Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2001-02. Published by the Egrervàry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres . ISSN 1587–4451.

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February 2001
Revised July 2001

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Abstract

By applying the matroid partition theorem of J. Edmonds [1] to a hypergraphic generalization of graphic matroids, due to M. Lorea [3], we obtain a generalization of Tutte's disjoint trees theorem for hypergraphs. As a corollary, we prove for positive integers k and q that every (kq)-edge-connected hypergraph of rank q can be decomposed into k connected sub-hypergraphs, a well-known result for $q = 2$. Another by-product is a connectivity-type sufficient condition for the existence of k edge-disjoint Steiner trees in a bipartite graph.

Keywords: Hypergraph; Matroid; Steiner tree

1 Introduction

An undirected graph $G = (V, E)$ is called **connected** if there is an edge connecting X and $V - X$ for every nonempty, proper subset X of V. Connectivity of a graph is equivalent to the existence of a spanning tree. As a connected graph on t nodes contains at least $t - 1$ edges, one has the following alternative characterization of connectivity.

Proposition 1.1. *A graph $G = (V, E)$ is connected if and only if the number of edges connecting distinct classes of $\mathcal{P}$ is at least $t-1$ for every partition $\mathcal{P} := \{V_1, V_2, \ldots, V_t\}$ of V into non-empty subsets.*

*Department of Operations Research, Eötvös University, Kecskeméti u. 10-12, Budapest, Hungary, H-1053 and Traffic Lab Ericsson Hungary, Laborc u. 1, Budapest, Hungary H-1037. e-mail: `frank@cs.elte.hu`. Supported by the Hungarian National Foundation for Scientific Research, OTKA T029772. Part of research was done while this author was visiting the Institute for Discrete Mathematics, University of Bonn, July, 2000.

**Department of Operations Research, Eötvös University, Kecskeméti u. 10-12, Budapest, Hungary, H-1053. e-mail: `tkiraly@cs.elte.hu`. Supported by the Hungarian National Foundation for Scientific Research, OTKA T029772.

***Institut für Mathematics, Universität Hannover, Welfengarten 1, D-30167 Hannover, Germany. e-mail: `kriesell@math.uni-hannover.de`.

February 2001. Revised July 2001

In [7] W.T. Tutte investigated the problem of decomposing a graph into a given number of connected subgraphs (spanning V), which problem is equivalent to that of finding k edge-disjoint spanning trees of G. He proved the following fundamental result.

Theorem 1.2 (Tutte). *An undirected graph $G = (V, E)$ contains k edge-disjoint spanning trees (or G can be decomposed into k connected spanning subgraphs) if and only if*

$$e_G(\mathcal{P}) \geq k(t-1) \tag{1}$$

holds for every partition $\mathcal{P} := \{V_1, V_2, \ldots, V_t\}$ of V into non-empty subsets where $e_G(\mathcal{P})$ denotes the number of edges connecting distinct classes of $\mathcal{P}$.

The goal of this note is to investigate possible generalizations of this result to hypergraphs. By a hypergraph we mean a pair $H = (V, \mathcal{E})$ where V is the node-set of H and $\mathcal{E}$ is a collection of not necessarily distinct nonempty subsets of V. The usual way to define a hypergraph H **connected** is to require the existence of a hyperedge of H intersecting both X and $V - X$ for every non-empty proper subset X of V. But the property formulated in Proposition 1.1 may also serve as a basis of an alternative concept of hypergraph connection. We say H to be **partition-connected** if $e_{\mathcal{E}}(\mathcal{P}) \geq |\mathcal{P}| - 1$ holds for every partition $\mathcal{P} = \{V_1, \ldots, V_t\}$ of V into non-empty classes where $e_{\mathcal{E}}(\mathcal{P})$ denotes the number of hyperedges intersecting at least two classes.

Partition-connectivity of hypergraphs clearly implies connectivity, and Proposition 1.1 states their equivalence for graphs. For general hypergraphs, however, a connected hypergraph need not be partition-connected since a partition-connected hypergraph must have at least $|V| - 1$ hyperedges while the hypergraph consisting of the single hyperedge $\{V\}$ is connected.

Therefore in an attempt to generalize Tutte's Theorem 1.2 for hypergraphs, there are (at least) two possibilities. We will show that the problem of decomposing a hypergraph into k (spanning) connected sub-hypergraphs is NP-complete for every integer $k \geq 2$. On the other hand, as a direct generalization of Tutte's theorem, a good characterization will be derived for hypergraphs which can be decomposed into k partition-connected sub-hypergraphs.

