3. Definition. *Möbius-ladders* are the graphs of the set $\mathcal{M} := \big\{LM_{4k+2} :\ k \in \{1, 2, \dots\}\big\}$. One can imagine LM_{4k+2} as a ladder with $2k + 1$ steps drawn on a Möbius-strip.

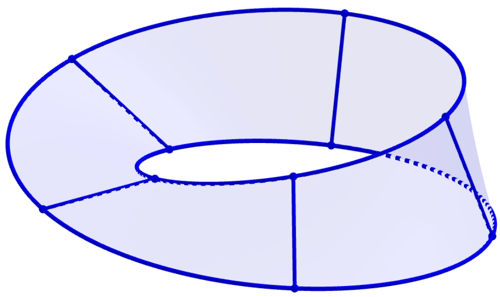

Figure 3.4: M_{10}.

3.2.4. Definition. *Biwheels* are the bipartite graphs of the set $\mathcal{B} = \{B_8, B_{10}, \dots, B_{2n}, \dots\}$ such that $B_{2n} = (S, T, E)$ where:

$$S := \{s_1, s_1, \dots, s_n\},$$
$$T := \{t_1, t_1, \dots, t_n\},$$
$$E := \big\{s_i t_{i+1}, s_{n-1} t_1, s_j t_j, s_j t_n, s_n t_j : i, j \in \{1, \dots, n-1\}, i \neq n-1\big\}.$$

In other words B_{2n} is the graph of the type of polyhedron shown in Figure 3.5.

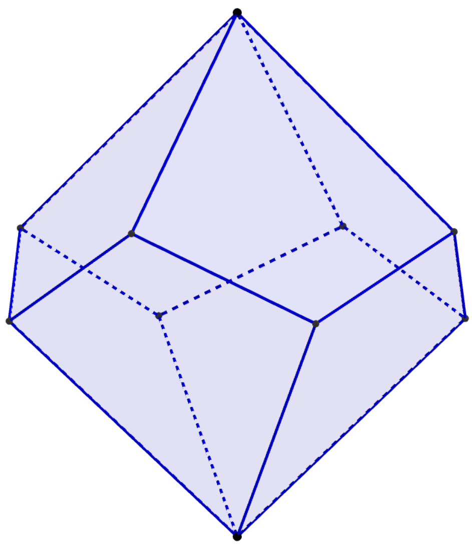

Figure 3.5: B_{10}.

3.2.5. Theorem. (W. McCuaig [9]) *The set of braces is constructively characterizable by* $\mathcal{S} = \mathcal{M} \cup \mathcal{L} \cup \mathcal{B}$ *and the following operations*[5]*:*

o_1: Add a new edge e between S and T.

o_2: Let (U, V) be either (S, T) or (T, S). For a vertex $u_1 \in U$ with degree at least 4, do the following:

- Split the edges incident to u_1 into two groups so that each group contains at least two edges.
- Substitute u_1 with a path $u_1' v u_1''$ so that any edge xu_1 from the first group is substituted by an edge xu_1', and any edge yu_1 from the second group is substituted by an edge yu_1''. Let us call this bipartite vertex splitting[6].
- Connect v to an other vertex $u_2 \in U$, see Figure 3.6.

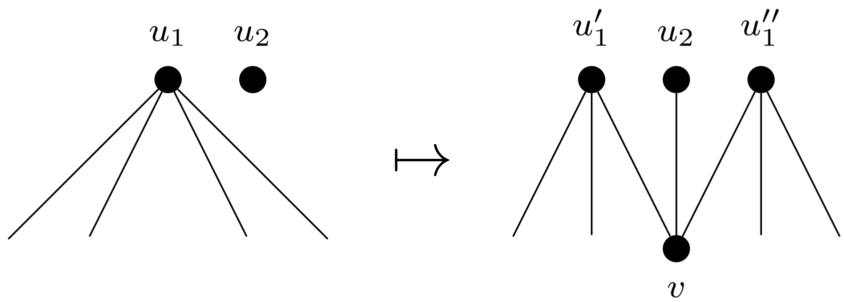

Figure 3.6: o_2: bipartite vertex splitting and adding edge $u_2 v$.

o_3: For vertices $s \in S$ and $t \in T$ both with degree at least 4 do the following:

- Perform bipartite vertex splitting on s into $s' u s''$ and on t into $t' v t''$.
- Connect u and v.

