Colouring bottomless rectangles and arborescences

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Abstract

We study problems related to colouring bottomless rectangles. Our main result shows that for any m, there is no semi-online algorithm that can colour bottomless rectangles from below with a bounded number of colours so that any m-fold covered point is covered by at least two colours. This is, surprisingly, a corollary of a stronger result for arborescence colourings. Any semi-online colouring algorithm that colours an arborescence in leaf-to-root order with a bounded number of colours produces arbitrarily long monochromatic paths. This is complemented by optimal upper bounds given by simple colouring algorithms from other direction. We present an improved lower bound for the *polychromatic colouring number*, $m_k \geq 2k - 1$.

1 Introduction

The systematic study of polychromatic colourings and cover-decomposition of geometric ranges was initiated by Pach over 30 years ago [15, 16]. The field has gained popularity in the new millennium, with several breakthrough results; for a (slightly outdated) survey, see [17], or see the up-to-date interactive webpage `http://coge.elte.hu/cogezoo.html` (maintained by Keszegh and Pálvölgyi).

A family of geometric regions $\mathcal{F}$ and a point set P in some $\mathbb{R}^d$ naturally define a hypergraph, $H(P, \mathcal{F})$. The vertex set of H is the points of P, and the edges are the sets $P' \subset P$ such that for some $F' \in \mathcal{F}$, $F' \cap P = P'$. We are interested in the dual hypergraph, $H(\mathcal{F}, P)$, on the vertex set $\mathcal{F}$, and $\mathcal{F}' \subset \mathcal{F}$ is an edge if for some $p \in P$, p is covered by exactly the regions in $\mathcal{F}'$. We are primarily interested in the case when $\mathcal{F}$ consists of polygons in $\mathbb{R}^2$.

We can then define the chromatic number $\chi_{\mathcal{F}}$ of any family $\mathcal{F}$ of geometric regions. This is the minimum number of colours needed to colour any finite point set P so that any region containing at least two points contains at least two colours. The dual chromatic number, $\chi^*_{\mathcal{F}}$, is defined analogously as the number of colours needed to properly colour any *finite* subfamily of $\mathcal{F}$. In this paper, we will study the *polychromatic colouring numbers*. The k-th polychromatic colouring number $m_k(\mathcal{F})$ is the smallest number needed to k-colour any finite point set P so that any region containing at least $m_k(\mathcal{F})$ points contains all k colours. This is the *primal* colouring problem. Similarly, its dual $m^*_k(\mathcal{F})$ is the smallest number needed to k-colour any finite subfamily of $\mathcal{F}$ so that any point covered by m^*_k regions is covered by all k colours. This is referred to as the *dual* colouring problem.

The polychromatic colouring problem is partly motivated by the sensor cover problem; given a set of sensors covering an area, can we decompose them into k sets so that any area covered by m^*_k sensors is covered in each of these sets? When $k = 2$, this is called the cover-decomposability problem. In particular, we say a set $P \subset \mathbb{R}^2$ is *cover-decomposable* if when $\mathcal{F}_P$ consists of all *translates* of P, $m^*_2(\mathcal{F}_P) < \infty$. In [16] it was shown that every centrally-symmetric open convex polygon is cover-decomposable, and this was extended to all open convex polygons in [19]. The bound $m^*_k(\mathcal{F}_P) = O(k)$ was proved for any convex polygon P in [7].

The problem becomes more complicated if we consider *homothets* of a convex polygon. For example, if $\mathcal{F}_\square$ denotes the family of axis-parallel squares in the plane, $m_2(\mathcal{F}_\square) \leq 215$ [1]. On the other hand, for any number m there is a family $\mathcal{F}_m$ of axis-parallel squares such that (1) each point in the plane is covered by at least m squares, but (2) any 2-colouring of $\mathcal{F}_m$ produces a point covered by squares of exactly one colour [13]. Furthermore, if $\mathcal{F}_{\sqsubset\!\sqsupset}$ denotes the family of axis-parallel rectangles in the plane, $m_2(\mathcal{F}_{\sqsubset\!\sqsupset}) = \infty$ [5]. Consequently, $m_k(\mathcal{F}_{\sqsubset\!\sqsupset}) = \infty$ for any k. The dual $m^*_k(\mathcal{F}_{\sqsubset\!\sqsupset})$ is infinite as well; there is a constant $C > 0$ such that for any numbers $n \geq r \geq 2$, there is a family of n axis-parallel rectangles for which and any colouring with at most $C \log n (r \log r)^{-1}$ colours produces a point covered by r monochromatic axis-parallel rectangles[18].

This paper focuses on one particular family: *bottomless rectangles*. A subset of $\mathbb{R}^2$ is called a (closed) bottomless rectangle if it is of the form $\{(x, y) : \ell \leq x \leq r, y \leq t\}$. We simply refer to a bottomless rectangle by these paramaters (ℓ, r, t). These range spaces were first defined by Asinowski et al. [2], who showed that for any positive integer k, any finite set of points in $\mathbb{R}^2$ can be k-coloured such that any bottomless rectangle with at least $3k - 2$ points contains all k colors. They also showed that the optimal number that can be written in place of $3k - 2$ in the above statement is at least $1.67k$. In our language, if $\mathcal{F}$ denotes the family of all bottomless rectangles in the plane, $1.67k \leq m_k(\mathcal{F}) \leq 3k - 2$.

Our paper studies the dual problem: we would like to determine the optimal $m_k^*(\mathcal{F})$. About this question much less is known; $m_2^* = 3$ [10] the best upper bound $m_k^* = O(k^{5.09})$ is a corollary of a more general result [4] about *octants* (combined with an improvement of the base case [11] that slightly lowered the exponent). The general conjecture, however, is that $m_k^* = O(k)$ for any family [17]. It was also proved in [4] that there is no semi-online algorithm "from above" for colouring bottomless rectangles. Our main result is a generalisation of this negative statement.

Theorem 1. *For any k and m, and any semi-online algorithm that k-colours bottomless rectangles from below (resp. from above, from the right, or from the left), there is a family of bottomless rectangles such that the algorithm will produce an m-fold covered point that is covered by at most one colour.*

Our proof is much more complicated than the one in [4]; while they use an Erdős-Szekeres type incremental argument [6], we need a certain diagonalisation method. In particular, we reduce the semi-online bottomless rectangle colouring problem to a question about semi-online colourings of arborescences, which is interesting in its own right.

Theorem 2. *For any k and m, and any semi-online colouring algorithm that k-colours the vertices of an arborescence in a leaf-to-root order, there is an arborescence such that the algorithm will produce a directed path of length m that contains at most one colour.*

We apply this theorem to four natural configurations of bottomless rectangles to show that for each configuration, there is a direction from which a semi-online algorithm fails. This is complemented by optimal upper bounds given by online algorithms from the other directions.

In order to avoid writing the superscript * for the entirety of this paper, we will simply refer to this as m_k, and specify the family we are referring to.