The problem of finding k disjoint spanning trees in a graph is a special case of that of finding k disjoint bases of a matroid and therefore Tutte's theorem may easily be derived from Edmonds' matroid partition theorem [1]:

Theorem 1.3 (Edmonds). *Let $M_i = (S, \mathcal{I}_i)$ be matroids on a common groundset S for $i = 1, \ldots, k$. Then the family $\mathcal{I}_\Sigma := \{I_1 \cup I_2 \cup \ldots \cup I_k : I_i \in \mathcal{I}_i\}$ forms the family of independent sets of a matroid M_Σ whose rank-function r_Σ is given by the following formula:*

$$r_\Sigma(Z) = \min\{\sum_{i=1}^{k} r_i(X) + |Z - X| : X \subseteq Z\}. \tag{2}$$

The matroid M_Σ defined in the theorem is called the **sum** of matroids $M_1, \ldots, M_k$. Our approach also makes use of this result and is based on an observation that the notion of circuit-matroids of graphs can be generalized to hypergraphs. This was done by M. Lorea [3].

In matroid theory a prime example is the class of of graphic matroids and in this light it is a bit strange that Lorea's pretty extension did not get any attention in the literature. Having not been aware of his construction, we also introduced hypergraphic matroids in the first version of the present work. It was András Recski who kindly drew our attention to Lorea's result, and hereby this is gratefully acknowledged.

We close this introductory section by listing some definitions and notation. By a **subpartition** $\mathcal{F}$ of U we mean a set of disjoint non-empty subsets of U. The members of $\mathcal{F}$ are called its **classes.** If the union of the classes is the ground-set U, we speak of a **partition** of U.

It will be convenient to associate a bipartite graph $G_H = (V, U_\mathcal{E}; F)$ with every hypergraph $H = (V, \mathcal{E})$ as follows. The elements of $U_\mathcal{E}$ correspond to the hyperedges of H (the elements of $\mathcal{E}$) so that a node $u_K \in U_\mathcal{E}$ corresponding to a hyperedge $K \in \mathcal{E}$ is adjacent to $v \in V$ in G_H if and only if $v \in K$. We will abbreviate $U_\mathcal{E}$ by U and sometimes will not distinguish between the set $\mathcal{E}$ of hyperedges and the corresponding set $U = U_\mathcal{E}$. Clearly, $|K|$ is the degree of u_K in G_H and $|U| = |\mathcal{E}|$. For a subset $\mathcal{F} \subseteq \mathcal{E}$ of hyperedges, let $U_\mathcal{F}$ denote the subset of nodes corresponding to the elements of $\mathcal{F}$ (in particular, $U_\mathcal{E} = U$). For a subset $X \subseteq U$, the subset of hyperedges corresponding to the elements of X is denoted by $\mathcal{E}_X$.

For $X \subseteq U$ let $\Gamma(X) := \{v \in V : v$ is adjacent in G_H to an element of $X\}$. The union of the hyperedges in $\mathcal{F}$ is denoted by $\cup(\mathcal{F})$. (Therefore $\Gamma(X) = \cup(\mathcal{E}_X)$). For a partition $\mathcal{P}$ of V, let $e_X(\mathcal{P})$ denote the number of elements of X having neighbours in at least two classes of $\mathcal{P}$. Similarly, for $\mathcal{F} \subseteq \mathcal{E}$ let $e_\mathcal{F}(\mathcal{P})$ denote the number of hyperedges in $\mathcal{F}$ intersecting at least two classes of $\mathcal{P}$. A hyperedge Z of H is **induced** by a subset X of V if $Z \subseteq X$. The number of hyperedges induced by X is denoted by $i_\mathcal{E}(X)$.

It is well-known (and easy to see anyway) that the set-function $|\Gamma(X)|$ is fully submodular. We say that the **strong Hall condition** holds for U in bipartite graph G_H if

$$|\Gamma(X)| \geq |X| + 1 \text{ for every non-empty subset } X \subseteq U. \quad (3)$$

A hypergraph $H = (V, \mathcal{E})$ is said to satisfy the strong Hall condition if the cardinality of the union of any $j \geq 1$ distinct hyperedges of H is at least $j + 1$.