[5]Let the general brace be in the form of $G = (S, T, E)$.

[6]In the article it is called vertex expansion.

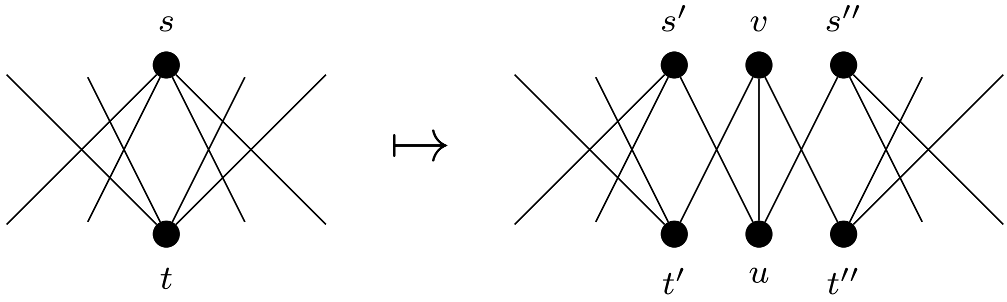

Figure 3.7: Example for o_3.

3.3 Generating 2-digraphs

Now let us take a look at how all this could be translated onto 2-digraphs.

3.3.1 The structure on small examples

In order to see things more clearly, first let us look at the correspondence on small instances. Figure 3.8 shows all the 2-digraphs on at most $n = 5$ vertices that we can get from almost[7] all the braces on at most $2n = 10$ nodes.

For $n = 3$ there is not much going on. We only have the brace $M_6 = = K_{3,3}$, and the corresponding doubly directed triangle. We cannot do any of the operations yet, because $K_{3,3}$ is a complete bipartite graph, and o_2 and o_3 need a vertex with degree at least 4.

For $n = 4$ we start with the brace $B_8 = L_8$, which is the graph of a cube. There are two corresponding 2-digraphs. We cannot perform o_2 or o_3 on the cube, we need to add an edge first. Luckily the only edge we can add is a main diagonal. Now we can perform o_2 to get to the next layer. There are two non-isomorphic braces we can get to this way. Let us denote them with G_1 and G_2.

For $n = 5$ we have braces B_{10} and M_{10} and the braces we can get to from B_8. There are also other braces we can get to by adding new edges.

[7] Braces we can get by only adding new edges are excluded.

One would think that adding an edge to a brace would simply translate to adding an arc to the corresponding 2-digraph, but sadly this is not the case. As we can see on Figure 3.8, we can get G_1 from M_{10} by adding an edge, but we cannot get all of the corresponding 2-digraphs by adding an arc. This is because although a new brace edge directly translates to an arc in the corresponding 2-digraph, it can potentially be used in a new perfect matching. Consequently all we can say is that a new brace edge translates to 2-digraphs, which we can get by adding a new arc and maybe applying some brace-equivalent operations.

3.3.2 The starting set

We need to define the analogue of ladders, Möbius-ladders, and bi-wheels.

3.3.1. Definition. Let us call the set of all 2-digraphs which correspond to a graph in an $\mathcal{X}$ set of braces the *dual* of $\mathcal{X}$, and denote it by $\mathcal{X}^*$.

We could use $\mathcal{S}^* = \mathcal{L}^* \cup \mathcal{M}^* \cup \mathcal{B}^*$ as a starting set, but we can reduce the size of this if we include the brace-equivalent operations.

