A quick proof for the cactus representation of mincuts

Tamás Fleiner* and András Frank**

Abstract

A short and simple proof is given for an elegant theorem of E.A. Dinits, A.V. Karzanov and M.V. Lomonosov on representing all of the minimum cuts of an undirected graph by a cactus, a graph built up from edge-disjoint circuits in a tree-like manner.

Keywords: minimum cut, cactus graph, crossing sets

Let $G = (V, E)$ be a connected graph. For a subset $\emptyset \subset X \subset V$ of nodes, the set $\Delta(X)$ of edges connecting X and $V - X$ is called a *cut* while X and $V - X$ are the *shores* of the cut. An easy exercise shows that in a connected graph a cut uniquely determines its shores, that is, if $\Delta(X) = \Delta(Y)$, then $X = Y$ or $X = V - Y$. A cut is a *star-cut* if one of its shores consists of a single node. Otherwise the cut is called *proper*. The degree of X is defined by $d(X) := |\Delta(X)|$. For two subsets X, Y of nodes, $d(X, Y)$ denotes the number of edges connecting $X - Y$ and $Y - X$ while $\bar{d}(X, Y)$ is the number of edges connecting $X \cap Y$ and $V - (X \cup Y)$, that is, $\bar{d}(X, Y) = d(V - X, Y)$. In particular, $d(x, y)$ denotes the number of parallel xy-edges for two distinct nodes x and y where an edge is called an *xy-edge* if its end-nodes are x and y. Two subsets X and Y of nodes are called *crossing* if none of $X - Y$, $Y - X$, $X \cap Y$, $V - (X \cup Y)$ is empty. Two cuts $\Delta(X)$ and $\Delta(Y)$ are *crossing* if X and Y are crossing.

For a positive integer k, $G = (V, E)$ is called *k-edge-connected* if every cut of G contains at least k edges. We call a cut of a k-edge-connected graph G a *mincut* if it has exactly k edges. A subset $T \subset V$ is *tight* if $d(T) = k$. A *proper tight* set is one for which $|T| \neq 1 \neq |V - T|$, that is, (proper) tight sets are the shores of (proper) mincuts. A tight set is said to cross another tight set if they are crossing. A mincut $\Delta(X)$ crosses another mincut $\Delta(Y)$ if X crosses Y. A family $\mathcal{F}$ of sets is *cross-free* if no two members of $\mathcal{F}$ cross each other.

*Department of Computer Science and Information Theory, Budapest University of Technology and Economics, Magyar Tudósok körútja 2, Budapest, H-1117. `fleiner@cs.bme.hu` Research was supported by MTA-ELTE Egerváry Research Group and the K60802 OTKA grant. E-mail:`fleiner@cs.bme.hu`

**MTA-ELTE Egerváry Research Group, Department of Operations Research, Eötvös University, Pázmány P. s. 1/c. Budapest, Hungary, H-1117. Research supported by the Hungarian National Foundation for Scientific Research, OTKA K60802 e-mail: `frank@cs.elte.hu`

Proposition 1. *Let $G = (V, E)$ be a k-edge-connected graph. If X and Y are two crossing tight sets, then each of $X - Y, Y - X, X \cap Y, X \cup Y$ is tight. Moreover, $d(X, Y) = 0 = \bar{d}(X, Y)$.*

Proof. It follows from

$$k + k = d(X) + d(Y) = d(X \cap Y) + d(X \cup Y) + 2d(X, Y) \geq k + k + 2d(X, Y)$$

that $d(X \cap Y) = k,\ d(X \cup Y) = k$, and $d(X, Y) = 0$. Similarly,

$$k + k = d(X) + d(Y) = d(X - Y) + d(Y - X) + 2\bar{d}(X, Y) \geq k + k + 2\bar{d}(X, Y)$$

implies that $d(X - Y) = k, d(Y - X) = k$, and $\bar{d}(X, Y) = 0$. □

Lemma 2. *Let $k \geq 1$ be an integer. Let $G = (V, E)$ be a k-edge-connected graph in which there is a proper mincut and every proper mincut is crossed by some proper mincut. Then k is even and G arises from a circuit by replacing each edge with $k/2$ parallel edges.*

Proof. By the assumption, there are two crossing tight sets X and Y. Then $\bar{d}(X, Y) = 0$ implies that $k = d(X \cap Y) = d(X - Y, X \cap Y) + d(Y - X, X \cap Y)$. If k were odd, then the two summands could not be equal. We may assume that $d(X - Y, X \cap Y) > d(Y - X, X \cap Y)$ but then $k = d(X) = d(X - Y) - d(X - Y, X \cap Y) + d(Y - X, X \cap Y) < k$, a contradiction. Therefore k is even.