We say that a sub-hypergraph $(V, \mathcal{E}')$ of a hypergraph $H = (V, \mathcal{E})$ is **spanning** if $V = \cup(\mathcal{E}')$. For a positive integer k, a hypergraph $H = (V, \mathcal{E})$ is called **k-edge-connected** (in short, **k-ec**) if $d_H(X) \geq k$ holds for every non-empty proper subset $X \subset V$ where $d_H(X)$ denotes the number of hyperedges intersecting both X and $V - X$. H is called **k-partition-connected** if $e_\mathcal{E}(\mathcal{P}) \geq k(|\mathcal{P}| - 1)$ holds for every partition $\mathcal{P} = \{V_1, \ldots, V_t\}$ of V. It is an easy exercise to show that H is k-partition-connected if and only if the deletion of at most j hyperedges results in at most $j/k + 1$ components.

A bipartite graph $G = (V, U; E)$ is called **elementary** if G is connected, $|V| = |U| \geq 1$, and $\Gamma(X) \geq |X| + 1$ holds for every nonempty proper subset X of V (which is equivalent to requiring the inequality for nonempty proper subsets of U.)

By a **hypercircuit** we mean a hypergraph the associated bipartite graph of which is elementary. Note that in the special case when the hypergraph is a graph, this notion coincides with the usual notion of a graph circuit. By a **hyperforest** we mean a hypergraph $H = (V, \mathcal{F})$ so that there is no subset $\mathcal{C} \subseteq \mathcal{F}$ for which $(\cup(\mathcal{C}), \mathcal{C})$ is a hypercircuit. This is equivalent to saying that H satisfies the strong Hall condition, or that there are at most $|X| - 1$ hyperedges of H included in X for every nonempty subset X of V. A hyperforest $H = (V, \mathcal{F})$ is a **spanning hypertree** if $V = \cup(\mathcal{F})$ and $|\mathcal{F}| = |V| - 1$.

Finally, we call a hypergraph H **forest representable** or **wooded** if it is possible to select two distinct elements from each hyperedge of H so that the chosen pairs, as graph edges, form a forest. If the representing forest may be chosen to be connected, then H is called **tree representable**.

2 Hypergraph connectivity and matroids

We start this section with a negative result.

Theorem 2.1. *The problem whether a hypergraph $H = (V, \mathcal{E})$ can be decomposed into k connected spanning sub-hypergraphs is NP-complete for every integer $k \geq 2$.*

Proof. Assume first that $k = 2$. Recall that the problem of colouring the nodes of a hypergraph by two colours so that no uni-coloured hyperedge arises is NP-complete. This implies that colouring the hyperedges of a hypergraph $H' = (V', \mathcal{E}')$ by red and blue so that every node belongs to a red and to a blue hyperedge is also NP-complete (that is, both the red and the blue hyperedges cover V). We show that this latter problem is polynomially solvable if there is a polynomial algorithm to decide decomposability of a hypergraph into two connected spanning sub-hypergraphs. To this end, let t be a new element, let $V := V' + t$, $\mathcal{E} := \{X + t : X \in \mathcal{E}'\}$ and $H = (V, \mathcal{E})$. Note that a sub-hypergraph $(V, \mathcal{F})$ of H is connected and spans V if and only if the corresponding sub-hypergraph $(V', \mathcal{F}')$ of H' covers the elements of V'. Therefore H can be decomposed into 2 connected spanning sub-hypergraphs if and only if H' can be decomposed into two sub-hypergraph each covering the ground-set.

The NP-completeness of the problem for $k \geq 3$ easily reduces to the special case $k = 2$. Let $H = (V, \mathcal{E})$ be a hypergraph and let H^+ denote the hypergraph arising from H by adding $k - 2$ copies of V as new hyperedges. As a hypergraph on ground-set V consisting of a single hyperedge V is connected, H^+ can be decomposed into k connected spanning sub-hypergraphs if and only if H can be decomposed into 2 connected spanning sub-hypergraphs. □

Let us turn our attention to decomposition of hypergraphs into partition-connected sub-hypergraphs. A basic fact in matroid theory is that the subforests of an undirected graph $G = (V, E)$ form the family of independent sets of a matroid on ground-set E, called the circuit-matroid of G. M. Lorea [3] extended this notion to hypergraphs.

P. Erdős had conjectured and L. Lovász [4] proved that the node set V of a hypergraph $H = (V, \mathcal{E})$ satisfying the strong Hall condition (that is, a hyperforest) can always be coloured by two colours so that there is no uni-coloured hyperedge. What Lovász actually proved (in a more general form) was the following result (which can also be derived from Edmonds' matroid intersection theorem).