3.3.2. Definition. Let *double cycles* be the 2-digraphs of the set $\mathcal{DC} = \{DC_3, DC_4, \ldots\}$, where $DC_n = (V, A)$, such that:

$$V := \{v_0, v_1, \ldots, v_{n-1}\},$$
$$A := \big\{v_i v_{i+1}, v_{i+1} v_i : i \in \{0, \ldots, n-1\}\big\}.$$

In other words DC_n is a doubly directed cycle.

Observe that $\mathcal{DC} \subset \mathcal{L}^* \cup \mathcal{M}^*$. Moreover each brace-class from $\mathcal{L}^* \cup \mathcal{M}^*$ is represented exactly once in $\mathcal{DC}$.

Directly defining a representant for each class in $\mathcal{B}^*$ is not so elegant, but it is doable.

3.3.3. Definition. Let $\mathcal{B}_0^* := \{B_4^*, B_5^*, \ldots\}$, where $B_n^* = (V, A)$, such that $V := \{v_1, v_2, \ldots, v_n\}$, and:

$$A := \big\{v_1 v_n, v_i v_{i+1}, v_j v_1, v_n v_j : i, j \in \{1, \ldots, n-1\}, j \neq 1\big\}$$

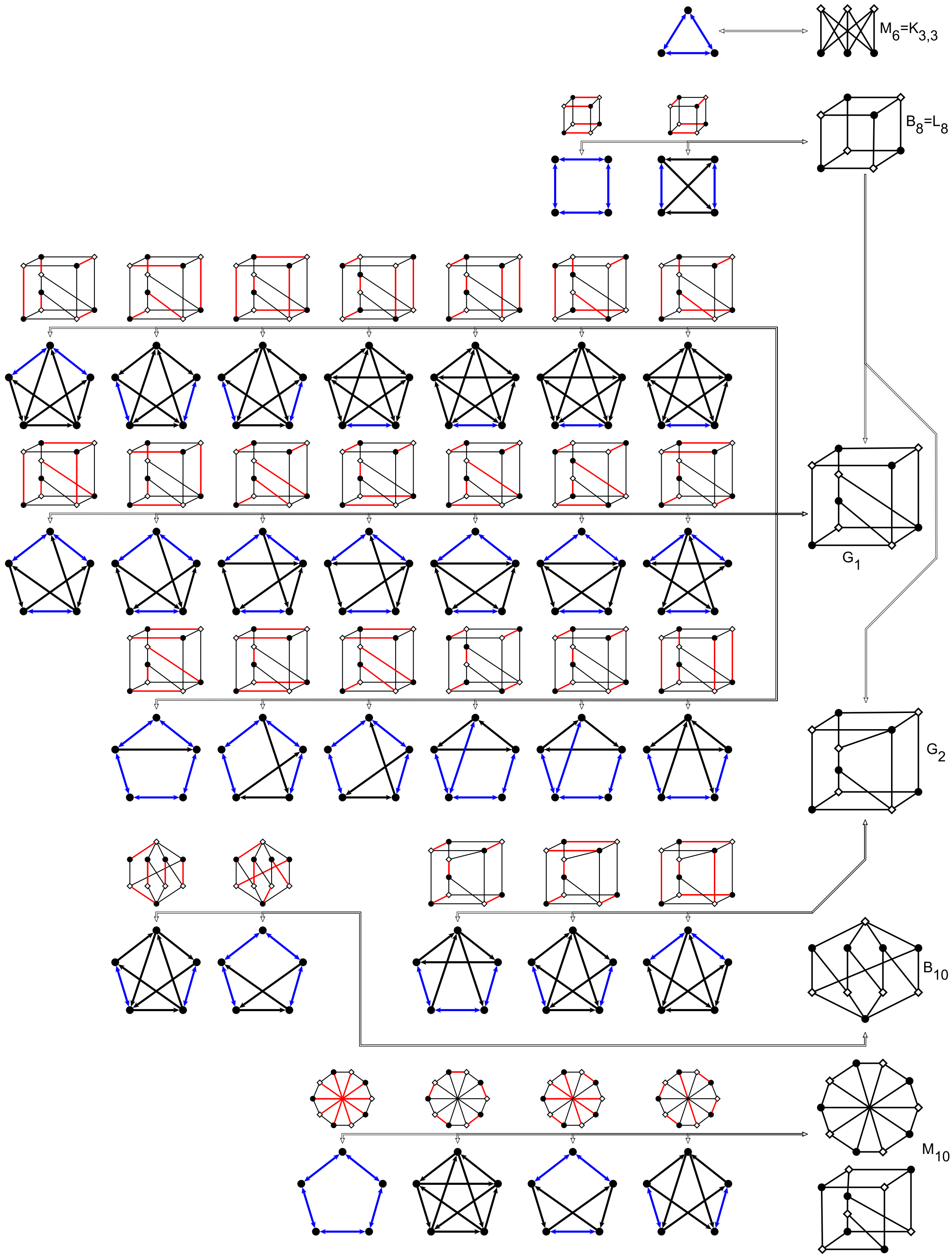

Figure 3.8: The characterization until $n = 5$.

The set $\mathcal{B}_0^*$ is constructed so, that each brace-class from $\mathcal{B}^*$ is represented exactly once in $\mathcal{B}_0^*$.

3.3.3 The characterization

3.3.4. Theorem. *The set of* 2*-digraphs is constructively characterizable by* $\mathcal{S^*}_0 = \mathcal{DC} \cup \mathcal{B}_0^*$ *and the following operations:*