In Section 2, we introduce these Erdős-Szekeres configurations, and define the colouring problem for them. In Section 3, we prove Theorem 2 for arborescences. In Section 4, we prove Theorem 1 along with upper bounds for certain families of bottomless rectangles, and improve the lower bound for general families to $m_k \geq 2k - 1$.

2 Erdős-Szekeres configurations

We would like to improve the upper bound $m_k = O(k^{5.09})$ for general families by classifying some configurations of bottomless rectangles, finding a colouring for each configuration, and combining these to obtain a good colouring for general families. To this end, we will use the classical result of Erdős and Szekeres [6] that any sequence of length $(k - 1)^2 + 1$ contains a monotone subsequence of length k.

Recall that we associated to each rectangle its parameters (ℓ, r, t). We refer to ℓ as its left-coordinate, r its right-coordinate, and t its height. Let p be a point covered by $(m - 1)^4 + 1$ rectangles. Ordering these rectangles by left endpoint, we find a subsequence of length $(m - 1)^2 + 1$ whose right endpoints are monotone. Applying the result of Erdős and Szekeres again, we find a (sub)subsequence of length m whose heights are monotone. This proves that any point that is contained in $(m-1)^4 + 1$ bottomless rectangles, is contained in m bottomless rectangles such that each of the three parameters of these m bottomless rectangles are in increasing or decreasing order. We name these configurations, respectively, *increasing/decreasing steps, towers* and *nested rectangles* (see Figure 1).

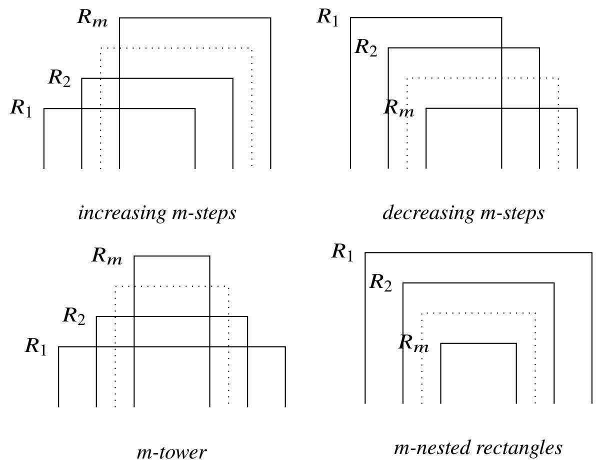

Figure 1: Erdős-Szekeres configurations

We are interested in colouring families with respect to a *fixed* configuration. For example, can we k-colour $\mathcal{F}$ so that any point covered by an m-tower is covered by all k colours? We refer to least such m as m_k *for towers*, and analogously for the other configurations.

Theorem 3. *$m_k = k$ for each fixed configuration.*

A result of Berge [3] shows that for any family $\mathcal{F}$ of geometric regions, $m_2(\mathcal{F}) = 2$ if and only if $m_k(\mathcal{F}) = k$ for all k. It is not hard to show that $m_2 = 2$ for each configuration, and then apply the result of Berge. Nevertheless, it will be valuable to see that there is simple online algorithm for each configuration.

An *online algorithm* is one where the vertices are presented in some order, and the algorithm must colour each vertex received immediately. Such algorithms have a large literature [8, 9, **?**, 14].

Another natural class to consider is that of *semi-online algorithms*. These algorithms need not colour a vertex immediately when it is presented, but once they colour a vertex, they cannot recolour it later. Moreover, at any stage the partial colouring must be such that for each hyperedge whose vertices have all appeared, the conditions of the colouring are satisfied. For example, in the case of proper colourings, when the last vertex of some hyperedge $\mathcal{E}$ is presented, the algorithm must colour the vertices so that at least two colours appear among the vertices of $\mathcal{E}$.

Proof. We first present a colouring algorithm for towers. We colour the rectangles in increasing order of height, i.e. from below, so that at every step the following property holds.

(*) If a point p is covered by a j-tower for $j \leq k$, then p is covered by at least j different colours.

At step 1, colour the rectangle of least height arbitrarily.

Suppose the first $t-1$ rectangles have been coloured so that (*) holds. We colour the rectangle R_t as follows.

For each $1 \leq i \leq k$, let y_i be the largest number so that if $p \in R_t$ has y-coordinate less than y_i, then p is covered by colour i. (This corresponds to a tallest rectangle S of colour i such that (S, R_t) is a tower.) If y_i does not exist for some colour i, colour R_t with colour i. Otherwise, suppose $y_1 > \cdots > y_k$, and colour R_t with colour k.

To see that (*) holds, let p be contained in a j-tower $R_1, \dots, R_{j-1}, R_t$. Then p is covered by at least $j-1$ colours by hypothesis, so at least $j-1$ values of y_i exist, and we coloured R_t with a different colour.

We use the same algorithm to colour k-nested sets, only we colour the rectangles from above. It is easy to check that with this ordering, the same property holds.

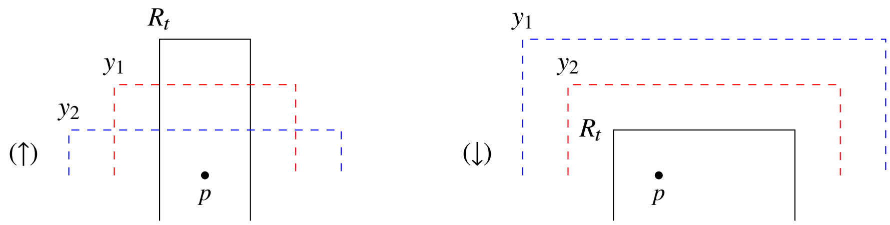

Figure 2: Colouring algorithms for k-towers and k-nested sets respectively

The algorithm for increasing k-steps is only slightly different. We colour the rectangles in decreasing order of right endpoint (from the right). At step t, for $1 \le i \le k$, let x_i be the least number so that if $p \in R_t$ has x-coordinate greater than x_i, then p is covered by a rectangle of colour i (corresponding to the leftmost rectangle S of colour i such that (R_t, S) form increasing steps). As earlier, if some x_i does not exist, give R_t colour i. Otherwise, if $x_1 < \cdots < x_k$, give R_t colour k.