Theorem 2.2. *A hypergraph H is a hyperforest if and only if it is wooded (or in other words, H meets the strong Hall condition if and only if H is forest representable).*

(Lovász' theorem may be formulated in terms of bipartite graphs: *A bipartite graph $(V, U; E)$ contains a subforest $F \subseteq E$ so that $d_F(v) = 2$ for every $v \in U$ if and only if the strong Hall condition holds for U.*)

This result clearly implies Erdős' conjecture since a forest is bipartite and a two-colouring of its nodes forms a required two-colouring of the hypergraph. [Note that the word *wooded* in Hungarian translates to *erdős*.]

Theorem 2.3 (Lorea). *Given a hypergraph $H = (V, \mathcal{E})$, the sub-hypergraphs of H which are hyperforests (or equivalently, the wooded sub-hypergraphs of H) form the family of independent sets of a matroid on ground-set $\mathcal{E}$.*

For completeness, we include here a proof.

Proof. We prove that hypercircuits satisfy the circuit axioms, that is, (a) no circuit may include another one and (b) the union of any two circuits remains dependent (that is, contains a circuit) even after discarding any element of their intersection.

The first axiom clearly holds. To derive the second one, let C_1 and C_2 be two subsets of $U_{\mathcal{E}}$ in the associated bipartite graph corresponding to two distinct hypercircuits $\mathcal{C}_1$ and $\mathcal{C}_2$ of H which has an element Z in common. We have $|C_1| + |C_2| = |\Gamma(C_1)| + |\Gamma(C_2)| \geq |\Gamma(C_1 \cap C_2)| + |\Gamma(C_1 \cup C_2)| \geq |C_1 \cap C_2| + 1 + |\Gamma(C_1 \cup C_2)|$ from which $|\Gamma(C_1 \cup C_2)| \leq |C_1| + |C_2| - |C_1 \cap C_2| - 1 = |C_1 \cup C_2| - 1$ and hence $\mathcal{C}_1 \cup \mathcal{C}_2 - \{Z\}$ cannot satisfy the strong Hall condition. Thus $\mathcal{C}_1 \cup \mathcal{C}_2 - \{Z\}$ must contain a hypercircuit. □

A matroid arising this way will be called the **circuit-matroid** of the hypergraph H and denoted by M_H. We call a matroid which is the circuit-matroid of a hypergraph a **hypergraphic** matroid. By choosing H to be the hypergraph consisting of a three-element groundset V and of four copies of V as hyperedges, we observe that the uniform matroid $U_{4,2}$ (the smallest non-binary matroid) is hypergraphic. We note that, unlike graphic matroids, hypergraphic matroids are not necessarily closed under contraction.

Let us recall another result of Edmonds. A set-function $b : 2^U \to \mathbf{Z}$ is called **intersecting submodular** if

$$b(X) + b(Y) \geq b(X \cap Y) + b(X \cup Y) \tag{4}$$

holds for every pair of subsets $X, Y \subseteq U$ with non-empty intersection. If (4) holds for disjoint subsets X and Y, as well, then b is called **fully** submodular. A set-function b is called monotone nondecreasing if $b(X) \geq b(Y)$ whenever $X \supset Y$. Throughout we will assume that a set-function is zero on the empty set. The other theorem of Edmonds [2] we need is as follows.

Theorem 2.4. *Let b be a non-negative, integer-valued, intersecting submodular set-function on a ground-set U. Then*

$$\mathcal{I}_b := \{I \subseteq U : b(Y) \geq |Y \cap I| \text{ for every } Y \subseteq U\} \tag{5}$$

forms the family of independent sets of a matroid $M = (U, \mathcal{F}_b)$ whose rank-function is given by

$$r_b(Z) = \min\{\sum b(X_i) + |Z - (X_1 \cup X_2 \cup \ldots \cup X_t)| : \{X_1, \ldots, X_t\} \text{ a subpartition of } U\}. \tag{6}$$

Furthermore, if b is monotone nondecreasing, then

$$\mathcal{I}_b = \{I \subseteq U : b(Y) \geq |Y| \text{ for every } Y \subseteq I\} \tag{7}$$

and

$$r_b(Z) = \min\{\sum b(X_i) + |Z - (X_1 \cup X_2 \ldots \cup X_t)| : \{X_1, \ldots, X_t\} \text{ a subpartition of } Z\}. \tag{8}$$