Since the complement of a tight set is also tight, the hypothesis implies that

$$\text{for any proper tight set } T \text{ and for any node } v \in V, \text{ there is a tight set crossing } T \text{ and containing } v. \tag{1}$$

Claim 3. *The degree of every node of G is k.*

Proof. Suppose indirectly that $d(v) > k$ for a node v. Consider a minimal proper tight set T containing v. By (1), there is a tight set X crossing T which contains v. But then $T \cap X$ is tight by Proposition 1 and hence $|T \cap X| \geq 2$ by $d(v) > k$ and this contradicts the minimality of T. □

Claim 4. *If $T = \{x, y\}$ is a tight set, then the number $d(x, y)$ of parallel xy-edges is $k/2$.*

Proof. $k = d(T) = d(x) + d(y) - 2d(x, y) = k + k - 2d(x, y)$, that is, $d(x, y) = k/2$. □

Claim 5. *Let v be a node and T a proper tight set containing v. Then T includes a two-element tight set containing v.*

Proof. Induction on the cardinality of $|T|$. As T itself will do if $|T| = 2$, we assume that $|T| \geq 3$. By (1), there is a tight set X crossing T and containing v. Then $T \cap X$ is also tight and in case $|T \cap X| \geq 2$ we are done by induction. Suppose now that $T \cap X = \{v\}$. By Proposition 1, $T' := T - X\ (= T - v)$ is a proper tight set. By (1), there is a proper tight set X' crossing T' and containing v. Then either $X' \subset T$ or else X' and T are crossing. In both cases, $T \cap X'$ is tight and $|T| > |T \cap X'| \geq 2$ from which we are done again by induction. □

Claim 6. *For every node v, there are two two-element tight sets containing v.*

Proof. It follows from Claim 5 that there is a two-element tight set $T_1 = \{v, x\}$. By (1), there is a tight set T' crossing T_1 that contains v. A second application of Claim 5 (with T' in place of T) implies that there is a two-element tight subset T_2 of T' which contains v, and this differs from T_1. □

Suppose that $\{v, x\}$ and $\{v, y\}$ are tight sets. Since there are exactly $k/2$ parallel vx-edges and $k/2$ parallel vy-edges, we conclude that every node v of G has exactly two distinct neighbours. As G is connected, it arises from a circuit by replacing each edge with $k/2$ parallel copies. □

We call a loopless and 2-edge-connected graph $C = (U, F)$ a *cactus* if each edge belongs to exactly one circuit. This is equivalent to saying that all blocks are circuits (allowing two-element circuits). For example, a cactus may be obtained by duplicating each edge of a tree. A more general cactus is shown in figure 1.

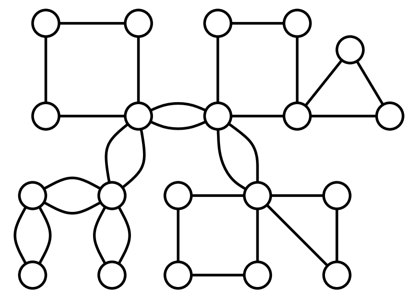

Figure 1: A cactus graph

Note that the mincuts of a cactus C are exactly those pairs of edges which belong to the same circuit of C. The following result states that the mincuts of an arbitrary graph have the same structure as the mincuts of a cactus. For algorithmic aspects and related results, see [4].

Theorem 7 (Dinits, Karzanov, and Lomonosov, [1])**.** *Let $k \geq 1$ be an integer and $G = (V, E)$ a loopless graph for which the minimum cardinality of a cut is k. There is a cactus $C = (U, F)$ and a mapping $\varphi : V \to U$ so that the preimages $\varphi^{-1}(U_1)$ and $\varphi^{-1}(U_2)$ are the two shores of a mincut of G for every 2-element cut of C with shores U_1 and U_2. Moreover, every mincut of G arises this way. Concisely: X is a tight set of G if and only if $\varphi(X)$ is a tight set of C.*

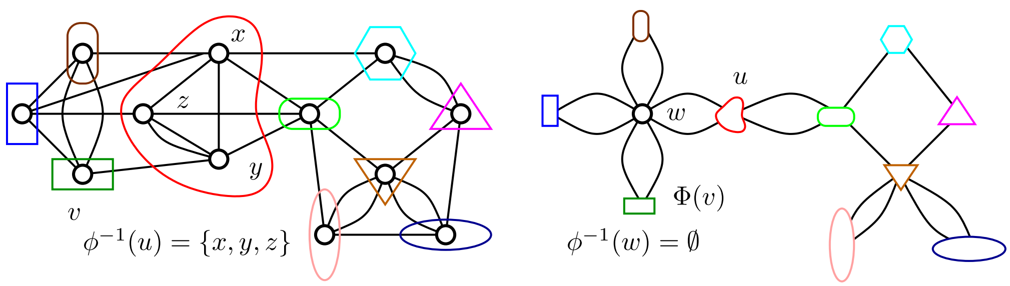

Figure 2: A graph and the cactus of its mincuts.

