On various notions of universally Baire sets and their applications to Haar meagerness

MSc Thesis

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Contents

1 Introduction **3**

2 Notations and preliminaries **4**

2.1 Descriptive set theoretical preliminaries 4

2.2 Topological and set theoretical preliminaries 6

3 Universally Baire sets **8**

3.1 The case of general Polish spaces 8

3.2 The case of σ-compact Polish spaces 23

4 Application to Haar meagerness **31**

4.1 The right definition and basic properties 32

4.2 Preliminary results 37

4.3 The theory of generalized Haar meager sets 39

5 Acknowledgement **46**

6 References **47**

1 Introduction

In what follows, G will always denote a Polish group, i.e. a separable topological groups whose topology can be induced by a complete metric. In 1970, Christensen introduced the so-called Haar null sets in [2] (for a definition of a Borel measure see Definition 2.1.7):

Definition 1.0.1. *$A \subseteq G$ is called* Haar null*, if there exists a Borel set B with $B \subseteq A$ and a Borel probability measure μ on G such that $\mu(gBh) = 0$ for every $g, h \in G$.*

Haar null sets were introduced by Christensen as a generalization of the so-called Haar measure zero sets to groups that are non-locally-compact. Haar null sets form a σ-ideal (see Definition 2.2.7), which is a natural expectation for a notion of any "null set". It turned out that it is a fruitful concept in mathematics and Haar null sets have numerous applications, see e.g. [5].

Sometimes it is hard to get a natural Borel covering set B in the definition of a Haar null set. To get a wider class of covering sets, we may use the so-called universally measurable sets (see Definition 2.1.8) and we arrive at the definition of generalized Haar null sets:

Definition 1.0.2. *$A \subseteq G$ is called* generalized Haar null*, if there exists a universally measurable set B with $B \supseteq A$ and a Borel probability measure μ on G such that $\mu(gBh) = 0$ for every $g, h \in G$.*

It is natural to ask what the Baire category analogues of the Haar null and generalized Haar null sets are. Meager sets (see Definition 2.1.11) make sense in every topological space, but it turned out that the best category analogue of the Haar null sets are not the meager sets. The following notion turned out to be a very good category analogue of Haar null sets.

Definition 1.0.3. *$A \subseteq G$ is called* Haar meager*, if there exists a Borel set B with $B \supseteq A$ and a compact metric space K with a continuous function $f : K \longrightarrow G$ such that $f^{-1}(gBh)$ is meager in K for every $g, h \in G$.*

The aim of this thesis is to introduce the generalized Haar meager sets as a category analogue of generalized Haar null sets. To do this, we need a category analogue of the universally measurable sets. For this purpose, we will introduce the universally Baire sets as a nice category analogue for the universally measurable sets first. It turned out that there are various mathematical objects called universally Baire sets, so we will start with collecting various notions of universally Baire sets in Section 3 that appeared in the literature and we will systematically investigate them. In Section 4, we will choose our definition of a universally Baire set, which will serve as a nice category analogue of the universally measurable sets and as an application we will introduce the generalized Haar meager sets.

2 Notations and preliminaries

2.1 Descriptive set theoretical preliminaries

Definition 2.1.1. *A topological space X is called a* Polish space*, if it is separable and completely metrizable.*

Notation 2.1.2. *Let A be a nonempty set and $n \in \omega$. We denote by A^n the set of finite sequences $s = (s_0, ..., s_{n-1})$ of length n from A (for $n = 0$ we have that $A^0 = \{\emptyset\}$).The length of a finite sequence s is denoted by $length(s)$. If $s \in A^n$ and $m \leq n$, we denote by $s|_m$ the restriction of s of length m, that is, $s|_m = (s_0, ..., s_{m-1})$. If $a \in A$ and $s \in A^n$, then $s^\frown a$ denotes the sequence $(s_0, ..., s_{n-1}, a)$. We say that the finite sequence t extends the finite sequence s if there exists a $m \leq length(t)$ such that $t|_m = s$, in symbols $s \prec t$. We will denote by $A^{<\omega}$ the set of all finite sequences from A, that is, $A^{<\omega} = \bigcup_{n\in\omega} A^n$.*

Notation 2.1.3. *When λ is a cardinal, we also consider it as a topological space with the discrete topology and when we write λ^ω, we consider λ^ω as a topological space equipped with the product topology. In the case $\lambda = \omega$, we will write $\mathbb{N}^\omega$, which is called the* Baire space *and the topological space 2^ω is called the* Cantor space.

Notation 2.1.4. *When X is a topological space, we will denote the Borel sets of X by $\mathcal{B}(X)$.*

Fact 2.1.5. *The Baire space and the Cantor space are Polish spaces.*

Theorem 2.1.6. *(see [12, Exercise 3.4]) The Baire space is homeomorphic to the the irrationals and the Cantor space is homeomorphic to the standard triadic Cantor set. The Baire space is also homeomorphic to the irrationals in the unit interval $[0, 1]$.*

Definition 2.1.7. *Let X be a Polish space, a measure μ on X is called a* Borel measure*, if it is the completion of a measure which is defined on the Borel σ-algebra of X.*

Definition 2.1.8. *Let X be a Polish space and $A \subseteq X$. The set A is called a* universally measurable set *if it is measurable with respect to any σ-finite Borel measure.*

Remark 2.1.9. *It is easy to see that if we only require A to be μ measurable for every finite Borel measure μ, then A is already a universally measurable set.*

Definition 2.1.10. *Let Y be a topological space, then $A \subseteq Y$ is called a* nowhere dense set*, if for every nonempty open set $U \subseteq Y$ there exists a nonempty open set $V \subseteq U$, such that $V \cap A = \emptyset$.*

Definition 2.1.11. *Let Y be a topological space, then $A \subseteq Y$ is called a* meager set*, if it is the countable union of nowhere dense sets. The set $B \subseteq Y$ is called* residual *or* comeager*, if B^c is meager.*

Notation 2.1.12. *For a topological space Y we denote the meager subsets of Y by $\mathcal{M}(Y)$.*

Notation 2.1.13. *For sets A, B we denote by $A \triangle B$ the* symmetric difference *of A and B, that is $A \triangle B = (A \setminus B) \cup (B \setminus A)$.*

Definition 2.1.14. *Let Y be a topological space, a set A has the* property of Baire *if there exists an open set U such that $U \triangle A$ is meager.*

Theorem 2.1.15. *(see [12, Proposition 8.22.]) The family of sets that have the property of Baire forms a σ-algebra, and it is the smallest σ-algebra containing all open sets and meager sets. Moreover, every set with property of Baire can be written as the union of a Borel set and a meager set*

Definition 2.1.16. *Let Y, Z be topological spaces. A function $f : Y \longrightarrow Z$ is* Baire measurable *if whenever $U \subseteq Z$ is open then $f^{-1}(U)$ has the property of Baire.*

Theorem 2.1.17. *(Baire category theorem, see [12, Theorem 8.4.]) Let Y be a complete metric space, then the countable intersection of dense open sets is a dense set.*

Definition 2.1.18. *A topological space Y is said to be* perfect*, if it has no isolated points.*

Definition 2.1.19. *A subset of a topological space is called* perfect *if it is closed and perfect with respect to the subspace topology.*

Theorem 2.1.20. *(see [12, Theorem 6.4.]) Let X be a Polish space, then X can be uniquely written as $X = P \cup C$, where P and C are disjoint and P is a perfect subset of X and C is a countable open set.*

Theorem 2.1.21. *(see [12, Theorem 3.11.]) A subspace of a Polish space is completely metrizable if and only if it is a $\mathcal{G}_\delta$ subspace. Equivalently, a subspace of a Polish space is Polish if and only if it is a $\mathcal{G}_\delta$ subspace.*

Definition 2.1.22. *Let X be a Polish space. The set $A \subseteq X$ is called* analytic *if there exists a Polish space Y and a continuous $f : Y \longrightarrow X$ such that $f(Y) = A$ or $A = \emptyset$. The set $A \subseteq X$ is called* coanalytic *if A^c is analytic.*

2.2 Topological and set theoretical preliminaries

Convention: Whenever we consider a topological space Z, we do *not* assume that Z is Hausdorff and we will carefully emphasize when the space Z is assumed to be Hausdorff.

Notation 2.2.1. *When Z is a topological space with $A \subseteq Z$, then we will denote the* interior *of A by* $\operatorname{int} A$*, the* closure *of A will be denoted by $\overline{A}$ and the* boundary *of A will be denoted by ∂A.*

Notation 2.2.2. *Let (Z, d) be a metric space and take an arbitrary $A \subseteq Z$ and $r > 0$. We will denote by $N_r(A)$ the open r-neighborhood of A, that is, $N_r(A) = \{x \in Z \mid \exists z \in A : \ d(x, z) < r\}$.*

Definition 2.2.3. *Let Z be a topological space and let $U \subsetneq Z$ be an open set, then U is called a* regular open set*, if $\operatorname{int} \overline{U} = U$. We will say that Z has a regular open basis, if it has a basis such that every element of the basis is a regular open set.*

Theorem 2.2.4. *(see [15, Theorem 4.6.]) Let X be a topological space with $A \subseteq X$ having the property of Baire. There exists a regular open set U and a meager set M such that $A = U \triangle M = A$.*

Definition 2.2.5. *A topological space Z is called* Čech-complete*, if Z is homeomorphic to a $\mathcal{G}_\delta$ subset of a compact Hausdorff space.*

Definition 2.2.6. *A topological space X is called* extremally disconnected *if the closure of every open set is open.*

Definition 2.2.7. *A system $\mathcal{I}$ (of subsets of some set) is called a* σ-ideal *if*

(i) $\emptyset \in \mathcal{I}$,

(ii) $A \in \mathcal{I}, B \subseteq A \Rightarrow B \in \mathcal{I}$ and

(iii) if $A_n \in \mathcal{I}$ for all $n \in \omega$, then $\bigcup_n A_n \in \mathcal{I}$.

Definition 2.2.8. *We will denote by* $\operatorname{add}(\mathcal{M})$ *the* additivity *of the σ-ideal of the meager sets of $\mathbb{N}^\omega$, that is the least cardinal κ, such that there exist meager sets $\{M_\alpha \mid \alpha < \kappa\}$, such that $\bigcup\limits_{\alpha<\kappa} M_\alpha$ is not meager.*

Definition 2.2.9. *We will denote by* $\operatorname{add}(\mathcal{N})$ *the* additivity *of the σ-ideal of the Lebesgue null sets of $\mathbb{R}$, that is the least cardinal κ, such that there exist Lebesgue null sets $\{N_\alpha \mid \alpha < \kappa\}$, such that $\bigcup\limits_{\alpha<\kappa} N_\alpha$ is not a Lebesgue null set.*

Fact 2.2.10. *The cardinal* $\operatorname{add}(\mathcal{N})$ *is equal to the least cardinal κ such that there exist Lebesgue null sets $\{N_\alpha \mid \alpha < \kappa\}$ such that for every $\alpha < \kappa$ we have that $N_\alpha \subseteq [0, 1]$ and $\bigcup\limits_{\alpha<\kappa} N_\alpha$ is not a Lebesgue null set.*

3 Universally Baire sets

3.1 The case of general Polish spaces

We start with the various definitions of a universally Baire sets.

Definition 3.1.1. *Let $\mathcal{F}$ be a class of topological spaces, Z be an arbitrary topological space and let $A \subseteq Z$. The set A is called $\mathcal{F}$-universally Baire, if $f^{-1}(A)$ has the property of Baire in Y for every $Y \in \mathcal{F}$ and for every continuous function $f : Y \longrightarrow Z$.*

We are mainly interested in the case, when Z is a Polish space and we will consider the cases, when $\mathcal{F}$ = topological spaces, compact Hausdorff spaces, Polish spaces, compact metric spaces, λ^ω for every cardinal λ, Čech-complete spaces, the Baire space or the Cantor space. We will define now each type of the previously mentioned universally Baire sets and give references where we can where they appeared.

In [10] the following was called a universally Baire set: $A \subseteq Z$ is universally Baire if $f^{-1}(A)$ has the property of Baire in Y whenever Y is a Čech-complete space (see Definition 2.2.5) and $f : Y \longrightarrow Z$ is a continuous function. Let us formulate it in a definition, but let us use a different name:

Definition 3.1.2. *Let $\mathcal{F}$ = Čech-complete spaces. Then $A \subseteq Z$ is called* Čech-complete-universally Baire *if A is $\mathcal{F}$-universally Baire. For brevity, we denote the Čech-complete-universally Baire sets of Z by $\mathcal{UB}_{\check{C}ech-complete}(Z)$.*

In [7] the following notions were introduced by Feng, Magidor and Woodin: let $A \subseteq \mathbb{N}^\omega$ and let λ be an infinite cardinal. The set A is called λ-universally Baire if whenever Y is a topological space with a regular open basis (see Definition 2.2.3) of cardinality $\leq \lambda$, $f^{-1}(A)$ has the property of Baire in Y for every continuous function $f : Y \longrightarrow \mathbb{N}^\omega$. Finally, $A \subseteq \mathbb{N}^\omega$ is called a universally Baire set if it is λ-universally Baire set for every infinite cardinal λ.