o_0^*: Perform a brace-equivalent operation, that is, rotate a cycle or reverse all the arcs.

o_1^*: Add a new arc.

o_2^*: For a vertex v with in-degree at least 3, do what we call *splitting vertex* v *inward* creating v':

- Split the arcs entering v into two groups, so that the first group contains at least one element and the second group contains at least two elements.
- Create a new vertex v' and replace each uv arc in the second group with uv'.
- Add the arc $v'v$.

After the splitting add the arc $v'w$ where w is different from v.

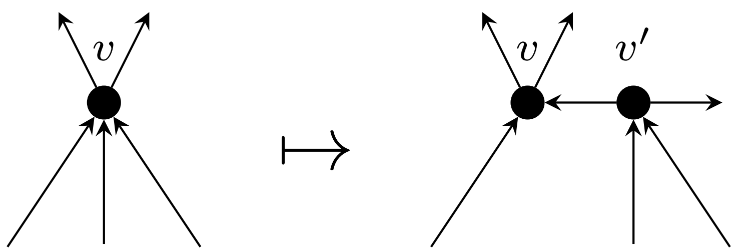

Figure 3.9: Inward vertex splitting and adding $v'w$.

$o_2^{*\prime}$: For a vertex v with out-degree at least 3, do the the analogue of o_2^*, where we call the analogue of inward vertex splitting *outward vertex splitting*.

This operation could be omitted thanks to o_0^*, but its inclusion will be useful later.

o_3^*: For vertex v_1 with in-degree at least 3 and v_2 with out-degree at least 3, where $v_1 = v_2$ is allowed, do the following:

- Split the vertex v_1 inward creating v_1'.
- Split the vertex v_2 outward creating v_2'.
- Add the arc $v_1'v_2'$.

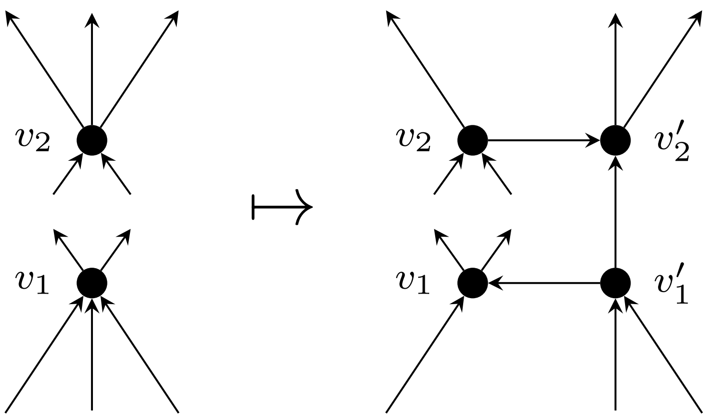

Figure 3.10: o_3^*.