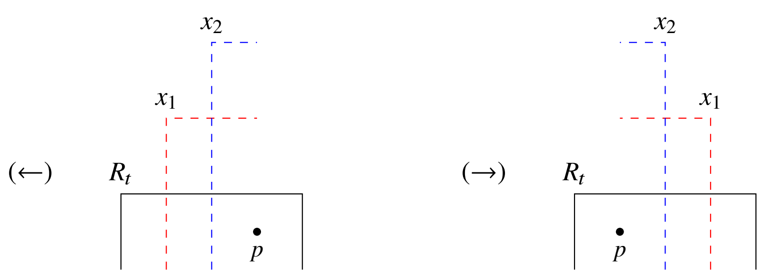

Figure 3: Colouring algorithms for increasing and decreasing k-steps respectively

□

Note that ordering a tower in increasing order of height (from below) is the same as ordering it in decreasing order of right endpoint (from the right), or increasing order of left endpoint (from the left). We may repeat the same algorithm for towers, colouring the rectangles from the left (or right) and we will still obtain a good colouring. Similarly, ordering a nested set from above is the same as ordering it from the left or right. Ordering increasing steps from the right (resp. decreasing steps from the left) is the same as ordering them from above.

	left (→)	right (←)	below (↑)	above (↓)
inc. steps	∞	$= k$	∞	$= k$
dec. steps	$= k$	∞	∞	$= k$
towers	$= k$	$= k$	$= k$	∞
nested	$= k$	$= k$	∞	$= k$

Table 1: m_k values for each configuration given by semi-online algorithms from different directions

The value ∞ indicates the non-existence of semi-online colouring algorithms, which we prove in the next section.

3 Arborescences

We would like to associate to each family $\mathcal{F}$ of rectangles a simple graph, and derive a polychromatic colouring of $\mathcal{F}$ from a suitable colouring of this graph. First we will define the family of graphs that we consider (arborescences), prove Theorem 2, then show how these graphs are obtained from bottomless rectangles.

An arborescence is a directed tree with a distinguished *root* vertex such that all edges are directed away from the root, i.e. there is a unique directed path from the root to any vertex. We denote the length of the shortest directed path from u to v, if it exists, by $dist(u, v)$. Recall that the length of a path is the number of edges, or one less than the number of vertices. A disjoint union of arborescences is called a *branching*. We say that an ordering of the vertices of a branching is *root-to-leaf* if every vertex is preceded by its in-neighbors and succeeded by its out-neighbors; in particular, from every component first the root is presented and last a leaf.

Claim 1. *The vertices of any branching can be k-coloured by an online algorithm in a root-to-leaf order such that any directed path on k vertices contains all k colours.*

Proof. If a root is presented, colour it with colour 1. Every time a new vertex v is presented in the component with root r, colour v according to the parity of $dist(v, r) \mod k$ (which can be determined from a root-to-leaf ordering). □

We call the reversal of a root-to-leaf ordering a *leaf-to-root* ordering; from each component, first a leaf is presented and last the root. Our main result, Theorem 2, shows that the converse of the above claim fails: any semi-online algorithm will in fact leave arbitrarily long monochromatic paths. In order to apply this result to bottomless rectangles, however, we will need a stronger condition on the leaf-to-root ordering.

For two vertices u and v of a branching, say $u < v$ if they are in the same component and there is a directed path from u to v. This forms a partial order where the roots are the minimal elements and the leaves the maximal. A leaf-to-root ordering is a linear extension of this partial order that presents the $<$-maximal element first.

If $u < v$ and there are no other vertices between them, i.e., uv is a directed edge, write $u \lessdot v$ and say that v is the *parent* of u. (Thus, somewhat contradicting the laws of nature, every vertex can have only one child, but several parents.) When presenting the vertices of a branching in a leaf-to-root order, the newly presented vertex u will always form a root, while its parents were all roots of the branching before u was presented.

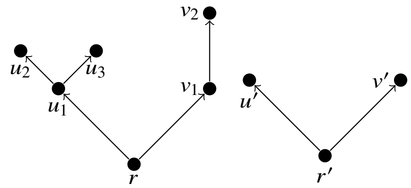

Figure 4: A branching with roots r and r'. In this example, $u_1 \gtrdot r$, i.e. u_1 is a parent of r, but u_2 is not a parent of r even though $u_2 > r$ (u_2 is a "grandparent" of r), and $v' \not> r$. A linear extension of this (or a leaf-to-root ordering) might present the vertices u', v' and r' before u_3, so it is not necessary that the roots of the branching are the last vertices presented.

Denote the roots of the branching before a new vertex u is presented by $v_1, v_2, \ldots$ indexed in the order in which they were presented. We say that a *leaf-to-root* ordering is *geometric* if the parents of u form an interval in this order, i.e., for every u, $\{v_i \mid u \lessdot v_i\} = \{v_i \mid \ell < i < r\}$ for some ℓ and r.

Now we state a stronger form of Theorem 2.

Theorem 4. *For any numbers m, k, there is no semi-online k-colouring algorithm that receives the vertices of an arborescence in a geometric leaf-to-root order and maintains that at every stage, all directed paths of length m contain all* 2 *colours.*

Call a semi-online k-colouring algorithm *m-proper* if any path on m vertices contains at least two colours. The theorem states there is no m-proper semi-online k-colouring algorithm for arborescences presented in geometric leaf-to-root order.

Proof. The idea of the proof is that for any vertex u, there are only finitely many possibilities for all directed paths of length m from u. However, we can always force the algorithm to produce a new "type" of path, leading to a contradiction.

Fix k colours, $C_1, \ldots, C_k$, a branching $\mathcal{F}$ with a geometric leaf-to-root order, a point $p \in V(\mathcal{F})$, and the time t at which p appears. To ease future notation, let us get some (many) definitions out of the way.

- p_u is a *u-parent* of p if there is a directed path $(p, p_1, \ldots, p_u)$, i.e., $dist(p, p_u) = u$ in the graph. We refer to the subpath $(p_1, \ldots, p_u)$ as the *chain* corresponding to p_u.
- A u-parent p_u of p is *in C_i* if p_u is a u-parent of p and every point in the chain $(p_1, \ldots, p_u)$ is coloured with C_i at time t. (Note that the colour of p need not be in C_i.)
- A u-parent p_u of p in C_i is *maximal* if there is no $p_{u+1} \gtrdot p_u$ that is also coloured with C_i at time t (note that this depends *only* on t, even if some such p_{u+1} is coloured later).
- Similarly, p_u is an *uncoloured* u-parent of p if every point of $(p_1, \ldots, p_u)$ is uncoloured, and it is a *maximal* uncoloured u-parent if there is no $p_{u+1} \gtrdot p_u$ that is also uncoloured.
- The *type* of p, $tp(p)$ is defined as the vector $(t_1, \ldots, t_k) \in \mathbb{N}^k$, where $t_i = \max\{u : p \text{ has a maximal } u\text{-parent in } C_i\}$.
- If two partially coloured trees, $\mathcal{T}_1$ and $\mathcal{T}_2$, are isomorphic, we write $\mathcal{T}_1 \cong \mathcal{T}_2$. Note that for the isomorphism we require that vertices coloured, say red, must be mapped to red vertices - we do not allow the isomorphism to permute the colours.

Now we are ready to formalise the notion of "possibilities for directed paths".

Let S_t be the set of points that have appeared by time t in the same connected component of $\mathcal{F}$ as p (or in the subtree rooted at p at time t). We now associate to p a tree $\mathcal{T}(p)$ by "trimming" the induced subgraph $\mathcal{F}[S_t]$ in the following steps. (See Figure 5.)