In order to determine the rank-function of the circuit matroid M_H, we need the following:

Alternative proof of Theorem 2.3. Let us define a set-function $b_H : 2^U \to \mathbf{Z}_+$ as follows. For a non-empty subset $X \subseteq U$ let

$$b_H(X) = |\Gamma(X)| - 1, \tag{9}$$

and let $b_H(\emptyset) := 0$. Since $|\Gamma(X)|$ is a fully submodular function, b_H is intersecting submodular. Obviously, b_H is monotone nondecreasing. Let us consider the matroid M_{b_H} defined in Theorem 2.4. In this a subset $I \subseteq U$ is independent if $|X| \leq b_H(X)$ holds for every subset $X \subseteq U$. This is equivalent to requiring that the sub-hypergraph $(V, \mathcal{I})$ meets the strong Hall condition, that is, by Lovász' theorem, $(V, \mathcal{I})$ is wooded. □

Theorem 2.5. *The rank function r_H of the circuit-matroid of a hypergraph H is given by the following formula.*

$$r_H(Z) = \min\{|V| - |\mathcal{P}| + e_Z(\mathcal{P}) : \mathcal{P} \text{ a partition of } V\}. \tag{10}$$

Proof. It suffices to prove the formula for the special case $Z = U$ since the value of $e_Z(\mathcal{P})$ does not change if the nodes in $U - Z$ are deleted from the representing bipartite graph G_H.

Let $H' = (V, \mathcal{E}')$ be a wooded sub-hypergraph of H (that is, $U_{\mathcal{E}'}$ an independent subset of matroid M_H). Then, for every partition $\mathcal{P}$ of V, there are at most $|V| - |\mathcal{P}|$ hyperedges in H' which are subsets of a class of $\mathcal{P}$. Therefore $|\mathcal{E}'|$ cannot be bigger than $|V| - |\mathcal{P}| + e_{\mathcal{E}'}(\mathcal{P})$, that is, $r_H(\mathcal{E}) \leq |V| - |\mathcal{P}| + e_{\mathcal{E}}(\mathcal{P})$. The proof will be completed by proving the existence of a partition for which equality holds.

By (8) we have

$$r_H(U) = \min\{\sum b_H(Z_i) + |U - \cup_i Z_i| : \{Z_1, \dots, Z_l\} \text{ a subpartition of } U\}. \quad (11)$$

A subpartition where the minimum is attained will be called a **minimizer** of (11). Let $\mathcal{F} := \{Z_1, \dots, Z_l\}$ be a minimizer of (11) so that $|\mathcal{F}|$ is as small as possible.

We claim that $\Gamma(Z_i) \cap \Gamma(Z_j) = \emptyset$ holds whenever $1 \leq i < j \leq l$. Indeed, as subset Z_i and Z_j are disjoint, $|\Gamma(Z_i \cup Z_j)| = |\Gamma(Z_i)| + |\Gamma(Z_j)| - |\Gamma(Z_i) \cap \Gamma(Z_j)|$. Therefore $\Gamma(Z_i) \cap \Gamma(Z_j) \neq \emptyset$ implies $|\Gamma(Z_i \cup Z_j)| \leq |\Gamma(Z_i)| + |\Gamma(Z_j)| - 1$, and hence $b_H(Z_i \cup Z_j) \leq b_H(Z_i) + b_H(Z_j)$. This is however impossible since by replacing Z_i and Z_j in $\mathcal{F}$ by their union we obtain another minimizer subpartition $\mathcal{F}'$ of $U_\mathcal{E}$ for which $|\mathcal{F}'| < |\mathcal{F}|$.

We claim furthermore that $\Gamma(u) \not\subseteq \Gamma(Z_i)$ for every node $u \in U_\mathcal{E} - \cup_i Z_i$. Indeed, if we had $\Gamma(u) \subseteq \Gamma(Z_i)$, then for $Z'_i := Z_i + u$ we would have $\Gamma(Z'_i) = \Gamma(Z_i)$ and then by replacing Z_i by $Z'_i := Z_i + u$ we would obtain a subpartition $\mathcal{F}'$ of $U_\mathcal{E}$ for which $\sum_{Z\in\mathcal{F}'} b_H(Z) + |U - \cup_{Z\in\mathcal{F}'} Z| < \sum_{Z\in\mathcal{F}} b_H(Z) + |U - \cup_{Z\in\mathcal{F}} Z|$ contradicting that $\mathcal{F}$ is a minimizer of (11).