Proof.

3.3.5. Lemma. *Operations $o_0^* - o_3^*$ keep 2-connectivity.*

Proof. Although this can be proved with the help of braces and Theorem 3.2.5, we prove it directly.

Reversing all the arcs keeps 2-connectivity, because all of the cuts get reversed, which means that no one-way cut can emerge.

Let us say that we got the digraph D' by rotating a directed cycle C to C'. The 2-connectivity of D' also follows from Lemma 3.1.6, but let us see the outline of an other proof.

We need to prove that if we remove any u vertex from D' then $D' - u$ is strongly connected. If $u \notin V(C')$ then $D' - u$ is obviously strongly connected. Let us suppose indirectly, that there is an $u \in V(C')$, such that $D' - u$ is not strongly connected. This means that there is a (V_1', V_2') one-way cut in $D' - u$. There is precisely one a arc in C' leaving V_1' and entering V_2'. Observe that if we rotate C' we get back to D. But on the other hand, if we leave out the v_a vertex corresponding to a we get a one-way cut in $D - v_a$.

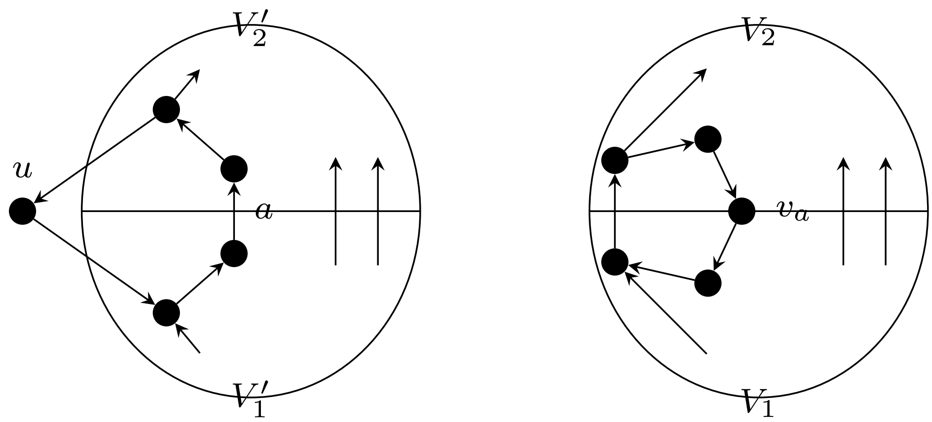

Figure 3.11: Cycle rotation keeps 2-connectivity.

Adding an arc obviously keeps 2-connectivity.

Let us say now that we got the digraph D' by performing o_2^* on D. Let v, v', and w be the same as in the definition of o_2^*. We need to prove that if we remove any u vertex from D' then $D' - u$ is strongly connected.

If $u = v$, then $D' - v - v' = D - v$, which means it is strongly connected. Also there are arcs leaving and entering v' in $D' - v$, which means that $D' - v$ is strongly connected too. We can make a similar argument if $u = v'$.

Even if u is another arbitrary vertex, v' has an y in-neighbour in $D' - u$ different from u (Figure 3.12). Let x be an arbitrary vertex in $D' - u - v - v' - y$. It is enough to show that we can reach x from v and

y from x, because we have the arcs yv' and $v'v$. Since $D - u$ is strongly connected, there is a path $p(z, z')$ for every $z, z' \in V(D - u)$. We can reach x from v, because $p(v, x) \subset D' - u$. We can also reach y from x, because $p(x, y) \subset D' - u$, or $p(x, y)$ meets v and we can modify $p(x, y)$ so that it uses v' too and it lies within $D' - u$.

We can make a similar argument for $o_2^{*\prime}$.