1. If q is uncoloured and $dist(p, q) > m$, delete q.

2. If q_1 and q_2 are both maximal t_i-parents in C_i for some remaining q, delete q_2 and all points that are $> q_2$.

3. For $i = 1, \ldots m$, if q is a $(m - i)$-parent of p, and $q_1 \gtrdot q$ and $q_2 \gtrdot q$ are such that the subtrees rooted at q_1 and q_2 are isomorphic, delete q_2.

The idea of this trimming process is to retain only the "essential" information about the colouring when p appears and reduce the number of possible $\mathcal{T}(p)$ to a bounded number of options. If we assume that the algorithm has produced a m-proper colouring until the time that p appears, then we can disregard vertices at distance $> m$ from p. If a vertex was not deleted during the trimming, we say that it was *preserved*.

We could modify step 1 to delete *all* points at distance $> m$ from p. However, in the proof we will use the fact that the type of any point at distance $\leq m$ from p is preserved (see the lemma). Of course, if the algorithm is good, then any directed path of length m contains at least 2 colours, so deleting only the uncoloured points is just a technical condition that simplifies notation. Finally, in step 3, we ensure that we do not have any "repetitions". For example, if all the branches rooted at p are isomorphic, by considering only one of them we do not lose any important information.

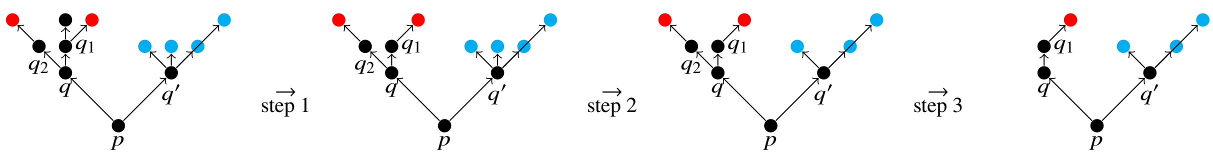

Figure 5: Example for trimming with $m = k = 2$. In step 1, we delete the uncoloured 3-parent of p, but preserve the red parent of q_1. In step 2, we "trim" the blue parents of q'. In step 3, the subtrees rooted at q_1 and q_2 are isomorphic, so we delete q_2.

We emphasise that $\mathcal{T}(p)$ depends only on the time at which p appears. For instance, in the above figure, even if q' is coloured blue at a later time, $\mathcal{T}(p)$ does not change.

We could define a trimming process in various other ways. The crucial point is that the following lemma is true.

Lemma 1. *Suppose that a semi-online colouring algorithm as in the statement of the theorem exists. Then the following hold.*

1. *The set $\{\mathcal{T}(p) : \mathcal{F}$ is a branching, $p \in V(\mathcal{F})\}$ is finite.*

2. *If $q \in S_t$ is preserved after the trimming, and q had an t_i-parent in C_i in $\mathcal{F}$, then q has an t_i-parent in C_i in $\mathcal{T}(p)$. In particular, the type of q is preserved.*

3. *Suppose $p' \lessdot p$ is presented, and q was an uncoloured u-parent of p in $\mathcal{T}(p)$ for $u < d$. If none of the points on the chain from p' to q are coloured when p' is presented, then q is preserved in $\mathcal{T}(p')$.*

Proof. Fix an m-proper semi-online k-colouring algorithm as in the statement of the theorem. The first claim says that there is a finite collection of branchings $(\mathcal{F}_i)_{i\in I}$, with fixed leaf-to-root orders, and points $p_i \in V(\mathcal{F}_i)$, so that for any other branching $\mathcal{F}$ presented in leaf-to-root order, for any $p \in V(\mathcal{F})$, there is a p_i such that $\mathcal{T}(p) \cong \mathcal{T}(p_i)$. Equivalently, we show that there are only finitely many possibilities for $\mathcal{T}(p)$. In step 1 of the trimming we delete uncoloured points at distance $> m$ from p. In step 2 we preserve only maximal parents in C_i of p for each colour C_i. Since the algorithm is m-proper, $\mathcal{T}(p)$ will have depth at most m. In step 2, we also delete "repetitions" so there are only finitely many possibilities for each of the branches above p. And in step 3, we delete isomorphic subtrees, so no two of the branches above p are isomorphic. Thus $\mathcal{T}(p)$ can take only finitely many values.

The second claim follows from our earlier argument.

For the third claim, we only need to consider the case when the algorithm produces an uncoloured u-parent q' such that one of q and q' must be trimmed (i.e., the subtrees rooted at q and q' are isomorphic). In this case, we can assume without loss of generality that q' is trimmed so the second property holds. □

Lemma 2. *At any stage of the algorithm, suppose that we have a collection of trees with roots $p_1, \ldots, p_s$ presented in this order such that no two $\mathcal{T}(p_i)$ and $\mathcal{T}(p_j)$ are isomorphic. Then presenting a vertex p with parents $p_1, \ldots, p_s$, will give a tree $\mathcal{T}(p)$ that is non-isomorphic to any $\mathcal{T}(p_i)$.*

Proof. Suppose for contradiction that for some p_i, $\mathcal{T}(p) \cong \mathcal{T}(p_i)$. Let $\varphi : \mathcal{T}(p) \to \mathcal{T}(p_i)$ be an isomorphism (preserving colourings). We prove by induction for all $u < m$ that there is a chain $p = r_0 \lessdot r_1 \lessdot \cdots \lessdot r_u$ in $\mathcal{T}(p)$ such that for all $i \le u$ we have $\varphi(r_{i-1}) = r_i$, and r_i is uncoloured.

First suppose p is coloured, say with C_1, in $\mathcal{T}(p)$, and let t_1 be maximal such that p has a t_1-parent in C_1. $r_1 = p_i = \varphi(p)$ was coloured with C_1 in $\mathcal{T}(r_1)$, and by the isomorphism r_1 has a t_1-parent in C_1. Since we did not recolour any points, this produces a $(t_1 + 1)$-parent in C_1 of p in $\mathcal{T}(p)$, contradicting the maximality of t_1.

So p must be uncoloured in $\mathcal{T}(p)$, which implies that r_1 was uncoloured in $\mathcal{T}(r_1)$. To complete the base case of the induction hypothesis, we need to show that r_1 remains uncoloured in $\mathcal{T}(p)$, i.e., when p appears. Let t_1 be as earlier, and suppose again that r_1 is coloured with C_1 in $\mathcal{T}(p)$. Then p has an $(t_1 + 1)$-parent in C_1 in $\mathcal{T}(p)$, again a contradiction.