Let $\mathcal{P}$ be the following partition of V. For each member Z_i of $\mathcal{F}$, let $\Gamma(Z_i)$ be a member of $\mathcal{P}$, and for each element $z \in V - \cup_i \Gamma(Z_i)$, let $\{z\}$ be a member of $\mathcal{P}$. By the claims above $e_\mathcal{E}(\mathcal{P}) = |U - \cup Z_i|$. Since $|\mathcal{P}| = |\mathcal{F}| + |V - \cup_i \Gamma(Z_i)|$ we have $r_H(\mathcal{E}) = \sum_i b_H(Z_i) + |U - \cup Z_i| = \sum_i (|\Gamma(Z_i)| - 1) + e_\mathcal{E}(\mathcal{P}) = \sum_i |\Gamma(Z_i)| - |\mathcal{F}| + e_\mathcal{E}(\mathcal{P}) = |V| - |\mathcal{P}| + e_\mathcal{E}(\mathcal{P})$, as required. ☐

Corollary 2.6. *The rank of the circuit matroid of a hypergraph $H = (V, \mathcal{E})$ is $|V| - 1$ if and only if H is partition-connected.*

Proof. By definition $r_H(\mathcal{E}) \leq |V| - 1$ and it follows from (10) that equality holds precisely if $|V| - |\mathcal{P}| + e_{\mathcal{E}_Z}(\mathcal{P}) \geq |V| - 1$ holds for every partition $\mathcal{P}$ of V, that is, $e_{\mathcal{E}_Z}(\mathcal{P}) \geq |\mathcal{P}| - 1$, which in turn is equivalent to the partition connectivity of H. ☐

For a positive integer k let M_{kH} denote the sum of k copies of matroid M_H. This matroid is also defined on the set of hyperedges and a subset of hyperedges is independent by definition if it can be decomposed into k wooded sub-hypergraphs.

Theorem 2.7. *The rank-function r_{kH} of matroid M_{kH} is given by the following formula.*

$$r_{kH}(Z) = \min\{k(|V| - |\mathcal{P}|) + e_{\mathcal{E}_Z}(\mathcal{P}) : \mathcal{P} \textit{ a partition of } V\} \quad (12)$$

Proof. Again, it suffices to prove the formula for the special case $Z = U$. Also, as Theorem 2.5 contained the special case $k = 1$, we may assume that $k \geq 2$. As an independent set of M_{kH} may contain at most $k(|V_i| - 1)$ hyperedges which are subsets of some partition class $V_i \in \mathcal{P}$, we see that $r_{kH}(U)$ is at most $k(|V| - |\mathcal{P}|) + e_\mathcal{E}(\mathcal{P})$.

To see the reverse inequality we must find a partition $\mathcal{P}$ of V for which $r_{kH}(U) = k(|V| - |\mathcal{P}|) + e_\mathcal{E}(\mathcal{P})$ holds. Combining (2) and (10) we obtain that $r_{kH}(U) = \min_{F\subseteq U}[kr_H(F) + |U - F|] = \min_{F\subseteq U}[k(\min\{|V| - |\mathcal{P}| + e_F(\mathcal{P}) : \mathcal{P}$ a partition of $V\})\ + |U - F|]$. Let F be a set for which the minimum is attained and let $\mathcal{P}$ be a partition of V for which the inner minimum is attained.

We claim that $e_F(\mathcal{P}) = 0$. Indeed, if there were indirectly an element $z \in F$ which has neighbours in at least two classes of $\mathcal{P}$, then we would have $k\ e_{F'}(\mathcal{P}) + |U - F'| = k[e_F(\mathcal{P}) - 1] + [|U - F| + 1] = k\ e_F(\mathcal{P}) + |U - F| - (k-1)$ for $F' := Z - z$ and, as $k \geq 2$, this would contradict the assumption that F is a minimizer.

We also claim that there is no element $x \in U - F$ with $e_x(\mathcal{P}) = 0$ for otherwise $e_{F'}(\mathcal{P}) = e_F(\mathcal{P})$ would hold for $F' := F + x$ which would contradict, by $|U - Z'| = |U - F| - 1$, the assumption that F is a minimizer.