In [7, Theorem 2.1.] the following was proved:

Theorem 3.1.3. *Let λ be an infinite cardinal and $A \subseteq \mathbb{N}^\omega$. The set A is λ-universally Baire if and only if $f^{-1}(A)$ has the property of Baire in λ^ω for every continuous $f : \lambda^\omega \longrightarrow \mathbb{N}^\omega$.*

This motivates the following definition:

Definition 3.1.4. *Let $\mathcal{F}$=$\{\lambda^\omega \mid \lambda$ is a cardinal$\}$. Then $A \subseteq Z$ is called* cardinal-universally Baire *if A is $\mathcal{F}$-universally Baire. For brevity, we denote the cardinal-universally Baire sets of Z by $\mathcal{UB}_{\forall\lambda^\omega}(Z)$.*

The previously mentioned two notions of universally Baire sets are the most well-known ones, but the following are also natural and interesting notions, we start with the most restrictive definition:

Definition 3.1.5. *Let $\mathcal{F}$ = topological spaces. Then $A \subseteq Z$ is called* topological-universally Baire *if A is $\mathcal{F}$-universally Baire. For brevity, we denote the topological-universally Baire sets of Z by $\mathcal{UB}_{topological}(Z)$.*

In [17] a set $A \subseteq \mathbb{R}^n$ is called universally Baire, if $f^{-1}(A)$ has the property of Baire for every compact Hausdorff space Y and for every continuous function $f : Y \longrightarrow \mathbb{R}^n$. This motivates the following definition:

Definition 3.1.6. *Let $\mathcal{F}$ = compact Hausdorff spaces. Then $A \subseteq Z$ is called* compact Hausdorff-universally Baire *if A is $\mathcal{F}$-universally Baire. For brevity, we denote the compact Hausdorff-universally Baire sets of Z by $\mathcal{UB}_{compact\ Hausdorff}(Z)$.*

It is worth to note that the previous definitions of universally Baire sets are interesting in an arbitrary topological space Z, but the following definitions are mostly considered when the space Z is a Polish space.

In [13] a subset A of a Polish space Z is called universally Baire, if $f^{-1}(A)$ has the property of Baire for every Polish space Y and for every continuous $f : Y \longrightarrow Z$. This motivates the following definition:

Definition 3.1.7. *Let $\mathcal{F}$ = Polish spaces. Then $A \subseteq Z$ is called* Polish-universally Baire *if A is $\mathcal{F}$-universally Baire. For brevity, we denote the Polish-universally Baire sets of Z by $\mathcal{UB}_{Polish}(Z)$.*

Note that the following definition is closely related to the ω-universally Baire notion of Feng, Magidor and Woodin:

Definition 3.1.8. *Let* $\mathcal{F}=\{\mathbb{N}^\omega\}$*. Then* $A \subseteq Z$ *is called* $\mathbb{N}^\omega$-universally Baire *if* A *is* $\mathcal{F}$*-universally Baire. For brevity, we denote the* $\mathbb{N}^\omega$*-universally Baire sets of* Z *by* $\mathcal{UB}_{\mathbb{N}^\omega}(Z)$*.*

In [11] a subset A of a Polish space Z is called universally Baire, if $f^{-1}(A)$ has the property of Baire for every compact metric space Y and for every continuous $f : Y \longrightarrow Z$. This motivates the following definition:

Definition 3.1.9. *Let* $\mathcal{F}$ = *compact metric spaces. Then* $A \subseteq X$ *is called* compact metric-universally Baire *if* A *is* $\mathcal{F}$*-universally Baire. For brevity, we denote the compact metric-universally Baire sets of* Z *by* $\mathcal{UB}_{compact\ metric}(Z)$*.*

It is well-known that any Polish space is the continuous image of $\mathbb{N}^\omega$ (see [12, Theorem 7.9.]) and every compact metric space is the continuous image of the Cantor space (see [12, Theorem 4.18.]). Therefore the following definition is also of interest parallel to Definitions 3.1.7 and 3.1.8:

Definition 3.1.10. *Let* $\mathcal{F}=\{2^\omega\}$*. Then* $A \subseteq Z$ *is called* Cantor-universally Baire *if* A *is* $\mathcal{F}$*-universally Baire. For brevity, we denote the Cantor-universally Baire sets of* Z *by* $\mathcal{UB}_{Cantor}(Z)$*.*

Throughout this subsection whenever we write X for a space, then X is assumed to be a Polish space. The main result of Subsection 3.1 is the following theorem:

Theorem 3.1.11. *The relationship between the various notions of universally Baire sets in* X *are the containments of the diagram below and no more containments are provable in* ZFC*.*

$$
\begin{array}{ccccc}
\mathcal{UB}_{topological}(X) & = & \mathcal{UB}_{\forall\lambda^\omega}(X) & = & \mathcal{UB}_{\check{C}ech-complete}(X) \\
 & & \rotatebox{-90}{$\subseteq$}\ \rotatebox{-90}{$\not\supseteq$}\ Con & & \rotatebox{-90}{$\subseteq$}\ \rotatebox{-90}{$\not\supseteq$}\ Con \\
\mathcal{UB}_{\mathbb{N}^\omega}(X) & = & \mathcal{UB}_{Polish}(X) & \overset{Con}{\not\subseteq} \ \overset{Con}{\not\supseteq} & \mathcal{UB}_{compact\ Hausdorff}(X) \\
 & & \rotatebox{-90}{$\subseteq$}\ \rotatebox{-90}{$\not\supseteq$}\ Con & & \rotatebox{-90}{$\subseteq$}\ \rotatebox{-90}{$\not\supseteq$}\ Con \\
 & & \mathcal{UB}_{compact\ metric}(X) & = & \mathcal{UB}_{Cantor}(X)
\end{array}
$$

Diagram 3.1.11 (Implications in a general Polish space X.)

Since Diagram 3.1.11 is quite large, we prove Theorem 3.1.11 through several theorems and lemmas. We start with two simple lemmas, which often come up later:

Lemma 3.1.12. *Let Z be an arbitrary topological space with $A \subseteq \tilde{Z} \subseteq Z$, and $\mathcal{F}$ be a class of topological spaces. If A is $\mathcal{F}$-universally Baire in Z, then A is also $\mathcal{F}$-universally Baire in $\tilde{Z}$.*

Proof. We need to check that $f^{-1}(A)$ has the property of Baire in Y for every $Y \in \mathcal{F}$ and continuous $f : Y \longrightarrow \tilde{Z}$. But this is trivial, because every continuous $f : Y \longrightarrow \tilde{Z}$ can be considered as a continuous $f : Y \longrightarrow Z$ and A is $\mathcal{F}$-universally Baire in Z, hence $f^{-1}(A)$ has the property of Baire in Y. □

Lemma 3.1.13. *Let Z be a topological space and $A \subseteq Y \subseteq Z$. Suppose that Y has the property of Baire in Z and A has the property of Baire in Y, then A also has the property of Baire in Z.*

Proof. Since A has the property of Baire in Y, we have that $A = B \cup M$, where B is a Borel set in Y and M is a meager set in Y. Since B is a Borel set in Y, there exists a Borel set $\tilde{B}$ in Z, such that $B = \tilde{B} \cap Y$. It is easy to see that if N is a nowhere dense subset of Y, then it is a nowhere dense subset of Z, hence any meager set in Y is also a meager set in Z. To finish the proof, we only need to note that $A = (\tilde{B} \cap Y) \cup M$, which clearly has the property of Baire in Z. □

Remark 3.1.14. The converse of Lemma 3.1.13 is not true, because we can take a set A in the standard triadic Cantor set, which does not have the property of Baire in the Cantor set, but it will have the property of Baire in $\mathbb{R}$, because the standard triadic Cantor set is nowhere dense, hence A is nowhere dense, too.

We turn to the proof of the first row of Diagram 3.1.11:

Theorem 3.1.15. $\mathcal{UB}_{\forall\lambda^\omega}(X) = \mathcal{UB}_{topological}(X) = \mathcal{UB}_{\check{C}ech-complete}(X)$.

Proof. We start with recalling two theorems from [10, 1K Theorem, 1E Theorem]:

Theorem 3.1.16. *If Z is a metrizable space, then* $\mathcal{UB}_{\check{C}ech-complete}(Z) = \mathcal{UB}_{\forall\lambda^\omega}(Z)$.

We do not recall the proof of Theorem 3.1.16, we only apply it in the case of Polish spaces. Since every Polish space X is metrizable, from the previously mentioned theorem follows that $\mathcal{UB}_{\forall\lambda^\omega}(X) = \mathcal{UB}_{\check{C}ech-complete}(X)$. The second theorem we recall from [10]:

Theorem 3.1.17. *Let Z be a compact Hausdorff space. The following are equivalent for $A \subseteq Z$:*

(i) $A \in \mathcal{UB}_{topological}(Z)$,

(ii) $A \in \mathcal{UB}_{\check{C}ech\text{-}complete}(Z)$,

(iii) $f^{-1}(A)$ has the property of Baire in Y, whenever Y is an extremally disconnected compact Hausdorff space and $f : Y \longrightarrow Z$ is a continuous function.

We will recall the proof of Theorem 3.1.17 later, but at this point we postpone it till the end of Subsection 3.2.

The containment $\mathcal{UB}_{topological}(X) \subseteq \mathcal{UB}_{\check{C}ech-complete}(X)$ is trivial, so let us choose an arbitrary $A \in \mathcal{UB}_{\check{C}ech-complete}(X)$ and our goal is to prove that $A \in \mathcal{UB}_{topological}(X)$. Now the only problem with Theorem 3.1.17 to apply for X is that X is not necessarily compact. It turns out that this is not a big problem, since for an arbitrary Polish space X there exists a Polish compactification of X - that is a compact metric space $\tilde{X}$, which contains a dense homeomorphic copy of X (see [12, Theorem 4.14.]).

Let $\tilde{X}$ be a Polish compactification of X. Note that it would be enough to prove that $A \in \mathcal{UB}_{\check{C}ech-complete}(\tilde{X})$, because then Theorem 3.1.17 would imply that $A \in \mathcal{UB}_{topological}(\tilde{X})$ and Lemma 3.1.12 would imply that $A \in \mathcal{UB}_{topological}(X)$.

To prove that $A \in \mathcal{UB}_{\check{C}ech-complete}(X) \Longrightarrow A \in \mathcal{UB}_{\check{C}ech-complete}(\tilde{X})$, take an arbitrary Čech-complete space Y and a continuous $f : Y \longrightarrow \tilde{X}$ and we need to prove that $f^{-1}(A)$ has the property of Baire in Y. First notice that X is a $\mathcal{G}_\delta$ subspace of $\tilde{X}$, because it is a completely metrizable subspace of a Polish space (see Theorem 2.1.21), so $f^{-1}(X)$ is a $\mathcal{G}_\delta$ subspace of Y, hence $f^{-1}(X)$ is a Čech-complete space. If we denote by g the restriction of f to $f^{-1}(X)$, then g is a continuous function from the Čech-complete space $f^{-1}(X)$ to X, hence $g^{-1}(A) = f^{-1}(A)$ has the property of Baire in $f^{-1}(X)$.

By Lemma 3.1.13 $f^{-1}(A)$ has the property of Baire in Z, since $f^{-1}(X)$ has the property of Baire in Z. □

To continue the proof of the containments of Diagram 3.1.11, we mention that the containments from the first row to the second row are trivial.

Fact 3.1.18. *$\mathcal{UB}_{topological}(X) \subseteq \mathcal{UB}_{Polish}(X)$ and $\mathcal{UB}_{topological}(X) \subseteq \mathcal{UB}_{compact\ Hausdorff}(X)$.*

Let us continue with the proof of the containments of the second row of Diagram 3.1.11, we start with a lemma:

Lemma 3.1.19. *Every perfect Polish space contains a dense subspace homeomorphic to $\mathbb{N}^\omega$.*

Proof. Let X be an arbitrary perfect Polish space and d be a complete compatible metric on X.

Claim 3.1.20. *Let $U \subseteq X$ be an arbitrary nonempty open set, then for every n there exists an infinite sequence of pairwise disjoint nonempty open balls $\{B_i\}_{i\in\omega}$ such that $\forall i : \overline{B_i} \subseteq U$ with $\bigcup_{i\in\omega} B_i$ dense in U and the radii of the balls are at most $\frac{1}{n}$.*

Proof. We first enumerate a dense sequence $\{q_i\}_{i\in\omega}$ in U. Let B_0 be a ball centered at q_0 and with radius r_0, such that $q_1 \notin \overline{B(q_0, r_0)}$ and $r_0 \leq \frac{1}{n}$. If we have defined $B_0, B_1, \ldots B_{n-1}$ such that they are pairwise disjoint, having radius at most $\frac{1}{n}$ and $\overline{B_0}\cup\cdots\cup\overline{B_{n-1}} \neq U$, then $U\setminus\overline{B_0}\cup\cdots\cup\overline{B_{n-1}}$ is a nonempty open set, so we can choose a $q_j \in U\setminus\overline{B_0}\cup\cdots\cup\overline{B_{n-1}}$, with j minimal. By the perfectness of the Polish space, we can also choose $p \in U \setminus \overline{B_0} \cup \cdots \cup \overline{B_{n-1}}$, $p \neq q_j$. We will choose B_n to be a ball centered at q_j with radius $r_n \leq \frac{1}{n}$, such that $p \notin \overline{B_n}$.

By construction we have that the above defined balls are pairwise disjoint and they have radii at most $\frac{1}{n}$. We also have that if $q_j \notin \overline{B_0} \cup \cdots \cup \overline{B_{j-1}}$, then $q_j \in B_j$, so $\bigcup_{i\in\omega} B_i$ is dense in U. □

After the previous claim we can easily construct a dense subspace of X homeomorphic to $\mathbb{N}^\omega$. Let $B_\emptyset := X$ and on the first level, we set $U = X$ and $n = 1$. Applying Claim 3.1.20, we get an infinite sequence of pairwise disjoint nonempty balls $B_0, B_1, \ldots$ with radius at most 1, such that their union is dense in X. Suppose that we have defined open balls $B_s, s \in \mathbb{N}^{<\omega}$ for every s with $|s| \leq n$ with the following properties:

(i) if $s \prec t$, then $B_s \supset \overline{B_t}$,

(ii) B_s has radius at most $\frac{1}{|s|}$, when $s \neq \emptyset$,

(iii) for $|s| < n$ we have that $\bigcup_{i\in\omega} B_{s^\frown i}$ is dense in B_s.