Proof. We use induction on $|V|$. As the theorem is trivial when $|V| \leq 2$, we assume that $|V| \geq 3$.

Suppose first that each mincut is a star-cut and let $v_1, \ldots, v_h$ denote the nodes of degree k. Let $U = \{u_0, u_1, \ldots u_h\}$ be the node-set of cactus C in which u_0 and u_i are connected by two parallel edges for each $i = 1, \dots, h$. Let $\varphi : V \to U$ be defined by $\varphi(v_i) = u_i$ for $i = 1, \dots, h$ and $\varphi(v) = u_0$ for $v \in V - \{v_1, \dots, v_h\}$. Then C and φ satisfy the requirements of the theorem.

Suppose now that there is a proper mincut, so $|V| \geq 4$. If every proper mincut is crossed by a mincut, then the theorem immediately follows from Lemma 2. Therefore we may assume that there is a mincut B with shores V_1 and V_2 which is not crossed by any other mincut.

For $j = 1, 2$, let G_j denote the graph arising from G by shrinking V_j into a single new node v_j in the sense that V_j is replaced by v_j so that there are as many parallel uv_j-edges in G_j as the number of edges in G connecting V_j and u for every node $u \in V - V_j$. By induction, the mincuts of G_j can be represented by a cactus $C_j = (U_j, F_j)$ and a mapping φ_j. We assume that U_1 and U_2 are disjoint. Since $d_{G_j}(v_j) = k$, the node $u_j := \varphi_j(v_j)$ is of degree 2 in C_j and there is no other node v of G_j with $\varphi_j(v) = u_j$.

Let $C = (U, F)$ be a cactus arising from C_1 and C_2 by identifying u_1 and u_2. Define $\varphi : V \to U$ by $\varphi(v) := \varphi_1(v)$ if $v \in V_2$ and $\varphi(v) := \varphi_2(v)$ if $v \in V_1$. Since no mincut crosses B, each mincut of G is either a mincut of G_1 or a mincut of G_2 and hence C and φ provide the requested representation of the mincuts of G. □

Remark 1. The proof of Theorem 7 extends to the capacitated version of the theorem word by word. In this case a strictly positive capacity function $g : V \to \mathbf{R}_+$ is given on the edge set E and k denotes the minimum total capacity of a cut.

Remark 2. In the uncapacitated case the situation is much simpler when k is odd, since then no two mincuts may cross each other. Therefore Theorem 7 transforms into the following simplified form.

Corollary 8. *If the minimum cardinality k of a cut of G is odd, then there is a tree $H = (U, F)$ along with a map $\varphi : V \to U$ so that the mincuts of G and the edges of H are in a one-to-one correspondence: for every edge $e \in F$, the pre-images of the two components of $H - e$ are the shores of the corresponding mincut.*

Actually, one does not really need here Theorem 7 since the Corollary follows directly from the well-known and easy property that every cross-free family can be represented by a tree.

Remark 3. Dinits and Vainshtein extended Theorem 7, as follows. Let $G = (V, E)$ be a graph with a terminal set $S \subseteq V$ of at least two elements. We say that a cut B of G separates S if both shores of B intersects S. Suppose that the minimum cardinality of a cut separating S is k. A subset $\emptyset \subset T \subset S$ is S-tight if there is a subset $X \subset V$ for which $d(X) = k$ and $T = S \cap X$.

Theorem 9 (Dinits and Vainshtein, [2])**.** *The S-tight sets admit a cactus representation.*

The proof of Theorem 9 is an easy extension of that of Theorem 7.

Note that the family of all minimum cuts separating S cannot be represented by a cactus if $|S| = 2$ since the number of minimum cuts separating nodes s and t may be exponential in $|V|$ while the number of cuts represented by a cactus is always less than $|V|^2$.

References

[1] E.A. Dinits, A.V. Karzanov and M.V. Lomonosov, *On the structure of a family of minimal weighted cuts in graphs*, in: Studies in Discrete Mathematics (in Russian), ed. A.A. Fridman, 290-306, Nauka (Moskva), 1976.

[2] Y. Dinits and A. Vainshtein, *The connectivity carcass of a vertex subset in a graph and its incremental maintenance*, Proc. 26th Annual ACM Symp. on Theory of Computing, 716-725.

[3] Tamás Fleiner and Tibor Jordán, *Coverings and structure of crossing families*, Connectivity augmentation of networks: structures and algorithms (Budapest, 1994), Mathematical Programming, **84(3)**, 505–518 (1999)

[4] H. Nagamochi and T. Ibaraki, Algorithmic aspects of graph connectivities, Cambridge University Press, 2008, Encyclopedia of Mathematics and Its Applications 123. (ISBN-13: 9780521878647).