Suppose we have produced a chain $p = r_0 \lessdot r_1 \lessdot \cdots \lessdot r_{u-1}$ from the induction hypothesis. If $u - 1 = m$, then we are done. Otherwise, let $r_u = \varphi(r_{u-1})$. Then r_u is uncoloured in $\mathcal{T}(r_1)$. Since $r_{u-1} \gtrdot r_{u-2}$, $r_u \gtrdot \varphi(r_{u-2}) = r_{u-1}$, so $p = r_0 \lessdot r_1 \lessdot \cdots \lessdot r_u$ is a chain, and it remains to show that r_u is uncoloured in $\mathcal{T}(p)$. Suppose r_u is coloured in $\mathcal{T}(p)$ with C_1. If s_1 is maximal so that r_{u-1} has an s_1-parent in C_1 in $\mathcal{T}(p)$, then r_u has an s_1-parent in C_1 in $\mathcal{T}(p_1)$, producing an $(s_1 + 1)$-parent in C_1 for r_{u-1} in $\mathcal{T}(p)$. This contradicts the maximality of s_1.

This eventually produces a chain of m uncoloured points, which contradicts the correctness of the semi-online algorithm. □

From here we can finish the proof of Theorem 4 with an infinite descent argument as follows. Order the finite sequences of naturals, $\mathbb{N}^{<\omega}$, such that $(s_1, s_2, \ldots, s_\ell) > (s'_1, s'_2, \ldots, s'_{\ell'})$ if there is some i such that for all $j < i$ we have $s_j = s'_j$ but $s_i > s'_i$, or $\ell > \ell'$ and for all $j \le \ell'$ we have $s_j = s'_j$. For a branching $\mathcal{F}$, we define its *associated sequence* as follows. For each root p_i of $\mathcal{F}$, consider the sequence of trees $\mathcal{T}(p_i)$ in the order their roots were presented. Let i_1 be the smallest index such that for every $\mathcal{T}(p_i)$ there is an $i' \le i_1$ such that $\mathcal{T}(p_i) \cong \mathcal{T}(p_{i'})$. The number of different trees $\mathcal{T}(p_i)$ (same as the number of different trees up to i_1) is denoted by s_1. In general, after i_{j-1} has been defined, let i_j be the smallest index such that for every $\mathcal{T}(p_i)$ with $i > i_{j-1}$ there is an $i_{j-1} < i' \le i_j$ such that $\mathcal{T}(p_i) \cong \mathcal{T}(p_{i'})$. The number of different trees $\mathcal{T}(p_i)$ for $i_{j-1} < i \le i_j$ is denoted by s_j. We repeat this for N steps, where N denotes the number of possible different (i.e., non-isomorphic) trees $\mathcal{T}$, or until there are no more roots in $\mathcal{F}$. The numbers $(s_1, \dots, s_\ell)$ are the associated sequence of $\mathcal{F}$.

Note that there are finitely many associated sequences, as each $N \geq s_1 \geq s_2 \geq \cdots \geq s_\ell$, and also $\ell \leq N$. Applying Lemma 2 to the largest associated sequence that can be attained during the run of the semi-online algorithm, we get a contradiction as follows. Let $\mathcal{F}$ be a branching whose associated sequence, $(s_1, \dots, s_\ell)$, is the largest.

Case 1: If $s_1 = N$, then we present a new point p whose parents are the roots of $\mathcal{F}$, and by lemma 2 we produce a new tree, which is not possible.

Case 2: If $N > s_1 > \cdots > s_\ell$, then $\ell < N$. Introduce a new vertex disjoint from all vertices of $\mathcal{F}$. This will either increase an earlier s_i, or give a new $s_{\ell+1} = 1$, but both of these contradict the maximality of $(s_1, \dots, s_\ell)$.

Case 3: There is some j for which $s_j = s_{j+1}$. This is only possible if all the trees $\mathcal{T}(p_i)$ for $i_{j-1} < i \leq i_j$ have an isomorphic copy $\mathcal{T}(p_{i'})$ for some $i_j < i \leq i_{j+1}$. Introduce a new vertex p under all the roots p_i of $\mathcal{F}$ with index $i > i_j$ to obtain a new branching $\mathcal{F}'$. By Lemma 2, the tree $\mathcal{T}(p)$ is non-isomorphic to any $\mathcal{T}(p_i)$ with $i_{j-1} < i \leq i_j$. Therefore, the associated sequence of $\mathcal{F}'$ will be larger than $(s_1, \dots, s_\ell)$, contradicting its maximality.

In summary, it is not possible for a semi-online k-colouring algorithm to produce finitely many associated sequences, so it cannot be m-proper. $\square$

4 Bottomless rectangles

4.1 Non-existence of semi-online colouring algorithms

In this section, we apply Theorem 4 to semi-online colouring algorithms for Erdős-Szekeres configurations. We start with towers.

Corollary 1. *There is no semi-online colouring algorithm for towers from above, i.e., for any numbers k and m, for any semi-online algorithm that k-colours bottomless rectangles from above, there is a family of bottomless rectangles such that any two intersecting rectangles form a tower, and the algorithm produces an m-fold covered point that is covered by at most one colour.*

Proof. In order to apply Theorem 4, we need to show that any branching can be realised as a family of towers so that

1. ordering the rectangles from above corresponds to a geometric leaf-to-root order of the branching, and

2. a semi-online colouring algorithm for towers from above corresponds to an m-proper semi-online k-colouring algorithm for branchings in this order.

For any arborescence $\mathcal{F}$ in geometric leaf-to-root order, we show by induction on $|\mathcal{F}|$ that it can be realised as a family of towers with this order. For $|\mathcal{F}| = 1$ this is clear. For the inductive step, we will need to use the fact that the ordering is geometric. For example, suppose we have a non-geometric order and three roots p, q, r that are realised as disjoint rectangles, with q between p and r. Then if the next root s is presented with $s < p$ and $s < r$, but $s \not< q$, s cannot be realised as a rectangle.

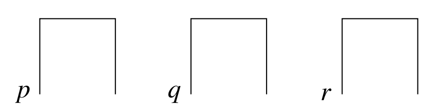

Figure 6: There is no way to present a new rectangle s that intersects p and r but not q.

Now we prove the induction step. Let $|\mathcal{F}| = n$, and r be the last element in the ordering of $V(\mathcal{F})$. Take any realisation of $\mathcal{F} \setminus \{r\}$ as a family of towers. If r is an isolated vertex in $\mathcal{F}$, present r as a disjoint rectangle to the right of the realisation $\mathcal{F} \setminus \{r\}$. Otherwise, since the order is geometric, r will only intersect some geometrically adjacent rectangles of $\mathcal{F}$ (by construction). Hence r can be realised as a minimal rectangle. □

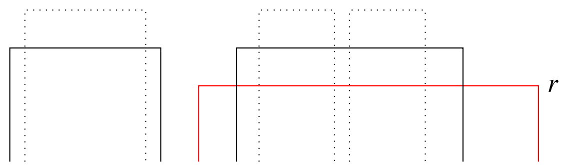

Figure 7: By the geometric ordering, we can realise r as a minimal element

The proof for nested rectangles from below, is analogous. □

Corollary 2. *There is no semi-online k-colouring algorithm from the left or from below for increasing steps. More precisely, for any integers k and m, there is no semi-online algorithm to k-colour rectangles from the left (or from below) so that at every step, any point covered by m-increasing steps is covered by at least 2 colours. Similarly, there is no semi-online colouring algorithm for decreasing steps from the right or from below.*

Note that this statement is slightly weaker than Theorem 1 or Corollary 1 because we do not exclude the other kind of configurations from the family.