We can conclude that $U - F$ consists of exactly those elements of U which have neighbours in at least two classes of $\mathcal{P}$, that is, $|U - F| = e_U(\mathcal{P})$. Hence we obtain that $r_{kH}(U) = k[|V| - |\mathcal{P}| + e_F(\mathcal{P})] + |U - F| = k[|V| - |\mathcal{P}| + 0] + e_U(\mathcal{P})$, as required. □

The following result is a direct generalization of Theorem 1.2 of Tutte.

Theorem 2.8. *A hypergraph $H = (V, \mathcal{E})$ is k-partition-connected if and only if H can be decomposed into k sub-hypergraphs each of which is partition-connected.*

Proof. If the decomposition exists, the hypergraph is clearly k-partition-connected. To see the other direction, observe first that, by Theorem 2.5, a sub-hypergraph $H' = (V, \mathcal{E}')$ of H is partition-connected if and only if $r_H(\mathcal{E}') = |V| - 1$.

Suppose now that H is k-partition-connected, that is, $e_{\mathcal{E}}(\mathcal{P}) \geq k(|\mathcal{P}| - 1)$ holds for every partition $\mathcal{P}$ of V. This is equivalent to requiring that $k(|V| - |\mathcal{P}|t) + e_{\mathcal{E}}(\mathcal{P}) \geq k(|V| - 1)$ that is, by Theorem 2.7, $r_{kH} = k(|V| - 1)$. Therefore the cardinality of every basis of matroid M_{kH} is $k(|V| - 1)$ and every basis is the union of k bases of M_H. It follows that the rank of M_H is $|V| - 1$ and that M_H admits k pairwise disjoint bases. By Corollary 2.6, a sub-hypergraph $H' = (V, \mathcal{E}')$ of H is partition-connected if and only if $r_H(\mathcal{E}') = |V| - 1$, the existence of k disjoint bases of M_H means that H can be decomposed into k partition-connected sub-hypergraphs. □

By the **rank** of a hypergraph we mean the cardinality of its largest hyperedge. A well-known corollary of Theorem 1.2 of Tutte is that a $(2k)$-edge-connected graph always contains k disjoint spanning trees. As a direct extension of this result we derive the following.

Corollary 2.9. *A (qk)-edge-connected hypergraph H of rank at most q can be decomposed into k partition-connected sub-hypergraphs and hence into k connected spanning sub-hypergraphs.*

Proof. By Theorem 2.8 it suffices to show that H is k-partition-connected. Let $\mathcal{P}$ be a partition of V. By the (qk)-edge-connectivity, there are at least qk hyperedges of H intersecting both V_i and $V - V_i$ for each class V_i of $\mathcal{P}$. Since one hyperedge may intersect at most q classes, we obtain that the total number of hyperedges intersecting more than one class is at least $(qk)|\mathcal{P}|/q > k(|\mathcal{P}| - 1)$, that is, H is indeed k-partition-connected, and therefore, by Theorem 2.8, H decomposes into k partition-connected sub-hypergraphs. □

We close this section by mentioning another corollary of Theorem 2.7. This is a direct generalization of Nash-Williams's well-known theorem on partitioning a graph into k forests [6].

Theorem 2.10. *The edge-set $\mathcal{E}$ of a hypergraph $H = (V, \mathcal{E})$ can be decomposed into k hyperforests if and only if for the number of hyperedges induced by X we have*

$$i_{\mathcal{E}}(X) \leq k(|X| - 1) \tag{13}$$

for every nonempty subset X of V.

Proof. Since one hyperforest may contain at most $|X| - 1$ hyperedges induced by X, (13) is necessary.

To see the sufficiency, let $\mathcal{P} := \{V_1, \ldots, V_t\}$ be a partition of V into nonempty classes. By (13) each class V_i includes at most $k(|V_i| - 1)$ hyperedges, so the number $e_{\mathcal{E}}(\mathcal{P})$ of hyperedges intersecting at least two classes of $\mathcal{P}$ is at least $|\mathcal{E}| - \sum_{j=1}^{t} k(|V_j| - 1) = |\mathcal{E}| - k(|V| - t)$. Therefore $k(|V| - |\mathcal{P}|) + e_{\mathcal{E}}(\mathcal{P}) \geq |\mathcal{E}|$ holds for every partition $\mathcal{P}$ of V and hence $\mathcal{E}$ is independent in matroid M_{kH} by formula (12), that is, $\mathcal{E}$ decomposes into k hyperforests. □