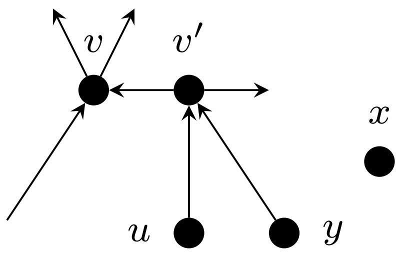

Figure 3.12: o_2^* keeps 2-connectivity.

Finally let us say that we got the digraph D' by performing o_3^* on D. Let us call an inward or outward vertex splitting a directed vertex splitting. Let v_1, v_1', v_2 and v_2' be the same as in the definition of o_3^*. We need to prove that if we remove any u vertex from D' then $D' - u$ is strongly connected.

Using the simple observation that directed vertex splitting keeps strong connectivity, we can show similarly as above, that $D' - u$ is strongly connected if $u \in \{v_1, v_1', v_2, v_2'\}$. If $u \notin \{v_1, v_1', v_2, v_2\}$ we can also make a similar argument as above, but now we need to show that we can reach x from v_1, v_2 from x, and y from z, where $y \neq u$ is an in-neighbour of v_1', $z \neq u$ is an out-neighbour of v_2', and x is an arbitrary node in $D' - u - v_1 - v_1' - v_2 - v_2' - y - z$ (Figure 3.13).

□

3.3.6. Lemma. *Let D be a 2-digraph. We can generate D from ${\mathcal{S}^*}_0$ using operations $o_0^* - o_4^*$.*

Proof. We prove this by induction on the number of steps needed to generate D^*[8] according to Theorem 3.2.5.

[8] D^* is the brace corresponding to D as in Definition 3.1.4

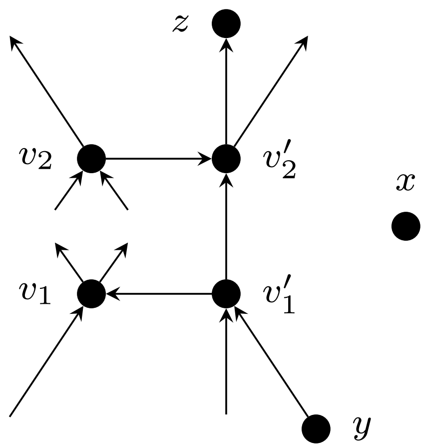

Figure 3.13: o_3^* keeps 2-connectivity.

If $D^* \in \mathcal{S} = \mathcal{M} \cup \mathcal{L} \cup \mathcal{B}$, then the number of brace generation steps needed is 0. Remember that we specifically chose the elements of ${\mathcal{S}^*}_0$, so that each brace-equivalence class in $\mathcal{S}^*$ is represented once in ${\mathcal{S}^*}_0$. Consequently in this case $D \in {\mathcal{S}^*}_0$, or we can get D form ${\mathcal{S}^*}_0$ by applying o_0^* a finite number of times.

Let us suppose we can generate every 2-digraph D, where the number of brace-generation steps needed to generate D^* is at most n. Let D_{n+1} be a 2-digraph such that we need $n+1$ brace-generation steps to generate $G_{n+1} := D_{n+1}^*$. We need to show that we can generate D_{n+1}.

Let o^{n+1} be the last operation in the generation of G_{n+1}. Let G_n be the brace we were at, before we applied o^{n+1} in the generation of G_{n+1}. Let $D_n = G_n^*(M, S)$ for some perfect matching M and an S color class of G_n, that is, D_n is a 2-digraph corresponding to the brace G_n. It suffices to show that $o_i^*(D_n) \in G_{n+1}^*$ for some $i \in \{1, 2, 3\}$, see Figure 3.14.

If $o^{n+1} = o_1$ then $G_{n+1} = G_n + e$, which directly corresponds to adding an arc to D_n.

Suppose now that $o^{n+1} = o_2$. Using the same notation as in Theorem 3.2.5, we can say that v and either u_1' or u_1'' (let us take u_1') are not covered by M. It is easy to check that we can get $G_{n+1}^*(M + u_1'v, S)$ with performing o_2^* or $o_2^{*\prime}$ on the vertex corresponding to the edge in M which covers u_2'.