Proof. We first prove the statement for increasing steps from the left. Again, we will prove the corollary by induction on $|\mathcal{F}|$. However, we also weaken our requirements for the colouring of the steps. That is, we need not assume that every directed path of length m in the branching corresponds to a point covered by exactly m increasing steps. It is easy to see that a semi-online colouring algorithm of $\mathcal{F}$ is m-proper if and only if when any point $p \in \mathcal{F}$ is presented, any directed path of length m *from* p contains at least 2 colours. So it suffices to prove the following by induction.

Any branching $\mathcal{F}$ with a geometric leaf-to-root order can be realised as a family of bottomless rectangles so that

1. when $p \in \mathcal{F}$ is presented, we realise p as a rectangle so that any directed path of length m from p corresponds to a point covered by exactly m increasing steps, and

2. any two rectangles intersect either as increasing or as decreasing steps.

The second assumption is a technical condition to ensure that q covers the top-right corner of r if and only if (r, q) form increasing steps, so we can choose the top-right corner of an appropriate rectangle as the point satisfying the first induction hypothesis.

The case $|\mathcal{F}| = 1$ is trivial. Let $|\mathcal{F}| = n$, and r be the last element in the ordering of $\mathcal{F}$. Take any realisation of $\mathcal{F} \setminus \{r\}$ satisfying the induction hypotheses. If r is an isolated vertex, let q be the last element presented (thus a root), and realise r as a rectangle so that (q, r) form decreasing 2-steps (see Figure 8). There are no directed paths of length m from r so both induction hypotheses are satisfied.

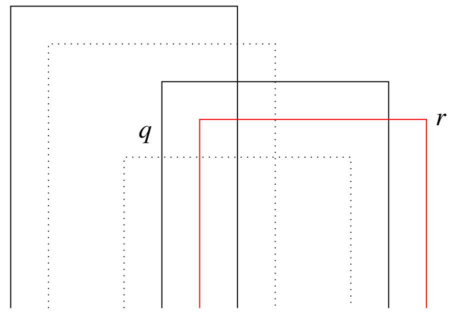

Figure 8: The rectangles in decreasing steps correspond to roots of the branching

Otherwise, since the ordering is geometric, r will only intersect the rightmost rectangles (by construction), thus can be realised as a rectangle that forms increasing steps with these rightmost roots, and decreasing steps with the other roots (see Figure 9).

To see that the first hypothesis is satisfied, consider the rectangles corresponding to any directed path of length m from r, say $(r_1, \ldots, r_{m-1}, r)$. Then the top-right corner of r_1 will not be covered by any rectangle other than the ones in this chain - this follows from the induction hypothesis and the fact that $\mathcal{F}$ is a branching, so r_2 is the unique child of r_1.

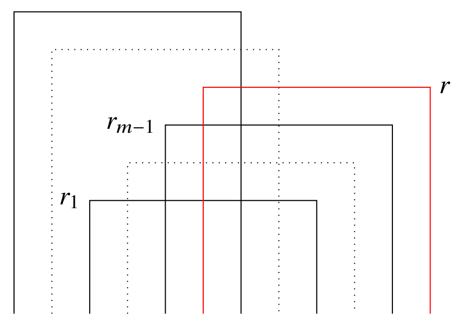

Figure 9: The new rectangle r is presented in increasing steps with r_{m-1}, and decreasing steps with the other minimal elements.

The proof for increasing steps from below follows analogously, except we change the second induction hypothesis to assume that any two rectangles intersect either as increasing steps or as a tower. In this construction, the roots of the branching at any time will correspond to a tower, and the geometric ordering ensures that a new root can be placed in increasing steps with the top-most rectangles of the tower.

The proof for decreasing steps is exactly the same, only interchanging left and right. □

4.2 Online and semi-online colouring algorithms

Recall that for each fixed configuration, $m_k = k$ (see Table 2), and this was given by online colouring algorithms from certain directions. The next question to ask is: how can we combine these colourings?

Nested rectangles seem to have the "simplest" structure of the four configurations. Indeed, ordering nested rectangles from above is the same as ordering it from the left or right. Further, if (R_1, R_2) are nested (R_1 contains R_2), and $\ell(R_1), r(R_1), y(R_1)$ denote the left endpoint, right endpoint, and height of R_1 respectively, then $p = (x_0, y_0) \in R_2$ is covered by $R_1 \iff y_0 < y(R_1) \iff x_0 > \ell(R_1) \iff x_0 < r(R_1)$. This shows that we can modify the algorithms for the other configurations to also colour nested rectangles.

Proposition 1. *$m_k = k$ if we k-colour $\mathcal{F}$ with respect to*

a) towers and nested sets

b) increasing steps and nested sets

c) decreasing steps and nested sets

Proof. We first present the algorithm for towers and nested sets. The precise statement is that $\mathcal{F}$ can be k-coloured so that any point contained in a k-tower or a k-nested set is covered by all k colours.

We colour the rectangles from the right (this can also be done from the left). We maintain the same property as earlier.

(*) If a point p is covered by a j-tower or a j-nested set for $j \leq k$, then p is covered by at least j different colours.

At step t, for $1 \leq i \leq k$, let y_i be the greatest number so that if $p \in R_t$ has y-coordinate less than y_i, then p is covered by a rectangle of colour i. As earlier, if some y_i does not exist, give R_t colour i. Otherwise, suppose $y_1 > \ldots > y_k$, and give R_t colour k.

To prove that (*) holds is not as straightforward as in Theorem 3. Let y denote the height of R_t. If $y_k > y$, or $y > y_1$, (*) holds by the same argument as in Theorem 3.

If not, we have $y_1 > \cdots > y_{\ell-1} > y > y_\ell > \cdots > y_k$. Suppose $p \in R_t$ is covered by a j-nested set $R_1, \ldots, R_{j-1}, R_t$. Since each y_i is maximal, (*) holds by the same argument as earlier. The only essentially different case is when $p \in R_t$ is covered by a j-tower. If we did not add a new colour to the set of rectangles containing p, this means that p was already covered by a rectangle of colour k. However, as y_k was chosen to be maximal, the y-coordinate of p must be less than y_k, so p is already covered by all k colours.

The algorithms for increasing and decreasing steps are modified in the exact same way. □

Proposition 2. *If $\mathcal{F}$ contains no towers and no nested sets, then $\mathcal{F}$ can be k-coloured so any point contained in $3k-2$ rectangles is covered by all k colours.*

Note that this is not the same as saying that *any* family can be k-coloured with respect to increasing and decreasing $(3k-2)$-steps: we impose a restriction on the family.