We continue the construction for $n+1$ as follows: we apply Claim 3.1.20 for every B_s with $|s| = n$ to get a countable sequence of balls $B_{s^\frown i}$ $(i \in \omega)$ such that they are pairwise disjoint with their union dense in B_s and radii at most $\frac{1}{|s|+1}$. By construction it is easy to see that $\bigcup_{|s|=m} B_s$ is dense in X for every $m \leq n$.

We claim that $N := \bigcap_{n=0}^{\infty} \bigcup_{s\in\mathbb{N}^n} B_s$ is a dense subspace homeomorphic to $\mathbb{N}^\omega$. First notice that N is dense, since by the Baire category theorem the countable intersection of dense open sets is dense. We also have to observe that $N = \bigcup_{x\in\mathbb{N}^\omega} \bigcap_{n\in\omega} B_{x|_n}$. By the construction we have that $\forall x \in \mathbb{N}^\omega$:

$$\bigcap_{n\in\omega} B_{x|_n} = \bigcap_{n\in\omega} \overline{B_{x|_n}}$$

which consists of exactly one element, because the diameters tend to zero and at each finite level $\overline{B_{x|_n}}$ is nonempty. Now the above argument gives rise to the natural function $f : \mathbb{N}^\omega \longrightarrow X$:

$$f(x) \in \bigcap_{n\in\omega} B_{x|_n}.$$

By construction it can be easily seen that this function is injective and surjective onto N. We claim that f is continuous, to see this, fix $x \in \mathbb{N}^\omega$ and $\varepsilon > 0$. If $y \in \mathbb{N}^\omega$ such that $x|_n = y|_n$, then $d(f(x), f(y)) \leq \frac{2}{n}$, so if we take n large enough, such that $\frac{2}{n} < \varepsilon$ and we set $U := \{y \in \mathbb{N}^\omega : y|_n = x|_n\}$, then U is an open set such that $x \in U$ and $f(U) \subseteq B(f(x), \varepsilon)$.

We show that f^{-1} is also continuous. We fix an arbitrary $f(x)$ and n and we want to find a neighbourhood U of $f(x)$, such that whenever $f(y) \in U$ we have that $x|_n = y|_n$. Clearly, such a neighbourhood is $B_{x|_n} \cap N$. Hence f is a homeomorphism between $\mathbb{N}^\omega$ and N.

□

Theorem 3.1.21. *For every Polish space X we have that $\mathcal{UB}_{Polish}(X) = \mathcal{UB}_{\mathbb{N}^\omega}(X)$.*

Proof. The containment $\mathcal{UB}_{Polish}(X) \subseteq \mathcal{UB}_{\mathbb{N}^\omega}(X)$ is obvious, since $\mathbb{N}^\omega$ is Polish, so let us suppose that $A \in \mathcal{UB}_{\mathbb{N}^\omega}(X)$ and our goal is to prove that $A \in \mathcal{UB}_{Polish}(X)$. For this let Y be an arbitrary Polish space and $f : Y \longrightarrow X$ be an arbitrary continuous function. We need to prove that $f^{-1}(A)$ has the property of Baire.

Take a partition of Y in a form $Y = P \cup C$, where P and C are disjoint and P is a perfect subset and C is a countable open set (such a decomposition exists, see Theorem 2.1.20). If Y is countable, then each of its subsets have the property of Baire. Hence, we can assume that Y is uncountable. In this case the above P is nonempty.

It is enough to prove that $f^{-1}(A) \cap P$ has the property of Baire in Y, because $f^{-1}(A) \cap C$ is countable and by Lemma 3.1.13 it is enough to prove that $f^{-1}(A) \cap P$ has the property of Baire in P.

All in all, we have that if g denotes the restriction of f to P, it is enough to prove that $g^{-1}(A)$ has the property of Baire in P. Let N be a dense subspace of P homeomorphic to $\mathbb{N}^\omega$, by Lemma 3.1.19 such a subspace exists. $P \setminus N$ is meager, because N is completely metrizable, hence it is a $\mathcal{G}_\delta$ subspace of P (see Theorem 2.1.21), which is also dense, so it is enough to prove that $g^{-1}(A) \cap N$ has the property of Baire in P. Again, by Lemma 3.1.13, it is enough to prove that $g^{-1}(A) \cap N$ has the property of Baire in N. Denoting by h the restriction of g to N, we only need to prove that $h^{-1}(A) = g^{-1}(A) \cap N = f^{-1}(A) \cap Y \cap N$ has the property of Baire in N. But the latter is trivial, because h is also a continuous function from the space N homeomorphic to $\mathbb{N}^\omega$, hence $h^{-1}(A)$ has the property of Baire in N. □

We turn to the third row of Diagram 3.1.11, let us first note that the containments from the second row to the third row are obvious:

Fact 3.1.22. *$\mathcal{UB}_{Polish}(X) \subseteq \mathcal{UB}_{compact\ metric}(X)$ and $\mathcal{UB}_{compact\ Hausdorff}(X) \subseteq \mathcal{UB}_{Cantor}(X)$.*

Finally, we prove the last unproved equality from Diagram 3.1.11, namely:

Theorem 3.1.23. *For every Polish space X we have that $\mathcal{UB}_{compact\ metric}(X) = \mathcal{UB}_{Cantor}(X)$.*

Proof. The containment $\mathcal{UB}_{compact\ metric}(X) \subseteq \mathcal{UB}_{Cantor}(X)$ is trivial, because 2^ω is a compact metric space. To prove the other direction, we start with recalling an important definition:

Definition 3.1.24. *Let Z and Y be arbitrary topological spaces. A continuous surjective $f : Y \longrightarrow Z$ is called* irreducible *if for every proper closed $F \subseteq Y$ the image $f(F)$ is a proper closed subspace of Z.*

In [18, Lemma 2.1.] the following was proved about irreducible maps:

Lemma 3.1.25. *Let K be a perfect compact metric space, then there exists an irreducible $f : 2^\omega \longrightarrow K$.*

Let us consider in a lemma one of the most fruitful properties of an irreducible mapping:

Lemma 3.1.26. *Let $f : Y \longrightarrow Z$ be an irreducible mapping, then for every nowhere dense $N \subseteq Y$ we have that $f(N)$ is nowhere dense in Z. If $M \subseteq X$ is meager then $f(M)$ is meager in Y.*

Proof. Let N be a nowhere dense subset of Y and we want to prove that $f(N)$ is a nowhere dense subset of Z. For this, let $U \subseteq Z$ be a nonempty open set. Since f is surjective and continuous, $f^{-1}(U)$ is a nonempty open set in Y. By the nowhere density of N there exists a nonempty $V \subseteq f^{-1}(U)$, such that $N \cap V = \emptyset$.

In this case V^c is a proper closed subset of Y, hence $f(V^c)$ is a proper closed subset of Y and $f(N) \subseteq f(V^c)$. From this we can see that $f(V^c)^c$ is a nonempty open set, which is disjoint from $f(N)$ and $f(V^c)^c \subseteq f(V) \subseteq U$ (the first containment follows from the surjectivity of f). All in all, we have proved that for every nonempty open set in Y there exists a nonempty open subset, which is disjoint from $f(N)$, so $f(N)$ is nowhere dense. The last assertion of the lemma is trivial from the first assertion. □

After this short detour about irreducible mappings, let us turn to the proof of the harder part of the theorem. Suppose that $A \in \mathcal{UB}_{Cantor}(X)$ and our goal is to prove that $A \in \mathcal{UB}_{compact\ metric}(X)$. Take an arbitrary compact metric space K and an arbitrary continuous $g : K \longrightarrow X$, we need to check that $g^{-1}(A)$ has the property of Baire in K.

If K is a countable, then the statement is trivial, so we can assume that K is uncountable. Take a partition of K into a perfect and a countable set, that is, $K = P \cup C$, where C is a countable open set and P is perfect set (such a decomposition exists, see Theorem 2.1.20). If K is uncountable then also P is uncountable, especially it is nonempty.

Since C is countable, it is enough to prove that $g^{-1}(A) \cap P$ has the property of Baire in K. By Lemma 3.1.13, it is enough to prove that $g^{-1}(A) \cap P$ has the property of Baire in P. We denote by $\tilde{g}$ the restriction of g to P. By Lemma 3.1.25, there exists an irreducible map $f : 2^\omega \to P$ and since $\tilde{g} \circ f$ is continuous, by assumption $f^{-1}(\tilde{g}^{-1}(A))$ has the property of Baire in 2^ω, so we can write $f^{-1}(\tilde{g}^{-1}(A)) = B \cup M$, where B is Borel and M is meager. Since f was also surjective, we have that $g^{-1}(A) \cap P = f(f^{-1}(\tilde{g}^{-1}(A))) =$

$f(B) \cup f(M)$. We have that $f(B)$ is an analytic set (see [12, Theorem 14.2.]), so it has the property of Baire (see [12, Theorem 21.6.]) and $f(M)$ is meager by Lemma 3.1.26, so $f(B) \cup f(M)$ has the property of Baire in P. □

Now we prove that none of the containments can be reversed in ZFC in Diagram 3.1.11 (apart from the equalities, where the containments are already reversed). We will need the following lemma:

Lemma 3.1.27. *Let $\kappa < \mathrm{add}(\mathcal{M})$ be a cardinal and let X be a Polish space and suppose that we are given a collection of subsets $\mathcal{H} = \{B_\alpha \mid \alpha < \kappa\}$ of X such that every member of $\mathcal{H}$ has the property of Baire, then $\bigcup_{\alpha<\kappa} B_\alpha$ has the property of Baire.*

Proof. First, we assume that $X = \mathbb{N}^\omega$. For every $\alpha < \kappa$ there exists an open set U_α and a meager set M_α such that $B_\alpha = U_\alpha \triangle M_\alpha$. It is easy to see that $\bigcup_{\alpha<\kappa} B_\alpha \triangle \bigcup_{\alpha<\kappa} U_\alpha \subseteq \bigcup_{\alpha<\kappa} M_\alpha$, but $\bigcup_{\alpha<\kappa} M_\alpha$ is a meager set, hence $\bigcup_{\alpha<\kappa} B_\alpha$ differs from the open set $\bigcup_{\alpha<\kappa} U_\alpha$ by a meager set so $\bigcup_{\alpha<\kappa} B_\alpha$ has the property of Baire.

Now we consider the case, when X is an arbitrary Polish space. First, if X is countable, then each of its subset has the property of Baire, especially $\bigcup_{\alpha<\kappa} B_\alpha$ also has the property of Baire, so we can assume that X is uncountable. We can take the partition of X into a nonempty perfect set P and a countable open set C (see Theorem 2.1.20). We will consider two cases, first we assume that $\operatorname{int} P = \emptyset$, then C is a nonempty open subset and P is a nowhere dense subset of X. Since $B_\alpha \cap C$ has the property of Baire in X and since C is an open set, it is easy to see that $B_\alpha \cap C$ has the property of Baire in C, too. Since C is countable we can conclude that $\bigcup_{\alpha<\kappa} (B_\alpha \cap C)$ has the property of Baire in C and by Lemma 3.1.13 $\bigcup_{\alpha<\kappa} (B_\alpha \cap C)$ has the property of Baire in X. Adding the nowhere dense set $\bigcup_{\alpha<\kappa} (B_\alpha \cap P)$ to $\bigcup_{\alpha<\kappa} (B_\alpha \cap C)$ we can see that $\bigcup_{\alpha<\kappa} B_\alpha$ has the property of Baire in X.

The second case is when we assume that $\operatorname{int} P \neq \emptyset$. By the first case, we can see that $\bigcup_{\alpha<\kappa} (B_\alpha \cap C)$ has the property of Baire and since ∂P is a nowhere dense set, it is enough to prove that $(\bigcup_{\alpha<\kappa} B_\alpha) \cap \operatorname{int} P = \bigcup_{\alpha<\kappa} (B_\alpha \cap \operatorname{int} P)$

has the property of Baire. By Lemma 3.1.13 it is enough to prove that $\bigcup_{\alpha<\kappa}(B_\alpha \cap \operatorname{int} P)$ has the property of Baire in the subspace $\operatorname{int} P$.

For brevity, let $D_\alpha := B_\alpha \cap \operatorname{int} P$. We first check that D_α has the property of Baire in $\operatorname{int} P$. Since D_α has the property of Baire in X, it can be written as $D_\alpha = F_\alpha \cup M_\alpha$, where F_α is a Borel set in X and M_α is a meager set in X. It is clear that F_α is also a Borel set in $\operatorname{int} P$ and if we can prove that M_α is meager in $\operatorname{int} P$, then D_α has the property of Baire in $\operatorname{int} P$.

To see that M_α is meager in $\operatorname{int} P$, it suffices to prove that whenever $N \subseteq \operatorname{int} P$ is a nowhere dense set in X, then N is also a nowhere dense set in $\operatorname{int} P$. Any nonempty open set of $\operatorname{int} P$ is of the form $V \cap \operatorname{int} P$ for some nonempty open set V in X, for this V we have that $V \cap \operatorname{int} P$ is a nonempty open set in X and since N is a nowhere dense set, there exists a nonempty open subset W of X, such that $W \subseteq V \cap \operatorname{int} P$, which is disjoint from N. This W is also a nonempty open subset of $\operatorname{int} P$ which is disjoint from N, hence N is also a nowhere dense subset of $\operatorname{int} P$. All in all, we have proved that D_α has the property of Baire in $\operatorname{int} P$.