3 Disjoint Steiner-trees

Let $G = (V_0, E)$ be an undirected graph with a so called terminal set $V \subseteq V_0$. By a **Steiner tree** of G (spanning V) we mean a subtree (V', E') for which $E' \subseteq E$ and $V \subseteq V' \subseteq V_0$. (No assumption is made on the minimality of $|E'|$.) The **disjoint Steiner trees** problem consists of finding k edge-disjoint Steiner trees of G. When $V = V_0$ we are interested in the existence of k disjoint spanning trees of G and in this case Theorem 1.2 of Tutte provides a characterization. When $|V| = 2$, a Steiner tree is a path connecting the two terminal nodes, and the edge-disjoint, undirected version of Menger's theorem gives an answer. For general V, however, the problem is known to be NP-complete, so deriving sufficient conditions for the existence of k disjoint Steiner trees may be of some interest. We say that G is **k-edge-connected in** V if every cut of G separating two elements of V has at least k edges. By Menger's theorem this is equivalent to requiring for every two elements u and v of V that there are k edge-disjoint paths in G connecting u and v.

Theorem 3.1. *Let $G = (V_0, E)$ be an undirected graph and $V \subset V_0$ a subset of nodes so that $U := V_0 - V$ is stable and G is $(3k)$-edge-connected in V. Then G contains k edge-disjoint Steiner trees spanning V.*

Proof. We use induction on the value $\mu_G := \sum(\max(0, d_G(v) - 3) : v \in U)$. Suppose first that μ_G is zero, that is, the degree of each node in U is at most 3. We may assume that V is also stable for otherwise each edge induced by V can be subdivided by a new node. Such an operation may add new nodes of degree two to the complement of V but it does not affect (qk)-edge-connectivity in V and k disjoint Steiner trees in the new graph determine k disjoint Steiner trees in G.

Let $H = (V, \mathcal{E})$ be the hypergraph corresponding to G, that is, for each element u of U, there is a corresponding hyperedge of H consisting of the neighbours of u in G. As the degree of each element of U is at most 3, the rank of H is at most 3.

For any non-empty, proper subset X of V let X' denote the set of those elements of U which have at least one neighbour in X and at most one neighbour in $V - X$ in the graph G. Since every degree in U is at most 3, we have $d_G(X \cup X') = d_H(X)$ and hence the $(3k)$-edge-connectivity of G implies the $(3k)$-edge-connectivity of H.

By Corollary 2.9, U can be partitioned into k disjoint subsets $U_1, \ldots, U_k$ so that $V \cup U_i$ induces a connected subgraph $G_i = (V \cup U_i, E_i)$ of G for each $i = 1, \ldots, k$. By choosing a spanning tree F_i from each G_i, we obtain the required edge-disjoint Steiner trees of G.

Suppose now that μ_G is positive and that the theorem holds for each graph G' with $\mu_{G'} < \mu_G$. Let $s \in U$ be a node with $d_G(s) \geq 4$. If there is a cut-edge e of G, then the elements of V belong to the same component of $G - e$ as G is at least k-edge-connected in V and then we may discard the other component of $G - e$ without destroying $(3k)$-edge-connectivity in V. Therefore we may assume that G is 2-edge-connected.

By Mader's undirected splitting theorem [5] there are two edges $e = vs$, $f = zs$ in E so that replacing e and f by a new edge vz the local edge-connectivities do not drop. In particular, the resulting graph G' continues to be $(3k)$-edge-connected in V. By induction there are k edge-disjoint Steiner trees in G'. If one of these trees contains the split-off edge vz, we replace it by e and/or f in order to obtain a Steiner tree of G. Therefore we have proved the existence of k edge-disjoint Steiner trees of G. □

In this proof we relied on the assumption made on the stability of U. It is not clear whether this assumption could possibly be left out. We do not know either if the hypothesis on $(3k)$-edge-connectivity can be weakened, perhaps to $(2k)$-edge-connectivity. This is certainly so for graphs in which the degree of each node in U is even and, in particular, for Eulerian graphs. The proof of this is almost identical to that of Theorem 3.1.

As far as algorithmic aspects are concerned, Edmonds' matroid partition algorithm may be used to compute a decomposition of a hypergraph into k partition-connected sub-hypergraphs or to compute a deficient partition to show that such a decomposition does not exist. Therefore Theorem 3.1 can be used for an approximation algorithm to compute the maximum number ν of disjoint Steiner trees. V is clearly ν edge-connected in G so, by Theorem 3.1, one can compute $\nu/3$ disjoint Steiner trees.

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