Proof. We use a proof of [1] that k-colours dynamically appearing points on a line, so that at any time, any $3k-2$ consecutive points have all k colours. More precisely, we have a set of points $P \subset \mathbb{R}$ such that each point of p "appears" at a different time. We can k-colour P so that at *any* time, any interval containing $3k-2$ points of P contains all k colours.

For each rectangle $R \in \mathcal{F}$ we associate its left endpoint $\ell(R)$, and suppose the left endpoints appear in decreasing order of height of the rectangles (so this is a colouring from above). Using the algorithm from [1], we k-colour the left endpoints. We claim that this is a good colouring of the rectangles.

Suppose at some time t, p is covered by $M \geq 3k-2$ rectangles, ordered by left endpoint $\ell(R_1) < \cdots < \ell(R_M)$. We claim that there are no other rectangles in this interval, so that the set of rectangles covering p contains all k colours.

If there is some rectangle S with $\ell(R_1) < \ell(S) < \ell(R_M)$, since $\mathcal{F}$ contains no towers and no nested sets, this implies that $r(R_1) < r(S) < r(R_M)$ (where this denotes the right endpoints). Further, as S has appeared before time t, the height of S must be greater than the y-coordinate of p. So S covers p, and this completes the proof. □

4.3 Unit width rectangles and ABA-free hypergraphs

We now pay closer attention to the colouring problem from Proposition 2. Our proof relied on reducing this to colouring a dynamically appearing finite point set P with respect to intervals on the line. If we map each point $x \in P$ that appears at time t to $(x, -t)$ in the plane, we see that this is exactly the same as colouring points with respect to bottomless rectangles. That is, colouring dynamically appearing points with respect to intervals (*d-pt/int*) is equivalent to colouring points with respect to bottomless rectangles (*pt/bottomless*). Similarly, the dual problems *d-int/pt* and *bottomless/pt* are equivalent. In particular, Proposition 2 shows that colouring steps with respect to points is a special case of the primal problem for bottomless rectangles, so any configuration realizable as steps/pt is also realizable as pt/bottomless (steps/pt $\subset$ pt/bottomless). We remark that this inclusion is strict, as $m_2 = 3$ for steps/pt, but $m_2 = 4$ for pt/bottomless.

Another natural extension of the primal problem is to consider the case when all the rectangles have the same fixed width, or unit width (*ptXunit bottomless*). Equivalently, given a fixed bottomless rectangle R and a finite point set P, we wish to k-colour P so that any translate of R containing m points of P contains all k colours. In this case, the primal colouring problem is equivalent to the dual, i.e. $m_{k(\text{pt/unit bottomless})} = m_{k(\text{unit bottomless/pt})}$ (this explains the notation ptXunit bottomless). We could refer to this as $m_{k(unit)}$, but the following proposition renders this unnecessary.

Proposition 3. *steps/pt = ptXunit bottomless, i.e., any hypergraph that can be realised as steps/pt can be also realised as ptXunit bottomless, and vice versa.*

Proof. The inclusion ptXunit bottomless $\subseteq$ steps/pt is easy to show: a family of rectangles of fixed width cannot contain any towers or nested sets.

We prove the reverse inclusion steps/pt $\subseteq$ ptXunit bottomless by our favourite method, induction on $|\mathcal{F}|$. This time, the base case $|\mathcal{F}| = 1$ is left as an exercise. However, we assume a stronger induction hypothesis. Suppose any family $\mathcal{F}$ of $n-1$ rectangles that do not contain towers or nested sets can be realised as a family $\mathcal{F}_{unit}$ of unit bottomless rectangles (with an isomorphic hypergraph), and that this realisation preserves heights and the ordering of left endpoints. That is, the height of a rectangle R in $\mathcal{F}$ is the same as its realisation in $\mathcal{F}_{unit}$, as is the order of the left endpoints.

Let $|\mathcal{F}| = n$, and let R be the leftmost rectangle in $\mathcal{F}$. Take any realisation of $\mathcal{F} \setminus R$ as a family of unit bottomless rectangles $\mathcal{G}$ Let $R_1, \ldots, R_m \in \mathcal{F}$ be the rectangles that intersect R, and $R'_1, \ldots, R'_m \in \mathcal{G}$ their realisations. Assume without loss of generality that $\ell(R_1) < \cdots < \ell(R_m)$.

In particular, $\ell(R) < \ell(R_i) < r(R)$ for each i (as they intersect), so $R_1, \ldots, R_m$ also intersect each other. This implies that the interval $[\ell(R'_1), \ldots, \ell(R'_m)]$ has length strictly less than 1. Thus for ϵ small enough, if we realise R as a unit width rectangle R' with $r(R') = \ell(R'_m) + \epsilon$ with the same height, then R' will intersect exactly the rectangles $R'_1, \ldots, R'_m$ (with the same hypergraph structure).

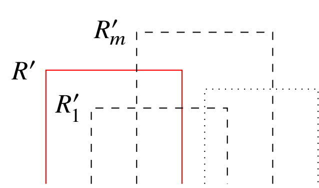

Figure 10: We ensure that the realisation of R preserves the hypergraph structure.

$\square$

So instead of considering colouring points with respect to unit bottomless rectangles, we may consider colouring steps with respect to points.

Proposition 4. *For steps/pt, $m_k \leq 2k - 1$.*

The proof of the proposition will use ABA-free hypergraphs [12]. We say a hypergraph $\mathcal{H}$ with an ordering $<$ of its vertex set is *ABA-free* if there are no hyperedges A and B and vertices $x < y < z$ with $x, z \in A \setminus B$ and $y \in B \setminus A$. For example, interval hypergraphs – where the vertices are points in $\mathbb{R}$ and the hyperedges are the subsets induced by some intervals – are ABA-free.

Proof. Let $\mathcal{F}$ be a family containing no nested sets or towers and P a finite point set. We claim that by ordering the rectangles by left endpoint, the resulting hypergraph on the vertex set $\mathcal{F}$ with edges induced by P is ABA-free.

Suppose for contradiction we have three rectangles with $\ell(R_1) < \ell(R_2) < \ell(R_3)$, and points p and q so that $p \in R_1, R_3$, $q \notin R_1, R_3$, and $q \in R_2$, $p \notin R_2$.

Recall that a point (x, y) is in a rectangle R if and only if $x \in [\ell(R), r(R)]$ and $y < y(R)$. Let $p = (x_p, y_p)$ and $q = (x_q, y_q)$. Then, $p \in R_1, R_3$ but $p \notin R_2$ implies,

$$\ell(R_1) < \ell(R_2) < \ell(R_3) < x_p < r(R_1) < r(R_2) < r(R_3), \text{ and}$$

$$y(R_1), y(R_3) > y_p > y(R_2).$$

And, $q \in R_2$ but $q \notin R_1, R_3$ implies,

$$\ell(R_1) < \ell(R_2) < x_q < r(R_2) < r(R_3), \text{ and}$$

$$y(R_3) > y_p > y(R_2) > y_q$$

$$q \notin R_3 \implies x_q < \ell(R_3).$$

Similarly,

$$y(R_1) > y_q, q \notin R_1 \implies x_q > r(R_1).$$

However, $r(R_1) > \ell(R_3)$, so this is a contradiction. □

We end this subsection by extending this to families that do not contain towers.