The last step to finish the proof is to prove that $\bigcup_{\alpha<\kappa} D_\alpha$ has the property of Baire in $\operatorname{int} P$. Since P is a perfect subset of X we have that $\operatorname{int} P$ also does not have isolated points and since $\operatorname{int} P$ is open, we have that $\operatorname{int} P$ is Polish by Theorem 2.1.21. By Lemma 3.1.19 there exists a dense subspace $\mathcal{N}$ homeomorphic to $\mathbb{N}^\omega$, which is automatically a $\mathcal{G}_\delta$ subspace of $\operatorname{int} P$ (see Theorem 2.1.21). Since $\mathcal{N}$ is a dense $\mathcal{G}_\delta$ subspace, $\operatorname{int} P \setminus \mathcal{N}$ is a meager subset of $\operatorname{int} P$, so it is enough to prove that $\bigcup_{\alpha<\kappa}(D_\alpha \cap \mathcal{N})$ has the property of Baire in $\operatorname{int} P$ and by Lemma 3.1.13 it is enough to prove that $\bigcup_{\alpha<\kappa}(D_\alpha \cap \mathcal{N})$ has the property of Baire in $\mathcal{N}$.

We are almost done, because in $\mathbb{N}^\omega$ we have seen that the union of κ many ($\kappa < \operatorname{add}(\mathcal{M})$) sets with the property of Baire also has the property of Baire, so we need to check that for every $\alpha < \kappa$ the set $D_\alpha \cap \mathcal{N}$ has the property of Baire in $\mathcal{N}$. This will be very simple, because D_α can be written as $D_\alpha = A_\alpha \cup L_\alpha$, where A_α is a Borel set in $\operatorname{int} P$ and L_α is a meager set in $\operatorname{int} P$. It is clear that $A_\alpha \cap \mathcal{N}$ is a Borel set in $\mathcal{N}$ and we will prove that $M_\alpha \cap \mathcal{N}$ is a meager set in $\mathcal{N}$ which is enough to conclude that $D_\alpha \cap \mathcal{N}$ has the property of Baire in $\mathcal{N}$. To see the meagerness of $M_\alpha \cap \mathcal{N}$, it is enough to prove that whenever $L \subseteq \mathcal{N}$ is a nowhere dense subset of $\operatorname{int} P$, then it is also a nowhere dense set in $\mathcal{N}$. Let $L \subseteq \mathcal{N}$ be a nowhere dense subset of $\operatorname{int} P$ and take an arbitrary nonempty open set U in $\mathcal{N}$. Since U is open in $\mathcal{N}$, there exists a nonempty open subset V of $\operatorname{int} P$ such that $U = V \cap \operatorname{int} P$. Since L is a nowhere dense set, there exists a nonempty open set W such that $W \subseteq V$ and W is disjoint from L, but $\mathcal{N}$ is a dense subspace of $\operatorname{int} P$, hence $W \cap \mathcal{N}$ is a nonempty open set in $\mathcal{N}$ such that $W \cap \mathcal{N} \subseteq U$ and it is disjoint from L, so L is a nowhere dense subset of $\mathcal{N}$. □

We give examples which show that no more containments are provable in ZFC in Diagram 3.1.11. For the definition of *add*($\mathcal{M}$) and *add*($\mathcal{N}$), see Definitions 2.2.8, 2.2.9 and see also Fact 2.2.10.

Theorem 3.1.28. *Suppose that* $\omega_1 = \operatorname{add}(\mathcal{N}) < \operatorname{add}(\mathcal{M}) = \omega_2$*, then we have that* $\mathcal{UB}_{Polish}([0,1]) \not\subseteq \mathcal{UB}_{\check{C}ech\text{-}complete}([0,1])$.

Proof. We first mention that there exists a model of set theory, where $\omega_1 = \operatorname{add}(\mathcal{N}) < \operatorname{add}(\mathcal{M}) = \omega_2$ holds (see [1, Model 7.6.9.]). Throughout the proof, we will denote by $\lambda(A)$ the Lebesgue measure of a set $A \subseteq \mathbb{R}$.

Since $\operatorname{add}(\mathcal{N}) = \omega_1$, there exists $\{N_\alpha \mid \alpha < \omega_1\}$ with $\lambda(N_\alpha) = 0$ for every $\alpha < \omega_1$ such that $N := \bigcup_{\alpha<\omega_1} N_\alpha$ has positive outer measure. We can cover each N_α by a Borel null set $\tilde{B}_\alpha$. We will make the $\tilde{B}_\alpha$-s disjoint by setting $B_\alpha := \tilde{B}_\alpha \setminus \bigcup_{\beta<\alpha} \tilde{B}_\beta$. Note, that for every $\alpha < \omega_1$ the set B_α is a Borel null set. Now our goal is to find $E \subseteq \omega_1$, such that $\bigcup_{\alpha\in E} B_\alpha$ is not measurable.

Suppose for a contradiction that $\bigcup_{\alpha\in E} B_\alpha$ is measurable for every $E \subseteq \omega_1$. We will show that in this case we can define a finite measure μ on all subsets of ω_1, which vanishes on singletons. In [15, Theorem 5.6.] it was proved that if μ is a finite measure on all subsets of ω_1, which vanishes on singletons, then μ is identically zero. If $E \subseteq \omega_1$, then we define the measure μ of E as:

$$\mu(E) := \lambda\Big(\bigcup_{\alpha\in E} B_\alpha\Big).$$

We clearly defined a finite measure μ on $\mathcal{P}(\omega_1)$ (the σ-additivity of this measure follows from the σ-additivity of the Lebesgue measure), which also vanishes on singletons. But $\mu(\omega_1) = \lambda(B) > 0$, because $\lambda(B)$ has positive outer measure, contradiction.

So there must be a set $E \subseteq \omega_1$, such that $C := \bigcup_{\alpha\in E} B_\alpha$ is not Lebesgue measurable. To see that $C \in \mathcal{UB}_{Polish}([0,1])$, take an arbitrary Polish space Y and a continuous $f : Y \longrightarrow [0,1]$, we need to check that $f^{-1}(C)$ has the property of Baire. But this is clear from Lemma 3.1.27, because $f^{-1}(B_\alpha)$ is a Borel set for every $\alpha < \omega_1$ and by assumption $\omega_1 < \mathrm{add}(\mathcal{M})$, hence $f^{-1}(C) = \bigcup_{\alpha\in E} f^{-1}(B_\alpha)$ also have the property of Baire.

On the other hand, it was proved in [10, 1C Proposition] that if $D \in \mathcal{UB}_{\text{Čech-complete}}(\mathbb{R})$, then D is Lebesgue measurable. It is easy to see that if $C \in \mathcal{UB}_{\text{Čech-complete}}([0,1])$, then also $C \in \mathcal{UB}_{\text{Čech-complete}}(\mathbb{R})$ holds, but this contradicts the previously mentioned result, so $C \notin \mathcal{UB}_{\text{Čech-complete}}([0,1])$. □

The following theorem is a consequence of the previous theorem:

Theorem 3.1.29. *For Polish spaces the following containments are* not *provable in ZFC:* $\mathcal{UB}_{Polish}(X) \subseteq \mathcal{UB}_{\check{C}ech-complete}(X)$, $\mathcal{UB}_{Polish}(X) \subseteq$ $\subseteq \mathcal{UB}_{compact\ Hausdorff}(X)$.

Proof. The first assertion of the theorem follows from the previous theorem and to see the second assertion, it is enough to note that in previous theorem the Polish space was $X = [0,1]$ and by Theorem 3.1.17 we have that $\mathcal{UB}_{\text{Čech-complete}}([0,1]) = \mathcal{UB}_{\text{compact Hausdorff}}([0,1])$. □

We will now prove that being a compact Hausdorff-universally Baire set cannot guarantee in ZFC to be a Polish-universally Baire set.

Theorem 3.1.30. *Assuming the Continuum Hypothesis, there exists an uncountable set $A \subseteq \mathbb{Z}^\omega$, such that $C \cap A$ is countable for every compact set $C \subseteq \mathbb{Z}^\omega$. For such a set A we have that for every compact space K and for every continuous $f : K \longrightarrow \mathbb{Z}^\omega$, $f^{-1}(A)$ is F_σ in K, but A does not have the property of Baire in $\mathbb{Z}^\omega$, especially A is a compact Hausdorff-universally Baire set, but not a Polish-universally Baire set.*

Proof. The construction is simply the so-called Sierpiński set. First, take any well-ordering of the nowhere dense closed subsets of $\mathbb{Z}^\omega$: $\{C_\alpha \mid \alpha < \omega_1\}$ (we used here the assumption $2^\omega = \omega_1$). It is easy to see that by transfinite recursion, we can pick for each $\alpha < \omega_1$ a point p_α, such that $p_\beta \neq p_\alpha$ for every $\beta < \alpha$ and $p_\alpha \notin \bigcup_{\beta \leq \alpha} C_\alpha$, we only need to remark that p_α can be chosen to be not an element of $\bigcup_{\beta \leq \alpha} C_\alpha \cup \bigcup_{\beta < \alpha} \{p_\beta\}$, but the latter set is meager so such a choice of p_α exists. Let $A := \{p_\alpha \mid \alpha < \omega_1\}$.

We have that A is uncountable but A has countable intersection with any of the sets C_α. Taking the union of countably many C_α-s still have countable intersection with A. We know that any meager set can be covered by countably many nowhere dense closed sets, so any meager set also has countable intersection with A. Using the well-known fact that every compact subspace C of $\mathbb{Z}^\omega$ is nowhere dense, we get the first part of the example (namely $C \cap A$ is countable for any compact $C \subseteq \mathbb{Z}^\omega$).

Take any compact space K and a continuous $f : K \longrightarrow \mathbb{Z}^\omega$. Then $f(K) \cap A$ is countable, since $f(K)$ is compact (it is the continuous image of a compact space). It is clear that $f^{-1}(A)$ has the property of Baire, because it is an F_σ set, so A is a compact Hausdorff-universally Baire set.

We also have that A cannot have the property of Baire, because if we suppose for a contradiction that $A = U \triangle M$ for some open U and meager M, then U must be empty (otherwise we could construct an uncountable closed nowhere dense set in U disjoint from M). On the other hand if A were a meager set, then A itself would be countable, as A has countable intersection with every meager set, but A is uncountable, a contradiction.

The identity function id on $\mathbb{Z}^\omega$ shows that $id^{-1}(A)$ does not have the property of Baire, so A is not a Polish-universally Baire set. □

The following theorem is a trivial consequence of the previous theorem:

Theorem 3.1.31. *For Polish spaces the following containments are* not *provable in ZFC:* $\mathcal{UB}_{\text{compact Hausdorff}}(X) \subseteq \mathcal{UB}_{Polish}(X)$, $\mathcal{UB}_{\text{compact Hausdorff}}(X) \subseteq \mathcal{UB}_{\check{C}ech-complete}(X)$.

3.2 The case of σ-compact Polish spaces

As we have seen before, Diagram 3.1.11 is quite complicated, in this subsection we will prove that when X is a σ-compact Polish space, the diagram becomes very simple, the aim of this subsection is to prove the following:

Theorem 3.2.1. *Let X be a σ-compact Polish space. Then the relationship between the various notions of universally Baire sets in X are the containments of the diagram below and no more containments are provable in ZFC.*

$$\begin{array}{ccccccc} \mathcal{UB}_{topological}(X) & = & \mathcal{UB}_{\forall\lambda^\omega}(X) & = & \mathcal{UB}_{\check{C}ech-complete}(X) & = & \mathcal{UB}_{compact\ Hausdorff}(X) \\ & & & \cap\!| \quad \not\vdash \quad Con & & & \\ \mathcal{UB}_{\mathbb{N}^\omega}(X) & = & \mathcal{UB}_{Polish}(X) & = & \mathcal{UB}_{compact\ metric}(X) & = & \mathcal{UB}_{Cantor}(X) \end{array}$$

Diagram 3.2.1 (Implications in a σ-compact Polish space X.)

We will prove Diagram 3.2.1 in two steps. First, we will assume that the space X is also compact, after that we will move towards the σ-compact case. We start with the first row of Diagram 3.2.1:

Theorem 3.2.2. *Let X be a compact metric space, then $\mathcal{UB}_{\check{C}ech-complete}(X) = \mathcal{UB}_{compact\ Hausdorff}(X)$.*

Proof. Since X is a compact Hausdorff space, by Theorem 3.1.17 we have that $A \in \mathcal{UB}_{\check{C}ech-complete}(X)$ is equivalent with $A \in \mathcal{UB}_{\text{compact Hausdorff}}(X)$. □

We will now weaken the compactness assumption:

Theorem 3.2.3. *Let X be a σ-compact Polish space, then $\mathcal{UB}_{\check{C}ech-complete}(X) = \mathcal{UB}_{compact\ Hausdorff}(X)$.*

Proof. The containment $\mathcal{UB}_{\check{C}ech-complete}(X) \subseteq \mathcal{UB}_{\text{compact Hausdorff}}(X)$ is always true, so let us turn to the reverse direction and assume that $A \in \mathcal{UB}_{\text{compact Hausdorff}}(X)$ and our goal is to derive that $A \in \mathcal{UB}_{\check{C}ech-complete}(X)$. Let Y be a Čech-complete space and take an arbitrary continuous $f : Y \longrightarrow X$, we need to check that $f^{-1}(A)$ has the property of Baire in Y.