Finally suppose that $o^{n+1} = o_3$. Using the same notation as in Theorem 3.2.5, we can say without loss of generality that u, v, s', and t' are not covered by M. It is easy to check that we can get $G^*_{n+1}(M + us' + + vt', S)$ with performing o^*_3 on the vertices corresponding to the edges in M[9] which cover s'' and t''. □

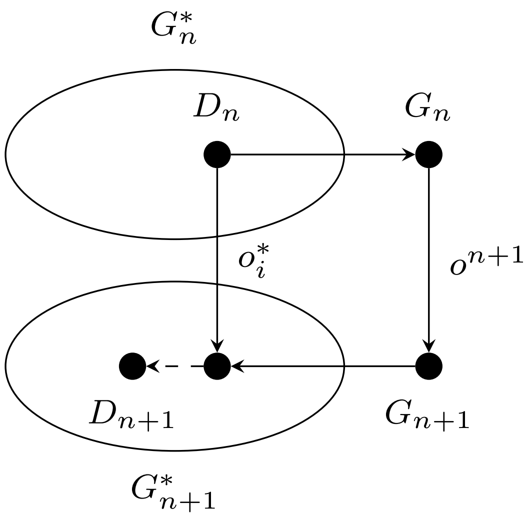

Figure 3.14: The outline of the proof.

Since the digraphs in S^*_0 are 2-connected, we had to show that the operations keep the 2-connectivity, and that every 2-digraph can be generated. □

3.3.7. Remark. We can strengthen the characterization given in Theorem 3.3.4 in the following sense. We can require the construction to be executed in two phases. In the first phase we only allow operations $o^*_1 - o^*_3$. In the second phase we only allow brace-equivalent operations.

Proof. Let us suppose indirectly that for a 2-digraph D there is no construction in the required form. Let us divide the generation of D into layers, such that operations $o^*_1 - o^*_3$ step into the next layer, but o^*_0 stays within the current layer. Let l be the index of the first layer in which we use o^*_0. Let l be maximal. According to our assumption l is not the index

[9]It is possible that $s''t'' \in M$, but this is not a problem, since $v_1 = v_2$ is allowed in operation o^*_3.

of the last layer. We will create another construction of D for which the index of the first layer in which we use o_0^* is larger than l (Figure 3.15). This will be a contradiction.

Let D_l be the 2-digraph which we got when we first stepped into layer l. Let o be the operation we did when we left layer l. Operation o was performed on some 2-digraph D_l' different from D_l.

Let p be the operation that if performed on D_l, it corresponds to the same brace-operation as o. The 2-digraph $D_{l+1}' := p(D_l)$ is in the same brace-class as $D_{l+1} := o(D_l')$. This means that if we perform p on D_l, and then perform some brace-equivalent operations, we can reach D_{l+1} without having to use o_0^* in layer l. □

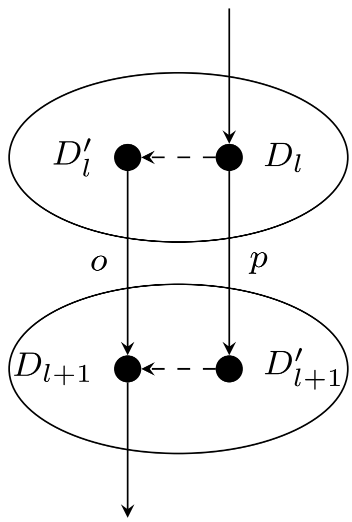

Figure 3.15: The outline of the proof.

Chapter 4

Open questions

We cannot leave out the brace equivalent operations entirely from the characterization in Theorem 3.3.4, because we need them to generate the members of $\mathcal{S}^*$. But this does not mean that if we used $\mathcal{S}^*$ as the starting set instead of ${\mathcal{S}^*}_0$, than o_0^* could not be omitted. The appeal of using ${\mathcal{S}^*}_0$ instead of $\mathcal{S}^*$ is that ${\mathcal{S}^*}_0$ is much easier to describe. This begs the following question.