Proposition 5. *For families $\mathcal{F}$ that do not contain towers, $m_k \leq 2k - 1$.*

Proof. As earlier, we want to show that the corresponding hypergraph is ABA-free. Suppose again that we have three rectangles with $\ell(R_1) < \ell(R_2) < \ell(R_3)$, and points p and q so that $p \in R_1, R_3$, $q \notin R_1, R_3$, and $q \in R_2$, $p \notin R_2$.

The previous proposition shows R_1, R_2, R_3 must contain at least one nested set. It is also easy to see that not all three of them can form a nested set, so exactly two of them do. Further, the condition $p \in R_1, R_3$ but $q \notin R_1, R_3$ implies that (R_1, R_3) must form a nested set (where R_1 contains R_3). In this case, it is easy to check that it is not possible to have a rectangle R_2 that forms steps with both R_1 and R_3, and contains q but not p. □

4.4 Lower bound constructions

Finally, we present an improved lower bound for general bottomless rectangle families, and a weaker lower bound that can be applied to the steps/pt problem.

Theorem 5. $m_k \geq 2k - 1$ *for bottomless rectangles.*

Proof. Our lower bound construction proceeds in two steps.

1. If $m_k < m_{k-1} + 2$, then every family has a polychromatic k-colouring that is proper.

2. There is a family so that no polychromatic k-colouring is proper.

This contradiction shows that $m_k \geq m_{k-1} + 2$, so by induction $m_k \geq 2k - 1$.

1. Suppose for some family $\mathcal{F}$, no polychromatic k-colouring of $\mathcal{F}$ is proper. Let $\mathcal{G}$ be a witness to the sharpness of m_{k-1}, i.e. any $(k-1)$-colouring of $\mathcal{G}$ produces a point covered by $m_{k-1} - 1$ rectangles but not all k colours. In a small interval around every 2-covered point in $\mathcal{F}$, we place a thin copy of $\mathcal{G}$ (see Figure 11).

   Any polychromatic colouring of this new family $\mathcal{F}'$ must induce a polychromatic colouring of $\mathcal{F}$, so some copy of $\mathcal{G}$ is covered by 2 rectangles of the same colour, say red.

   By hypothesis, any point in this copy of $\mathcal{G}$ covered by at least m_k rectangles is covered by all k colours. Since every such point is covered by exactly two red rectangles from $\mathcal{F}$, recolouring every red rectangle in $\mathcal{G}$ blue cannot ruin this property. However, this induces a $(k-1)$-colouring of $\mathcal{G}$ so that any point in $m_{k-1} - 1$ rectangles is covered by all $k - 1$ colours, a contradiction.

   So every family must have a polychromatic colouring that is proper.

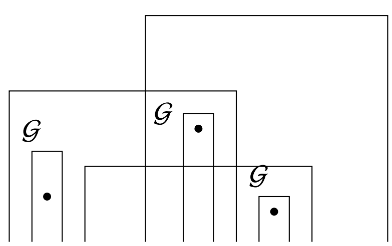

Figure 11: Place disjoint thin copies of $\mathcal{G}$ around every 2-covered point in $\mathcal{F}$.

2. Consider the family in Figure 12.

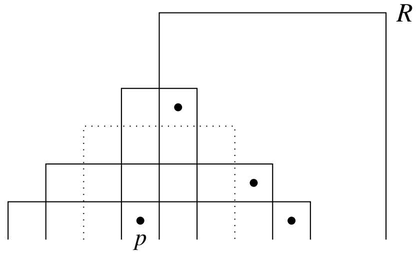

Figure 12: No polychromatic colouring of this family will be proper.

We have an m-tower (where m may be arbitrarily large), so that each rectangle from the tower meets R in a 2-covered point. Suppose without loss of generality that R is coloured red in some polychromatic k-colouring. For this colouring to be proper, no rectangle of the tower can be red - however the point p will then be covered by m rectangles, none of which are red, so the colouring cannot be polychromatic. This completes our proof.

□

This lower bound cannot be applied to ptXunit bottomless, as this construction relies heavily on towers. For these, we prove the following weaker lower bound.

Proposition 6. *$m_k \geq 2\lfloor \frac{2k-1}{3} \rfloor + 1$ for colouring the translates of a bottomless rectangle.*

Proof. This is a generalisation of the construction that shows that $m_k = 3$. Let $\mathcal{F}$ be a family of $2k - 1$ rectangles partitioned into 3 almost equal subfamilies, $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ as follows.

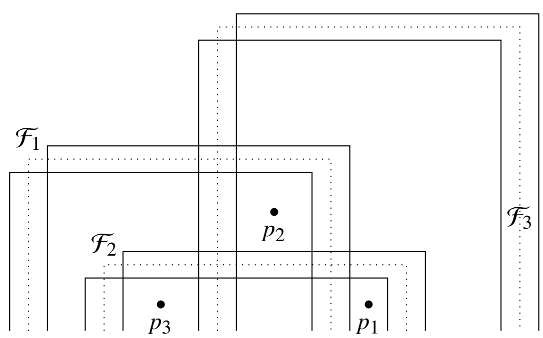

Figure 13: This construction shows that $m_k \geq 2\lfloor \frac{2k-1}{3} \rfloor + 1$

Consider any k-colouring of $\mathcal{F}$. Some colour, say red, is used at most once, so it appears in at most one of $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$, say $\mathcal{F}_i$. Then the point p_i is covered by the other two subfamilies, and no red rectangle. Since $\lfloor \frac{2k-1}{3} \rfloor \leq |\mathcal{F}_i| \leq \lceil \frac{2k-1}{3} \rceil$, this proves the lower bound. □

Note that the family in the figure does not contain any towers or nested sets. This gives a lower bound to complement Proposition 4, namely that for steps/pt, $m_k \geq 2\lfloor \frac{2k-1}{3} \rfloor + 1$.

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Figyelmeztetés! A kitöltött Nyilatkozat nem menthető el, csa Nyomtatás

NYILATKOZAT

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NEPTUN azonosító: ZV1CF9

Diplomamunka címe: Colouring bottomless rectangles and arborescences
Colouring bottomless rectangles and arborescences

A **diplomamunka** szerzőjeként fegyelmi felelősségem tudatában kijelentem, hogy a dolgozatom önálló szellemi alkotásom, abban a hivatkozások és idézések standard szabályait következetesen alkalmaztam, mások által írt részeket a megfelelő idézés nélkül nem használtam fel.

Budapest, 2020, May 30

a hallgató aláírása