Let $X = \bigcup_{n\in\omega} C_n$, where all the C_n-s are compact subspaces of X. We can assume that all the C_n-s are nonempty. First note that by Lemma 3.1.12 $C_n \cap A \in \mathcal{UB}_{\text{compact Hausdorff}}(C_n)$ for all n, hence by Theorem 3.2.2 $C_n \cap A \in \mathcal{UB}_{\text{Čech-complete}}(C_n)$.

Fix n and let f_n denote the restriction of f to $f^{-1}(C_n)$. Since C_n is a $\mathcal{G}_\delta$ subspace of X, $f^{-1}(C_n)$ is a $\mathcal{G}_\delta$ subspace of Y, hence $f^{-1}(C_n)$ is also a Čech-complete space, so $f_n^{-1}(C_n \cap A) = f^{-1}(C_n) \cap f^{-1}(A)$ has the property of Baire in $f^{-1}(C_n)$. Since $f^{-1}(C_n)$ is a $\mathcal{G}_\delta$ subspace, it has the property of Baire in Y, so by Lemma 3.1.13 $f^{-1}(C_n) \cap f^{-1}(A)$ has the property of Baire in Y, and since the sets with the property of Baire form a σ-algebra, $f^{-1}(A)$ has the property of Baire in Y. □

To see that $\mathcal{UB}_{Polish}(X) \subseteq \mathcal{UB}_{\text{compact Hausdorff}}(X)$ is not provable in ZFC, even when the space X is compact, we only need to remember that under certain extra set-theoretic assumption we constructed a set $A \subseteq [0,1]$ such that $A \in \mathcal{UB}_{Polish}([0,1])$ but $A \notin \mathcal{UB}_{\check{C}ech-complete}([0,1])$ in Theorem 3.1.28. On the other hand we know by Theorem 3.2.3 that $\mathcal{UB}_{\check{C}ech-complete}([0,1]) = \mathcal{UB}_{\text{compact Hausdorff}}([0,1])$.

Let us continue with the second row of Diagram 3.2.1:

Theorem 3.2.4. *Let X be a compact metric space, then $\mathcal{UB}_{Polish}(X) = \mathcal{UB}_{compact\ metric}(X)$.*

Proof. The containment $\mathcal{UB}_{Polish}(X) \subseteq \mathcal{UB}_{compact\ metric}(X)$ is trivial, so let us suppose that $A \in \mathcal{UB}_{compact\ metric}(X)$ and our goal is to prove that in this case $A \in \mathcal{UB}_{Polish}(X)$. Take an arbitrary Polish space Y and an arbitrary continuous $f : Y \longrightarrow X$, we need to check that $f^{-1}(A)$ has the property of Baire in Y.

First, we identify X with a compact (equivalently a closed) subspace of the Hilbert cube $[0,1]^\omega$ (this is a well-known result, see [12, Theorem 4.14.]). Then the function f has the form $f = (f_n)_{n\in\omega}$, where f_n is a continuous function $f_n : Y \to [0,1]$ for every n.

We choose a metric d on Y compatible with its topology with $d \leq 1$. Let

$(y_n)_{n\in\omega}$ be dense in Y and let g_n denote the continuous function

$$g_n : Y \to [0,1],\ g_n(y) := d(y, y_n).$$

It is easy to see that the g_n-s separate the points, that is $\forall y \neq y' \in Y\ \exists n$, such that $g_n(y) \neq g_n(y')$. We consider the following embedding of the space Y into the Hilbert cube $\imath : Y \to [0,1]^\omega$:

$$\imath :\ y \mapsto (f_0(y), g_0(y), f_1(y), g_1(y), \dots).$$

It is easy to see that $\imath$ is really an embedding (or see [12, Theorem 4.14.]), so from now on we identify Y with $\imath(Y)$.

Now here comes the beautiful observation, after this identification f takes the point $(f_0(y), g_0(y), f_1(y), g_1(y), \dots)$ to $(f_0(y), f_1(y), \dots)$, in other words, f is simply a projection on the even coordinates. So we can extend f to a $\tilde{f} : [0,1]^\omega \to [0,1]^\omega$ by the formula

$$\tilde{f} :\ (x_0, y_0, x_1, y_1, \dots) \mapsto (y_0, y_1, \dots)$$

which is clearly continuous. By the continuity of $\tilde{f}$ we have that $\tilde{f}(\overline{Y}) \subseteq \overline{f(Y)} \subseteq X$, where for the latter containment we used the fact that X is closed in $[0,1]^\omega$.

We will denote by h the restriction of the function $\tilde{f}$ to $\overline{Y}$. Since $\overline{Y}$ is compact and $h : \overline{Y} \longrightarrow X$ is continuous and $A \in \mathcal{UB}_{compact\ metric}(X)$, we have that $h^{-1}(A)$ has the property of Baire in $\overline{Y}$ and by the fact that Y is comeager in $\overline{Y}$ we have that $h^{-1}(A) \cap Y = f^{-1}(A)$ has the property of Baire in Y, which we wanted to prove. □

From compact Polish spaces we can easily go to σ-compact Polish spaces:

Theorem 3.2.5. *Let X be a σ-compact Polish space, then $\mathcal{UB}_{Polish}(X) = \mathcal{UB}_{compact\ metric}(X)$.*

Proof. The containment $A \subseteq \mathcal{UB}_{compact\ metric}(X)$ is trivial, so let us suppose that $A \in \mathcal{UB}_{compact\ metric}(X)$. To prove that in this case $A \in \mathcal{UB}_{Polish}(X)$ is also true, take an arbitrary Polish space Y and a continuous $f : Y \longrightarrow X$, we need to check that $f^{-1}(A)$ has the property of Baire.

Let $X = \bigcup_{n\in\omega} C_n$, where all the C_n-s are compact subspaces of X (we can assume that every C_n is nonempty). By Lemma 3.1.12 we have that $C_n \cap A$ is a compact metric-universally Baire set in C_n for every n, so by Theorem 3.2.4 $C_n \cap A$ is also Polish-universally Baire in C_n. Let f_n denote the restriction of f to $f^{-1}(C_n)$. Since $f^{-1}(C_n)$ is a closed set, it is a Polish subspace. Since $C_n \cap A$ is Polish-universally Baire, $f_n^{-1}(C_n \cap A) = f^{-1}(C_n) \cap f^{-1}(A)$ has the property of Baire in $f^{-1}(C_n)$, hence by Lemma 3.1.13 $f^{-1}(C_n) \cap f^{-1}(A)$ has the property of Baire in Y. □

With the proof of Theorem 3.2.5 we finished the proof of Diagram 3.2.1. We will now prove Theorem 3.1.17, which we promised in Subsection 3.1. We will use the theory of Boolean algebras and Stone spaces in the proof of Theorem 3.1.17 and we assume that the reader is familiar with these notions. We refer the reader to [8, Chapters 1-3]. We recall what the theorem says:

Theorem 3.2.6. *Let X be a compact Hausdorff space. The following are equivalent for $A \subseteq X$:*

(i) $A \in \mathcal{UB}_{\check{C}ech-complete}(X)$,

(ii) $A \in \mathcal{UB}_{topological}(X)$,

(iii) $f^{-1}(A)$ has the property of Baire in Z, whenever Z is an extremally disconnected compact Hausdorff space and $f : Z \longrightarrow X$ is a continuous function.

In fact, in [10] there is a fourth equivalent condition in the previous theorem, but we will give only the proof of the above mentioned three equivalent conditions. The fourth condition sounds as:

Theorem 3.2.7. *Let X be a compact Hausdorff space. For $A \subseteq X$ we have that $A \in \mathcal{UB}_{\check{C}ech-complete}(X)$ if and only if there exists a compact Hausdorff space K and a continuous surjection $f : K \longrightarrow X$, such that $f^{-1}(A) \in \mathcal{UB}_{\check{C}ech-complete}(K)$.*

Proof. (of Theorem 3.2.6) The implication $(ii) \Rightarrow (i)$ is trivial, because for an arbitrary topological space X we have that $\mathcal{UB}_{topological}(X) \subseteq \mathcal{UB}_{\check{C}ech-complete}(X)$.

The implication $(i) \Rightarrow (iii)$ follows from the simple fact that every compact Hausdorff space is Čech-complete.

We turn to the proof of implication $(iii) \Rightarrow (ii)$. Let W be an arbitrary topological space and $f : W \longrightarrow X$ be an arbitrary continuous function. We need to check that $f^{-1}(A)$ has the property of Baire in W.

Let $\mathcal{E} := \{A \subseteq W \mid \operatorname{int} \overline{\partial A} = \emptyset\}$, or in other words, $\mathcal{E}$ is the collection of subset of W with nowhere dense boundary. It is well-known and easy to check that the family $\mathcal{E}$ is a field, hence $\mathcal{E}$ forms a Boolean algebra with the usual set-theoretic operations $(\emptyset, W, \cup, \cap, ^c)$. Note that $\mathcal{E}$ contains the open and closed sets.

We denote by $RO(W)$ the regular open algebra of W and we will also need the space $Z := St(RO(W))$, the Stone space of the regular open algebra of W. Since $RO(W)$ is a complete Boolean algebra, in this case it is well-known that Z is an extremally disconnected space (see Definition 2.2.6, in fact the Stone space of a Boolean algebra is extremally disconnected if and only if the Boolean algebra is complete). We denote by $\Delta^0_1(Z)$ the clopen sets of Z and recall that if for $U \in RO(W)$ we introduce the following:

$$T_U := \{\mathcal{U} \mid \mathcal{U} \text{ is an ultrafilter on } RO(W) \text{ and } U \in \mathcal{U}\}, \tag{3.1}$$

then these sets form the usual clopen basis of Z.

We denote by α the canonical Boolean isomorphism between the regular open algebra $RO(W)$ and the algebra of clopen sets $\Delta^0_1(Z)$:

$$\alpha : RO(W) \longrightarrow \Delta^0_1(Z),\ U \mapsto T_U.$$

For $A \in \mathcal{E}$, let $A^* := \alpha(\operatorname{int}(\overline{A}))$. Note that the previous expression makes sense because the interior of a closed set is always a regular open set, hence we can apply the Boolean isomorphism α to $\operatorname{int}(\overline{A})$. We will use the well-known result that $A \mapsto A^*$ is a Boolean homomorphism between $\mathcal{E}$ and $\Delta^0_1(Z)$.

We now define $g \subseteq Z \times X$ and our goal is to show that actually g defines a continuous function $g : Z \longrightarrow X$. Set

$$g := \bigcap_{F \subseteq X,\, F \text{ closed}} (Z \times F) \cup ((Z \setminus (f^{-1}(F))^*) \times X) = \tag{3.2}$$

$$= \bigcap_{G \subseteq X,\, G \text{ open}} (Z \times (X \setminus G)) \cup ((f^{-1}(G))^*) \times X).$$

By the first (defining) equality, we can see that g is a closed subset of $Z \times X$. To see the second equality, we need to check that if $F \subseteq X$ is closed and $G = X \setminus F$, then $Z \setminus (f^{-1}(F))^* = (f^{-1}(G))^*$, but this is clear because * is a homomorphism between $\mathcal{E}$ and $\Delta_1^0(Z)$ and $f^{-1}(G) = W \setminus f^{-1}(F)$.

Claim 3.2.8. *For every $z \in Z$ there exists a unique $x \in X$ such that $(z, x) \in g$. In other words, g is a function (it is customary in set theory that we identify a function with its graph).*

Proof. We fix an arbitrary $z \in Z$ and we define the following subcollection of closed sets:

$$L_z := \{F \mid F \subseteq X \text{ closed and } z \in (f^{-1}(F))^*\}. \tag{3.3}$$

We will check that the elements of L_z have the finite intersection property, hence their intersection will be nonempty. Every element of L_z is nonempty, so it is enough to see that if $F_1, F_2 \in L_z$, then $F_1 \cap F_2 \in L_z$ and by induction we get that if $F_1, ..., F_n \in L_z$, also $F_1 \cap ... \cap F_n \in L_z$.

Suppose that $F_1, F_2 \in L_z$, to establish that $F_1 \cap F_2 \in L_z$, by definition we need that $z \in (f^{-1}(F_1 \cap F_2))^*$. Using that * is a homomorphism we get that $(f^{-1}(F_1 \cap F_2))^* = (f^{-1}(F_1) \cap f^{-1}(F_2))^* = (f^{-1}(F_1))^* \cap (f^{-1}(F_2))^*$.

We can see now that $\bigcap L_z$ is not empty because X is compact. We should now derive that for every $x \in L_z$ we have that $(z, x) \in g$. Recall that in the definition of g, we intersected elements of the form $(Z \times F) \cup ((Z \setminus (f^{-1}(F))^*) \times X)$, where F was closed. So by fixing such a closed $F \subseteq X$, if $x \in F$, we are done, so suppose that $x \notin F$, in this case we need to prove that $z \in Z \setminus (f^{-1}(F))^*$. Suppose for a contradiction that $z \in (f^{-1}(F))^*$, then

$F \in L_z$ so $x \in F$, contradiction.

So we have proved half of the lemma, namely that for any $z \in Z$ there exists some x (in fact, every $x \in L_z$ would be good by the above chain of ideas) such that $(z, x) \in g$. To show the uniqueness of this x, suppose for a contradiction that $\exists x_0 \neq x_1 \in X$ such that $(z, x_0), (z, x_1) \in g$.