4.0.1. Question. Is there such a characterization which has a nice starting set, but only contains operations which increase the number of arcs?

A straightforward question is the case of k-connected digraphs.

4.0.2. Definition. A connected graph G on at least $2k + 2$ vertices is called *k-extendible* or *k-elementary* if it has a perfect matching and any matching in G of size k can be extended to a perfect matching.

Braces are 2-extendible bipartite graphs. It can be shown that k-extendible bipartite graphs correspond to k-digraphs, in the same way that braces correspond to 2-digraphs in Lemmas 3.1.2 and 3.1.3. This means that if there was a characterization for k-extendible bipartite graphs, it cloud likely be translated into a characterization for k-digraphs.

4.0.3. Question. Is there a constructive characterization for k-extendible bipartite graphs similar to Theorem 3.2.5?

It can be shown, that every k-extendible graph is $k+1$-connected. This is a bit discouraging because there is no known useful characterization for 4-connected graphs. But 3-extendible digraphs are only a subset of 4-connected graphs, so maybe it is easier to characterize them.

Bibliography

[1] D. W. Barnette. Construction of 3-connected graphs. *Israel Journal of Mathematics*, 86:397–407, 1994. doi:10.1007/BF02773688.

[2] David W. Barnette and Branko Grünbaum. On steinitz's theorem concerning convex 3-polytopes and on some properties of planar graphs. In *The Many Facets of Graph Theory*, pages 27–40. Springer, Berlin, Heidelberg, 1969. doi:10.1007/BFb0060102.

[3] Joseph Cheriyan and S. N. Maheshwari. Finding nonseparating induced cycles and independent spanning trees in 3-connected graphs. *Journal of Algorithms*, 9:507–537, 1988. doi:10.1016/0196-6774(88)90015-6.

[4] András Frank. *Gráfelmélet*. 2014. URL: http://web.cs.elte.hu/~frank/jegyzet/graf/graf.2014.pdf.

[5] László Lovász. *Combinatorial Problems and Exercises*. Akadémiai Kiadó - North Holland, 1979.

[6] W. Mader. A reduction method for edge-connectivity in graphs. *Annals of Discrete Mathematics*, 3:145–164, 1978. doi:10.1016/S0167-5060(08)70504-1.

[7] W. Mader. Konstruktion aller n-fach kantenzusammenhängenden digraphen. *Europ. J. Combinatorics*, 3:63–67, 1982. URL: https://core.ac.uk/download/pdf/82371283.pdf.

[8] W. Mader. On vertices of outdegree n in minimally n-connected digraphs. *Journal of Graph Theory*, 39(2):129–144, 2002. doi:10.1002/jgt.10018.

[9] William McCuaig. Brace generation. *Journal of Graph Theory*, 38(3):124–169, 2001. doi:10.1002/jgt.1029.

[10] Erika R. Bérczi-Kovács and László A. Végh. Constructive characterization theorems in combinatorial optimization. *RIMS Kôkyûroku Bessatsu*, B23:147–169, 2010.

[11] H. E. Robbins. A theorem on graphs, with an application to a problem of traffic control. *The American Mathematical Monthly*, 46(5):281–283, 1939. doi:10.2307/2303897.

[12] Carsten Thomassen. Kuratowski's theorem. *Journal of Graph Theory*, 5(3):225–241, 1981. doi:10.1002/jgt.3190050304.

[13] W. T. Tutte. A theory of 3-connected graphs. *Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen. Series A: Mathematical sciences*, 64:441–455, 1961. URL: https://core.ac.uk/download/pdf/82624113.pdf.

[14] Hassler Whitney. Non-separable and planar graphs. *Transactions of the American Mathematical Society*, 34:339–362, 1932. doi:10.1090/S0002-9947-1932-1501641-2.