Since X was Hausdorff, there are closed sets F_0, F_1, such that $x_i \notin F_i$ and $F_0 \cup F_1 = X$. For such F_i-s we have that $W = f^{-1}(F_0) \cup f^{-1}(F_1)$ and $Z = (f^{-1}(F_0))^* \cup (f^{-1}(F_1))^*$. If $z \in (f^{-1}(F_i))^*$, then $(z, x_i) \notin g$ by (3.2), contradiction. □

We have proved that g is a $Z \longrightarrow X$ function. To see that it is continuous, recall the well-known fact (see [14, Exercise 8, page 171]) that if A, B are topological spaces with B compact Hausdorff and if $graph(h) \subseteq A \times B$ is closed for a function $h : A \longrightarrow B$, then h is continuous. The graph of g is closed because we defined g as an intersection of closed sets in (3.2). We are in a favourable position now, because we have a continuous function $g : Z \longrightarrow X$, where Z is an extremally disconnected compact Hausdorff space. By assumption of the theorem, we have that $g^{-1}(A)$ has the property of Baire in Z. We want to conclude from this that $f^{-1}(A)$ has the property of Baire in W.

We remind the reader that if we take the Stone space of a complete Boolean algebra, then the regular open sets of the Stone space are exactly the clopen sets. Since $g^{-1}(A)$ has the property of Baire, there exists a regular open set O (by Theorem 2.2.4), which is also clopen such that $O \triangle g^{-1}(A)$ is meager in Z. Hence, there exists an element $E \in RO(W)$, such that $O = E^*(= \alpha(E))$ and there are dense open sets $\{Z_n \mid n \in \omega\}$ in Z such that $(E^* \triangle g^{-1}(A)) \cap \bigcap_{n \in \omega} Z_n = \emptyset$.

Our goal is now to derive that $E \triangle f^{-1}(A)$ is meager in W. For this, let us define the following open sets in W:

$$V_n := \cup\{U \mid U \subseteq W,\ U \text{ is open and } U^* \subseteq Z_n\}. \tag{3.4}$$

Claim 3.2.9. *For every n we have that V_n is an open dense set in W.*

Proof. By definition, all V_n-s are open sets, so suppose for a contradiction that there is a nonempty open set U such that $U \subseteq W \setminus V_n$. It is easy to see that we can choose U to be a regular open set (take $\text{int}(\overline{U})$ instead of U). Since U is nonempty, U^* is also nonempty, therefore $U^* \cap Z_n$ is a nonempty open set. Since $U^* \cap Z_n$ is a nonempty open set, we can choose a standard basic clopen set $T_G \subseteq U^* \cap Z_n$ for some nonempty regular open set $G \in RO(W)$ (see (3.1) above).

Since $T_G = G^* \subseteq U^*$ and * is a Boolean isomorphism between $RO(W)$ and $\Delta^0_1(Z)$ we have that $G \subseteq U$, so G is disjoint from V_n. On the other hand $G^* \subseteq Z_n$, so by the definition of V_n we have that $G \subseteq V_n$, a contradiction. □

Claim 3.2.10. $(E \setminus f^{-1}(A)) \cap (\bigcap_{n\in\omega} V_n) = \emptyset$.

Proof. Suppose for a contradiction that there is an $x \in E \cap \bigcap_{n\in\omega} V_n$, but $x \notin f^{-1}(A)$. We consider the following collection of clopen sets in Z:

$$\mathcal{H} := \{U^* \mid U \subseteq W,\ U \text{ is open, } x \in U\}.$$

The elements of $\mathcal{H}$ have the finite intersection property, because if we take $U_1^*, ..., U_n^* \in \mathcal{H}$, then $x \in U_1 \cap ... \cap U_n$, so $(U_1 \cap ... \cap U_n)^* = U_1^* \cap ... \cap U_n^*$ is nonempty. Since Z is compact, we have that $\cap\mathcal{H} \neq \emptyset$. Pick an arbitrary $z \in \cap\mathcal{H}$, we must have that $z \in E^*$, because $x \in E$ and it means that $E^* \in \mathcal{H}$.

For all n we have that $x \in V_n$ and by (3.4) there exists an open set U such that $x \in U$ and $U^* \subseteq Z_n$, but the latter also means that $z \in Z_n$. So we have a z such that $z \in E^* \cap \bigcap_{n\in\omega} Z_n$ and we have that $z \in g^{-1}(A)$, because $(E^* \triangle g^{-1}(A)) \cap \bigcap_{n\in\omega} Z_n = \emptyset$.

From $x \notin f^{-1}(A)$ we get that $f(x) \neq g(z)$, so there are disjoint open sets G, H in X such that $f(x) \in G$ and $g(z) \in H$. We know that $x \in f^{-1}(G)$, so $z \in (f^{-1}(G))^*$, but we also claim that $z \in (f^{-1}(H))^*$. If $z \notin (f^{-1}(H))^*$, then $(z, g(z)) \in g$ would fail, because $(z, g(z)) \in g$ implies either $(z, g(z)) \in Z \times (X \setminus H)$ or $(z, g(z)) \in (f^{-1}(H))^* \times X$.

All in all we have that

$$z \in (f^{-1}(G))^* \cap (f^{-1}(H))^* = (f^{-1}(G) \cap f^{-1}(H))^* = \emptyset,$$

contradiction. □

Claim 3.2.11. $(\mathrm{int}(W \setminus E) \setminus f^{-1}(X \setminus A)) \cap (\bigcap_{n\in\omega} V_n) = \emptyset$.

Proof. The proof is similar to the proof of the previous claim, we have to define similarly $\mathcal{H}$ as before and do the same reasoning as before with the appropriate modifications. We leave it to the reader. □

By the previous two claims we get that $\mathrm{int}(W \setminus E) \setminus f^{-1}(X \setminus A)$ and $E \setminus f^{-1}(A)$ are disjoint from $\bigcap_{n\in\omega} V_n$, so they are meager in W. Summarizing,

$$E \triangle f^{-1}(A) = (E \setminus f^{-1}(A)) \cup (f^{-1}(A) \setminus E) \subseteq$$

$$\subseteq (E \setminus f^{-1}(A)) \cup (f^{-1}(A) \cap [((W \setminus E) \setminus \mathrm{int}(W \setminus E)) \cup \mathrm{int}(W \setminus E)]).$$

Since $f^{-1}(A) \cap \mathrm{int}(W \setminus E) = \mathrm{int}(W \setminus E) \setminus f^{-1}(X \setminus A)$ and $(W \setminus E) \setminus \mathrm{int}(W \setminus E)$ is a nowhere dense closed set (we used here that $W \setminus E$ is closed) we get that the above set is meager. So $E \triangle f^{-1}(A)$ is also meager. □

Remark 3.2.12. *It would be interesting to examine which of the seven consistent noncontainments in Diagrams 3.1.11 and 3.2.1 hold in ZFC.*

4 Application to Haar meagerness

Our goal in this section is to introduce the generalized Haar meager sets in Polish groups. To have a reasonable definition, one should have a reasonable definition of a universally Baire set, which serves as a category analogue of a universally measurable set. In the previous section we offered various ways how someone can choose a "universally Baire" notion. In the first subsection we are going to cast our vote to the Polish-universally Baire sets and we will argue why the Polish-universally Baire sets are the "best" choice as a category analogue to universally measurable sets. We are also going to investigate some basic properties of Polish-universally Baire sets.

In the second subsection we will collect theorems on Haar null, generalized Haar null and Haar meager sets. In the third subsection we will define the generalized Haar meager sets and we are going to investigate how the theorems collected in the second subsection can be carried out in the theory of generalized Haar meager sets.

4.1 The right definition and basic properties

We mentioned in the introduction that our purpose is to introduce the generalized Haar meager sets, but to do this we need the "right" notion of a universally Baire set which is the analogue in some sense to the universally measurable sets. Looking at the definition of Haar meager sets (see Definition 4.2.3) we could think that for our purpose we should choose the compact metric-universally Baire sets as "the universally Baire sets". On the other hand if we have a look at Theorem 3.1.30, we can see that consistently there exists a set which is compact metric-universally Baire, but does not have the property of Baire and one naturally expects a universally Baire set to have the property of Baire.

Remark 4.1.1. *If we define a set to be universally Baire if and only if it is compact metric-universally Baire, one could easily check that the set we have constructed in Theorem 3.1.30 would contradict Theorem 4.3.4.*

The next natural choice of a universally Baire set in Polish spaces would be the Polish-universally Baire sets. It turns out that in the sense of Theorem 4.1.3 and Corollary 4.1.6 Polish-universally Baire sets behave like universally measurable sets. Hence, from this point on we will call a Polish-universally Baire set simply just a universally Baire set, and we formulate it in a definition to emphasize this notion:

Definition 4.1.2. *Let X be a Polish space with $A \subseteq X$. The set A is called* universally Baire *if $f^{-1}(A)$ has the property of Baire for every Polish space*

Y and for every continuous $f : Y \longrightarrow X$. We denote the universally Baire sets of X by $\mathcal{UB}(X)$.

Let us make some easy observations about universally Baire sets. First, every universally Baire set has the property of Baire (because $id : X \longrightarrow X$ is continuous). We also have that universally Baire sets form a σ-algebra (because the sets with the property of Baire form a σ-algebra, see Theorem 2.1.15), which contains the Borel sets. We can say even more, every analytic set is universally Baire, because analytic sets are closed under continuous preimage (see [12, Proposition 14.4.]) and it is well known that analytic sets have the property of Baire (see [12, Theorem 21.6.]). It is also easy to see that universally Baire sets are closed under the Souslin operation (for a definition see [12, Definition 25.4.]), because the sets with the property of Baire are closed under the Souslin operation (see [12, Theorem 29.13.]).

Our goal is to show that universally Baire sets behave like the universally measurable sets. Recall the following important theorem about universally measurable sets:

Theorem 4.1.3. *(see [9, 434D]) Let X be a Polish space with $A \subseteq X$. The set A is universally measurable if and only if $f^{-1}(A)$ is Lebesgue measurable for every Borel $f : [0,1] \longrightarrow X$.*

We will show that a similar theorem holds for universally Baire sets in the sense of category.

Theorem 4.1.4. *Let X, Y be arbitrary Polish spaces with a Borel function $f : Y \longrightarrow X$. If $A \in \mathcal{UB}(X)$, then $f^{-1}(A) \in \mathcal{UB}(Y)$.*

Proof. Let Z be an arbitrary Polish space and let $g : Z \longrightarrow Y$ be an arbitrary continuous function, we need to check that $g^{-1}(f^{-1}(A))$ has the property of Baire. The function $f \circ g : Z \longrightarrow X$ is Baire measurable, so it is continuous on a dense $\mathcal{G}_\delta$ subspace $N \subseteq Z$ (see [12, Theorem 8.38.]). The subspace N is Polish, because it is a $\mathcal{G}_\delta$ subspace of a Polish space (see Theorem 2.1.21). If h denotes the restriction of $f \circ g$ to N we get that $h^{-1}(A)$ has the property of Baire in N. From Lemma 3.1.13 follow that $h^{-1}(A)$ has the property of Baire in Z. To see that $g^{-1}(f^{-1}(A))$ has the property of Baire, it is enough to note that $Z \setminus N$ is meager, hence $g^{-1}(f^{-1}(A)) = h^{-1}(A) \cup B$ for some meager set $B \subseteq Z \setminus N$. □

Corollary 4.1.5. *Let X be a Polish space with $A \subseteq X$. The set A is universally Baire if and only if $f^{-1}(A)$ has the property of Baire for every Borel function $f : \mathbb{N}^\omega \longrightarrow X$.*

Proof. If $A \in \mathcal{UB}(X)$, then the previous theorem guarantee that $f^{-1}(A)$ has the property of Baire for every Borel function $f : \mathbb{N}^\omega \longrightarrow X$.

Conversely, if $f^{-1}(A)$ has the property of Baire for every Borel function $f : \mathbb{N}^\omega \longrightarrow X$, then we get that as a special case that $f^{-1}(A)$ has the property of Baire for every continuous function $f : \mathbb{N}^\omega \longrightarrow X$. From Theorem 3.1.21 follow that $A \in \mathcal{UB}(X)$. □

The following corollary "justifies" our definition of a universally Baire set:

Corollary 4.1.6. *Let X be a Polish space with $A \subseteq X$. The set A is universally Baire if and only if $f^{-1}(A)$ has the property of Baire for every Borel function $f : [0,1] \longrightarrow X$.*

Proof. If $A \in \mathcal{UB}(X)$, then Theorem 4.1.4 guarantee that $f^{-1}(A)$ has the property of Baire for every Borel function $f : [0,1] \longrightarrow X$.

Conversely, suppose that $f^{-1}(A)$ has the property of Baire for every Borel function $f : [0,1] \longrightarrow X$. By Theorem 3.1.21 it is enough to prove that $g^{-1}(A)$ has the property of Baire for every continuous $g : \mathbb{N}^\omega \longrightarrow X$. By Theorem 2.1.6 we have that $\mathbb{N}^\omega$ is homeomorphic to $[0,1] \setminus \mathbb{Q}$ and throughout the proof we will identify $\mathbb{N}^\omega$ with $[0,1] \setminus \mathbb{Q}$.

If $A = X$, then it is clear that $A \in \mathcal{UB}(X)$ so we can assume that $A \neq X$ and we can fix an arbitrary $y \in X \setminus A$. Take an arbitrary continuous $g : \mathbb{N}^\omega \longrightarrow X$ and define the following function:

$$h(a) = \begin{cases} g(a) & \text{if } a \in [0,1] \setminus \mathbb{Q}, \\ y & \text{if } a \in [0,1] \cap \mathbb{Q}. \end{cases}$$

It is clear that h is a Borel function from $[0,1]$ to X, hence $h^{-1}(A) = g^{-1}(A)$ has the property of Baire in $[0,1]$. To see that $g^{-1}(A)$ has the property of Baire in $\mathbb{N}^\omega$ we can choose an open set U in $[0,1]$ and a meager set M in $[0,1]$ such that $U \triangle M = g^{-1}(A)$. It is easy to see that in this case $g^{-1}(A) = (U \cap \mathbb{N}^\omega) \triangle (M \cap \mathbb{N}^\omega)$ and $(U \cap \mathbb{N}^\omega)$ is open in $\mathbb{N}^\omega$ and $(M \cap \mathbb{N}^\omega)$ is meager in $\mathbb{N}^\omega$, hence $g^{-1}(A)$ has the property of Baire in $\mathbb{N}^\omega$. □

Theorem 4.1.4 tells us that we could define equivalently the universally Baire sets of a Polish space X in the following way: the set $A \subseteq X$ is universally Baire if and only if $f^{-1}(A)$ has the property of Baire for every Polish space Y and for every Borel function $f : Y \longrightarrow X$. Theorem 4.1.4 also gives us that being a universally Baire set only depends on the Borel structure in the following sense: if τ' is any other Polish topology on X for which τ and τ' has the same Borel sets, then A is universally Baire in (X, τ) if and only if A is universally Baire in (X, τ'). So we can set the definition of a universally Baire set for standard Borel spaces:

Definition 4.1.7. *Let $(X, \mathcal{S})$ be a standard Borel space with $A \subseteq X$. We will say that A is* universally Baire *in X if it is universally Baire for some Polish topology τ, where $\mathcal{B}(X, \tau) = \mathcal{S}$. We will denote by $\mathcal{UB}(X)$ the universally Baire sets of X for a standard Borel X.*

We will not deal with theory of universally Baire sets in standard Borel spaces, but we present here some easy theorems about the behavior of universally Baire sets in standard Borel spaces.

Theorem 4.1.8. *Let $(X, \mathcal{S})$ be a standard Borel space with $A \in \mathcal{UB}(X)$ and $B \in \mathcal{S}$. Then $A \cap B$ is universally Baire in $(B, \mathcal{S}\mid_B)$.*

Proof. If $A \cap B$ is empty then we are done, so we can assume that $A \cap B \neq \emptyset$ and we can fix an arbitrary $x \in A \cap B$. Let τ be a Polish topology on X such that $\tau|_B$ (i.e. the restriction of τ to B) is also Polish (such a topology exist, see [12, Theorem 13.1.]) and $\mathcal{B}(\tau) = \mathcal{S}$.

Let Y be an arbitrary Polish space and let f be an arbitrary continuous function from Y to B, we need to check that $f^{-1}(A \cap B)$ has the property of Baire in Y. Consider the following function $g : Y \to X$:

$$g(a) = \begin{cases} f(a) & \text{if } a \in B, \\ x & \text{if } a \notin B. \end{cases}$$

It is clear that g is a Borel function and $g^{-1}(A) = f^{-1}(A \cap B)$. By Theorem 4.1.4 $g^{-1}(A)$ has the property of Baire. □

Theorem 4.1.9. *Let $(X, \mathcal{S})$ be a standard Borel space with $A \subseteq X$. Suppose that $B_n \in \mathcal{S}$ for every n, $X = \bigcup_{n \in \omega} B_n$ and $A \cap B_n$ is a universally Baire subset of B_n for every n. Then $A \in \mathcal{UB}(X)$.*

Proof. Let τ be a Polish topology on X such that $\mathcal{B}(\tau) = \mathcal{S}$ and let Y be a Polish space with a continuous function $f : Y \to X$, we need to check that $f^{-1}(A)$ has the property of Baire. Let $H_n := f^{-1}(A \cap B_n)$ and let

$$I := \{n \in \omega \mid H_n \neq \emptyset\}.$$

For every $n \in I$ we fix an arbitrary point $x_n \in A \cap B_n$ and define a function $g_n : Y \to B_n$:

$$g_n(a) = \begin{cases} f(a) & \text{if } a \in f^{-1}(B_n), \\ x_n & \text{otherwise.} \end{cases}$$

It is clear that g_n is a Borel functions for every $n \in I$, hence $g_n^{-1}(A \cap B_n)$ has the property of Baire. By the definition of the g_n-s we have that

$$f^{-1}(A) = \bigcup_{n \in A} g_n^{-1}(A \cap B_n) \setminus f^{-1}(B_n^c)$$

and from the right side of the above equation it is clear that $f^{-1}(A)$ has the property of Baire. □

Theorem 4.1.10. *Let X_n be standard Borel spaces for all $n \in \omega$ and let A_n be universally Baire sets in X_n. Then $\bigtimes_{n \in \omega} A_n \subseteq \bigtimes_{n \in \omega} X_n$ is universally Baire.*

Proof. Let τ_n be Polish topology on X_n compatible with the Borel structure of X_n. Let Y be an arbitrary Polish space and take an arbitrary continuous

$f : Y \longrightarrow \bigtimes_{n\in\omega} X_n$. We need to check that $f^{-1}(\bigtimes_{n\in\omega} A_n)$ has the property of Baire. Let f_n denote the coordinate functions of f, that is $f = (f_n)_{n\in\omega}$, where $f_n : Y \longrightarrow X_n$ is continuous. It is easy to see that $f^{-1}(\bigtimes_{n\in\omega} A_n) = \bigcap_{n\in\omega} f_n^{-1}(A_n)$, hence $f^{-1}(\bigtimes_{n\in\omega} A_n)$ has the property of Baire. □

4.2 Preliminary results

In this subsection we recall the definition of Haar null sets, generalized Haar null sets and Haar meager sets and we also recall some basic theorems that we are going to investigate in Subsection 4.3 in light of generalized Haar meager sets. Throughout Subsections 4.2 and 4.3 whenever we use G, it will denote a Polish group.

Definition 4.2.1. *$A \subseteq G$ is called* Haar null*, if there is a Borel set B with $B \supseteq A$ and a Borel probability measure μ on G such that $\mu(gBh) = 0$ for every $g, h \in G$. The system of Haar null sets will be denoted by $\mathcal{HN}(G)$.*

Definition 4.2.2. *A set $A \subseteq G$ is called* generalized Haar null*, if there is a universally measurable set B with $B \supseteq A$ and a Borel probability measure μ on G such that $\mu(gBh) = 0$ for every $g, h \in G$. The system of generalized Haar null sets will be denoted by $\mathcal{GHN}(G)$.*

Definition 4.2.3. *A set $A \subseteq G$ is called* Haar meager*, if there is a Borel set B with $B \supseteq A$ and a compact metric space K with a continuous function $f : K \longrightarrow G$ such that $f^{-1}(gBh)$ is meager in K for every $g, h \in G$. The system of Haar meager sets will be denoted by $\mathcal{HM}(G)$.*

Maybe the most important theorem about Haar null, generalized Haar null and Haar meager sets is the following:

Theorem 4.2.4. *(see [5, Theorem 3.2.5, Theorem 3.2.6]) The systems of Haar null, generalized Haar null and Haar meager sets form σ-ideals.*

The following theorem gives us the relation between meager and Haar meager sets:

Theorem 4.2.5. *(Darji, Doležal-Rmoutil-Vejnar-Vlasák, see [5, Theorem 3.3.12]) Every Haar meager set is meager, that is,* $\mathcal{HM}(G) \subseteq \mathcal{M}(G)$.

We also recall the connection between Haar meager sets and meager sets when the group G is locally compact:

Theorem 4.2.6. *(Darji, Doležal-Rmoutil-Vejnar-Vlasák, see [5, Theorem 3.3.13]) In a locally compact Polish group* G *we have that Haar meager and meager sets coincide, that is,* $\mathcal{HM}(G) = \mathcal{M}(G)$.

On the other hand we have the following theorem, when the group G is non-locally-compact and admits a two-sided invariant metric:

Theorem 4.2.7. *(Darji, Doležal-Rmoutil-Vejnar-Vlasák, see [5, Theorem 3.3.14]) In a non-locally-compact Polish group* G *that admits a two-sided invariant metric meagerness is a strictly weaker notion than Haar meagerness, that is,* $\mathcal{HM}(G) \subsetneq \mathcal{M}(G)$.

We recall the following theorem about Haar meager sets:

Theorem 4.2.8. *(Doležal-Vlasák / Darji, see [5, Theorem 4.1.11] For a Borel set* $B \subseteq G$ *the following are equivalent:*

(1) *there exists a (nonempty) compact metric space* K *and a continuous function* $f : K \to G$ *such that* $f^{-1}(gBh)$ *is meager in* K *for every* $g, h \in G$ *(i.e.* B *is Haar meager),*

(2) *there exists a continuous function* $f : 2^\omega \to G$ *such that* $f^{-1}(gBh)$ *is meager in* 2^ω *for every* $g, h \in G$,

(3) *there exists a (nonempty) compact set* $C \subseteq G$, *a continuous function* $f : C \to G$ *such that* $f^{-1}(gBh)$ *is meager in* C *for every* $g, h \in G$,

In the following theorem we will assume that G is a locally compact Polish group. It is well-known that one can introduce the so-called Haar measure(s) on a locally compact Hausdorff group and one can introduce the so-called Haar null sets, which are well-defined (see [3]). In the following theorem we denote by $\mathcal{N}(G)$ the system of sets with Haar measure zero.

Theorem 4.2.9. *(Christensen, Mycielski, see [5, Theorem 3.3.7]) If G is a locally compact Polish group, then the system of sets with Haar measure zero is the same as the system of Haar null sets and is the same as the system of generalized Haar null sets, that is, $\mathcal{N}(G) = \mathcal{HN}(G) = \mathcal{GHN}(G)$.*

Recall the following theorem about an equivalent characterization for a universally measurable set to be a generalized Haar null set:

Theorem 4.2.10. *(Hunt-Sauer-Yorke, see [5, Theorem 4.1.5]) For a universally measurable set $B \subseteq G$ the following are equivalent:*

(i) there exists a Borel probability measure μ on G such that $\mu(gBh) = 0$ for every $g, h \in G$, that is, B is generalized Haar null.

(ii) there exists a Borel probability measure μ on G and a generalized Haar null set $N \subseteq G$ such that $\mu(gBh) = 0$ for every $g, h \in G \setminus N$.

4.3 The theory of generalized Haar meager sets

Finally, we are in the position to define the generalized Haar meager sets:

Definition 4.3.1. *A set $A \subseteq G$ is called a* generalized Haar meager set*, if there exists a universally Baire set B with $B \supseteq A$ and a compact metric space K with a continuous $f : K \longrightarrow G$ such that $f^{-1}(gBh)$ is meager in K for every $g, h \in G$. The system of generalized Haar meager sets will be denoted by $\mathcal{GHM} = \mathcal{GHM}(G)$.*

We call a function f a witness function (for a set $A \subseteq G$) if it satisfies the assumptions of the above definition. We will now prove that $\mathcal{GHM}$ forms a σ-ideal, the proof is essentially the same as the proof that Haar meager sets form a σ-ideal.

Theorem 4.3.2. *The system $\mathcal{GHM}$ of generalized Haar meager sets is a σ-ideal.*

Proof. It is trivial that $\emptyset \in \mathcal{GHM}$ and if $B \in \mathcal{GHM}$ then every subset of B is also a generalized Haar meager set. So we need to check that $\mathcal{GHM}$ is closed under countable union.

Let A_n be generalized Haar meager for all $n \in \omega$. By definition there are universally Baire sets $B_n \subseteq G$, compact metric spaces K_n with continuous functions $f_n : K_n \to G$ such that $f_n^{-1}(gB_nh)$ is meager in K_n for every $g, h \in G$. Let also d be a complete metric on G such that d is compatible with the topology of G.

We construct for all $n \in \omega$ a compact metric space $\tilde{K}_n$ with a continuous function $\tilde{f}_n : \tilde{K}_n \to G$ satisfying that $\tilde{f}_n^{-1}(gB_nh)$ is meager in $\tilde{K}_n$ for every $g, h \in G$ (i.e. $\tilde{f}_n$ is a witness function) and the size of the images $\tilde{f}_n(\tilde{K}_n) \subseteq G$ decreases quickly.

First, take neighborhoods U_n of 1_G such that if $u \in U_n$, then $d(k \cdot u, k) < 2^{-n}$ for every k in the compact set $f_0(K_0)f_1(K_1) \cdots f_{n-1}(K_{n-1})$. Note that this is possible from the well-known fact that if $C \subseteq V$, where C is compact and V is open in a topological group G, then there exists a neighborhood U of 1_G, such that $C \cdot U \subseteq V$. Setting $C = f_0(K_0)f_1(K_1) \cdots f_{n-1}(K_{n-1})$ and $V := N_{2^{-n}}(f_0(K_0)f_1(K_1) \cdots f_{n-1}(K_{n-1}))$ gives us an appropriate U_n.

Let $x_n \in f_n(K_n)$ be an arbitrary element and $\tilde{K}_n = \overline{f_n^{-1}(x_nU_n)}$. The set $\tilde{K}_n$ is compact (because it is a closed subset of a compact set) and nonempty. Let $\tilde{f}_n : \tilde{K}_n \to G$, $\tilde{f}_n(k) = x_n^{-1}f_n(k)$. It is clear that $\tilde{f}_n$ continuous. Of course $\tilde{f}_n(k)$ can be extended to K_n by the formula $\tilde{f}_n(k) = x_n^{-1}f_n(k)$ and we will also write $\tilde{f}_n(k)$ for this function.

Claim 4.3.3. *For every $n \in \omega$ and $g, h \in G$ we have that $\tilde{f}_n^{-1}(gB_nh)$ is meager in $\tilde{K}_n$.*

Proof. Fix $n \in \omega$ and $g, h \in G$. The set $\tilde{f}_n^{-1}(U_n)$ is open in K_n and because f_n is a witness function, the set $\tilde{f}_n^{-1}(gB_nh) = f_n^{-1}(x_ngB_nh)$ is meager in K_n. This means that $\tilde{f}_n^{-1}(U_n) \cap \tilde{f}_n^{-1}(gB_nh)$ is meager in $\tilde{f}_n^{-1}(U_n)$.

Since each open subset of K_n is comeager in its closure and the closure of $\tilde{f}_n^{-1}(U_n) = f_n^{-1}(x_nU_n)$ is $\overline{f_n^{-1}(x_nU_n)} = \tilde{K}_n$ and an easy formal calculations yield that $\tilde{f}_n^{-1}(gB_nh) \cap \tilde{K}_n$ is meager in $\tilde{K}_n$. □

Let K be the compact set $\prod_{n\in\omega} \tilde{K}_n$ and define the continuous function $\psi_n : K \to G$ such that for $k = (k_0, k_1, \cdots) \in K$ we have that:

$$\psi_n(k) = \tilde{f}_0(k_0) \cdot \tilde{f}_1(k_1) \cdot \ldots \cdot \tilde{f}_{n-1}(k_{n-1}).$$

By the choice of U_n and $\tilde{f}_n(\tilde{K}_n) \subset \overline{U_n}$ we obtain that $d(\psi_{n-1}(k), \psi_n(k)) \leq 2^{-n}$ for every $k \in K$. Using the completeness of d this means that the sequence of functions $(\psi_n)_{n\in\omega}$ is uniformly convergent. Let $f : K \to G$ be the limit of this sequence. The function f is continuous because it is the uniform limit of continuous functions.

We claim that f witnesses that $A = \bigcup_{n\in\omega} A_n$ is generalized Haar meager. Note that A is contained in the universally Baire set $B = \bigcup_{n\in\omega} B_n$, so it is enough to show that $f^{-1}(gBh)$ is meager in K for every $g, h \in G$. As meager subsets of K form a σ-ideal, it is enough to show that $f^{-1}(gB_nh)$ is meager in K for every $g, h \in G$ and $n \in \omega$.

Fix $g, h \in G$ and $n \in \omega$ and also fix $k_j \in \tilde{K}_j$ for every $j \neq n$. Claim 4.3.3 yields that

$$\begin{aligned}
&\{k_n \in \tilde{K}_n : f(k_0, k_1, \dots, k_n, \dots) \in gB_nh\} = \\
&= \{k_n \in \tilde{K}_n : \tilde{f}_0(k_0) \cdot \tilde{f}_1(k_1) \cdot \dots \cdot \tilde{f}_n(k_n) \cdot \dots \in gB_nh\} = \\
&= \tilde{f}_n^{-1}\left(\left(\tilde{f}_0(k_0) \cdot \dots \tilde{f}_{n-1}(k_{n-1})\right)^{-1} \cdot gB_nh \cdot \left(\tilde{f}_{n+1}(k_{n+1}) \cdot \tilde{f}_{n+2}(k_{n+2}) \cdot \dots\right)^{-1}\right)
\end{aligned}$$

is meager in $\tilde{K}_n$. Applying the Kuratowski-Ulam theorem (see [12, Theorem 8.41.]) in the product space $\left(\prod_{j\neq n} \tilde{K}_j\right) \times \tilde{K}_n$ we get that the set $\tilde{f}_n^{-1}(gB_nh)$ is meager. □

We know that every Haar meager set is meager by Theorem 4.2.5. We show that the same is true for generalized Haar meager sets:

Theorem 4.3.4. *Every generalized Haar meager set is meager, that is,* $\mathcal{GHM}(G) \subseteq \mathcal{M}(G)$.

Proof. Let A be a generalized Haar meager subset of G. By definition there is a universally Baire set $B \supseteq A$ and a compact metric space K with a continuous function $f : K \to G$ such that $f^{-1}(gBh)$ is meager in K for every $g, h \in G$.

Consider the set

$$S = \{(g, k) : f(k) \in gB\} \subseteq G \times K,$$

which has the Baire property, because it is the preimage of B under the continuous map $(g, k) \mapsto g^{-1} \cdot f(k)$. For every $g \in G$ we have that the g-section of S, that is $S_g = \{k \in K : f(k) \in gB\} = f^{-1}(gB)$ is meager in K by assumption. Hence, by the Kuratowski-Ulam theorem S is meager in $G \times K$. Using the Kuratowski-Ulam theorem again, there exist comeager many $k \in K$ such that the section $S^k = \{g \in G : f(k) \in gB\} = f(k) \cdot B^{-1}$ is meager in G. Since K is compact, there is at least one such k. Then the homeomorphism $x \mapsto x^{-1} \cdot f(k)$ maps the meager set S^k to B and this shows that B is meager. □

As an analogue to Theorem 4.2.9 we have the following theorem when G is locally compact Polish:

Theorem 4.3.5. *If G is a locally compact Polish group, then the system of Haar meager sets is the same as the system of generalized Haar meager sets and the same as the the system of meager sets, that is, $\mathcal{M}(G) = \mathcal{HM}(G) = \mathcal{GHM}(G)$.*

Proof. The containment $\mathcal{HM}(G) \subseteq \mathcal{GHM}(G)$ is obvious from the fact that every Borel set is a universally Baire set.

The containment $\mathcal{GHM}(G) \subseteq \mathcal{M}(G)$ is Theorem 4.3.4 and $\mathcal{M}(G) \subseteq \mathcal{HM}(G)$ is Theorem 4.2.6. □

Theorem 4.2.7 tells us that in a non-locally-compact Polish group which admits a two-sided invariant metric meagerness is a strictly weaker notion than Haar meagerness. We show that with the same assumptions meagerness is a strictly weaker notion than generalized Haar meagerness:

Theorem 4.3.6. *In a non-locally-compact Polish group G that admits a two-sided invariant metric meagerness is a strictly weaker notion than generalized Haar meagerness, that is $\mathcal{GHM}(G) \subsetneq \mathcal{M}(G)$.*

Proof. Almost the same proof can be done as in [5, Theorem 3.3.14]. We know that $\mathcal{GHM}(G) \subseteq \mathcal{M}(G)$ by Theorem 4.3.4, to construct a meager but not generalized Haar meager set, we will use the first theorem from [16]:

Theorem 4.3.7. *(Solecki) Assume that G is a non-locally-compact Polish group that admits a two-sided invariant metric. Then there exists a closed set $F \subseteq G$ and a continuous function $\varphi : F \to 2^\omega$ such that for any $x \in 2^\omega$ and any compact set $C \subseteq G$ there is a $g \in G$ with $gC \subseteq \varphi^{-1}(\{x\})$.*

The sets $\{\varphi^{-1}(\{x\}) : x \in 2^\omega\}$ are disjoint closed sets. Since G is separable, only countable many $\varphi^{-1}(\{x\})$ can have nonempty interior. Fix an $x_0 \in 2^\omega$ such that $M := f^{-1}(\{x_0\})$ has empty interior. It is clear that M is closed, hence nowhere dense. On the other hand M is not a generalized Haar meager set because for every compact metric space K and continuous function $f : K \to G$ there exists a $g \in G$ such that $gf(K) \subseteq M$, thus $f^{-1}(g^{-1}M) = K$. □

We now give an equivalent characterization for a universally Baire set to be a generalized Haar meager set analogously to Theorem 4.2.10.

Theorem 4.3.8. *For a universally Baire set $B \subseteq G$ the following are equivalent:*

(i) there exists a compact metric space K and a continuous function $f : K \longrightarrow G$, such that $f^{-1}(gBh) = 0$ for every $g, h \in G$, in other words, B is a generalized Haar meager set.

(ii) there exists a compact metric space K, a continuous function $f : K \longrightarrow G$ and a generalized Haar meager set $N \subseteq G$ such that $\mu(gBh) = 0$ for every $g, h \in G \setminus N$.

Proof. $(i) \Rightarrow (ii)$ is trivial from the definition, so let us turn to the proof of $(ii) \Rightarrow (i)$. Since N is generalized Haar meager, we can cover it by a generalized Haar meager set which is also universally Baire. Without loss of generality we may assume that N itself is a generalized Haar meager set, which is also universally Baire. Let L be a compact metric space and $j : L \longrightarrow G$ continuous, which witnesses the generalized Haar meagerness of N, that is, for every $g, h \in G$, $j^{-1}(gNh)$ is meager in L. Let α denote the inverse function, that is $\alpha : G \longrightarrow G, \quad g \mapsto g^{-1}$.

We will prove that

$$m : L \times K \times L \longrightarrow G, \ (a, b, c) \mapsto j(a)^{-1} \cdot f(b) \cdot j(c)^{-1}$$

witnesses that B is generalized Haar meager. Note that $L \times K \times L$ is compact and m is continuous.

Fix $g, h \in G$, we need to prove that $m^{-1}(gBh)$ is meager in $L \times K \times L$. For brevity, we will denote $m^{-1}(gBh)$ by H.

$$m^{-1}(gBh) = \{(a, b, c) \mid f(b) \in j(a)gBhj(c)\}$$

We want to use Kuratowski-Ulam theorem in the following setup: there are comeager many $(a, c) \in L \times L$, such that the section of H at (a, c) $(= H^{(a,c)})$ is meager in K. And we can easily guarantee comeager many such pairs, because if $j(a) \cdot g \notin N$ and $h \cdot j(c) \notin N$, then the section at (a, c) is meager in K. So all the sections at (a, c) for which $(a, c) \in (j^{-1}(Ng^{-1}))^c \times (j^{-1}(h^{-1}N))^c$ is meager, and $(j^{-1}(Ng^{-1}))^c \times (j^{-1}(h^{-1}N))^c$ is comeager, because j witnesses that N is generalized Haar meager. □

To continue the theory of generalized Haar meager sets we recall the following lemma from [5, Lemma 4.1.10]:

Lemma 4.3.9. *If (K, d) is a compact metric space, then there exists a continuous function $\varphi : 2^\omega \to K$ such that if M is meager in K, then $\varphi^{-1}(M)$ is meager in 2^ω.*

With the previous lemma we can do the analogue of Theorem 4.2.8 to generalized Haar meager sets:

Theorem 4.3.10. *For a universally Baire set $B \subseteq G$ the following are equivalent:*

(1) there exists a compact metric space K with a continuous function $f : K \to G$ such that $f^{-1}(gBh)$ is meager in K for every $g, h \in G$ (i.e. B is generalized Haar meager),

(2) there exists a continuous function $f : 2^\omega \to G$ such that $f^{-1}(gBh)$ is meager in 2^ω for every $g, h \in G$,

(3) there exists a compact set $C \subseteq G$ with a continuous function $f : C \to G$ such that $f^{-1}(gBh)$ is meager in C for every $g, h \in G$.

Proof. The implication (1) $\Rightarrow$ (2) is an easy consequence of Lemma 4.3.9. If K and f satisfies the requirements of (1) and φ is the function granted by Lemma 4.3.9, then $\tilde{f} = f \circ \varphi : 2^\omega \to G$ will satisfy the requirements of (2) because $\tilde{f}$ is continuous and for every $g, h \in G$ the set $f^{-1}(gBh)$ is meager in K, hence $\varphi^{-1}(f^{-1}(gBh)) = \tilde{f}^{-1}(gBh)$ is meager in 2^ω.

The implication (2) $\Rightarrow$ (3) is obvious when G is countable, since the only generalized Haar meager subset of G is the empty set. In this case any compact $C \subseteq G$ and continuous function $f : C \to G$ are sufficient. If G is not countable then it is well known that there is a compact set $C \subseteq G$ that is homeomorphic to 2^ω. Composing the witness function $f : 2^\omega \to G$ granted by (2) with this homeomorphism yields a function that satisfies our requirements (together with C).

The implication (3) $\Rightarrow$ (1) is trivial. □

As a final word we want to mention here some interesting results on coanalytic hulls. In [5, Theorem 4.1.1 and Theorem 4.1.8] it was proved that the Borel hull in the definition of Haar null sets and Haar meager sets can be replaced by an analytic hull. The following theorems show that it cannot be replaced by a coanalytic hull. These theorems were proved in the abelian case, but they can be generalized to groups with a two-sided invariant metric.

In [6, Theorem 1.8] the following theorem was proved:

Theorem 4.3.11. *(Elekes-Vidnyánszky) If G is a non-locally-compact abelian Polish group, then there exists a coanalytic set $A \subseteq G$ that is not Haar null, but there is a Borel probability measure μ on G such that $\mu(gAh) = 0$ for every $g, h \in G$.*

Since every analytic and coanalytic set is universally measurable (see [12, Theorem 21.10]) we get that the set A is generalized Haar null, but not Haar null:

Corollary 4.3.12. *If G is a non-locally-compact and abelian Polish group, then $\mathcal{GHN}(G) \supsetneq \mathcal{HN}(G)$.*

In the case of Haar meager sets, the following theorem is known (see [4, Theorem 13]):

Theorem 4.3.13. *(Doležal-Vlasák) If G is a non-locally-compact abelian Polish group, then there exists a coanalytic set $A \subseteq G$ that is not Haar meager, but there is a compact metric space K and a continuous function $f : K \to G$ such that $f^{-1}(gAh)$ is meager in K for every $g, h \in G$.*

Since every analytic and coanalytic set has the property of Baire (see [12, Theorem 21.6]) we get that the set A is generalized Haar meager, but not Haar meager.

Corollary 4.3.14. *If G is a non-locally-compact and abelian Polish group, then $\mathcal{GHM}(G) \supsetneq \mathcal{HM}(G)$.*

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