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C_4–free 2-factors in bipartite graphs

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Abstract

D. Hartvigsen [H] recently gave an algorithm to find a C_4–free 2–factor in a bipartite graph and using this algorithm he proved several nice theorems. Now we give a simple inductive proof for a generalization of his Tutte-type theorem, and prove the corresponding Ore-type theorem as well. The proof follows the idea of the inductive proof for the Hall's theorem given by Halmos and Vaughn [HV].

Keywords: graphs; factors

1 Introduction

Let $G = (V, E)$ be a bipartite graph with bipartition $V = X \cup Y$, and let $f : V \to \{1, 2\}$ be a function on the vertices. For a vertex $v \in V$ we call the value $f(v)$ the weight of v. For a set $V' \subseteq V$ we write $f(V')$ for $\sum_{v \in V'} f(v)$. A subgraph $G' \subseteq G$ is called an f–factor if $\deg_{G'}(v) = f(v)\ \forall v \in V$. An f-factor is called C_4–free if it does not contain the four-cycle as a subgraph.

For $Z \subseteq V$ let $G-Z$ denote the subgraph of G induced by $V \setminus Z$. Let $I(G-Z)$ denote the set of isolated vertices in $G-Z$, $K(G-Z)$ denote the set of two-vertex components in $G-Z$ with total weight 4, and $C(G-Z)$ denote the set of C_4 components in $G-Z$ with total weight 8. Moreover let $q(Z) = f(I(G-Z)) + 2|K(G-Z)| + 2|C(G-Z)|$.

We call a subset Z *violating* if $f(Z) < q(Z)$ and *tight* if $f(Z) = q(Z)$. A subset Z is called *non-trivial* if it intersects both X and Y.

2 The theorem

Theorem 2.1. *The bipartite graph G has a C_4–free f–factor iff*

$$\text{for all } Z \subseteq V: \quad f(Z) \geq q(Z). \qquad (*)$$

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Proof. The necessity is trivial: at most $f(Z)$ edges of an f–factor can leave Z but at least $q(Z)$ edges of a C_4–free f–factor must leave $V \setminus Z$.

Now we prove the sufficiency by induction on $f(V)$. If there are no edges connecting weight-two vertices then the theorem follows easily from Hall's theorem.

Induction step: Suppose G satisfies (*) and has an edge with weight-2 endvertices. As $Z = X$ and $Z = Y$ are not violating clearly $f(X) = f(Y)$. Our aim is to either find a tight set and along this set cut the problem into two independent smaller problems; or reduce the graph to a smaller one, find a factor in that by induction and construct a good factor of the original graph. Unlike in the well known proof of Halmos and Vaughn [HV] for Hall's theorem, here the second case is more difficult to handle. First we define two procedures.

Edge−reduction(x, y): if $xy \in E, f(x) = f(y) = 2$ then delete the edge xy and let $f'(x) = f'(y) = 1$.

C_4*−reduction*(x_1, y_1, x_2, y_2): if $x_i \in X,\ y_i \in Y,\ x_iy_j \in E$ and $f(x_i) = f(y_i) = 2$ for $i, j = 1, 2$ then contract x_1 and x_2 into a new vertex x, contract y_1 and y_2 into a new vertex y, delete the parallel edges as well as the edge xy, and let $f'(x) = f'(y) = 1$.

Take any edge uv with $f(u) = f(v) = 2$. Let $H_1, H_2, \dots, H_t$ denote the C_4–subgraphs of G containing the edge uv and having total weight 8. If $t > 0$, call the C_4–reduction procedure with $V(H_1) = \{x_1, y_1, x_2, y_2\}$. If the resulting graph G_1 has a C_4–free f'–factor F' then it can be easily extended to a C_4–free f–factor F of G: if F' has edges originally going to x_i and y_j then extend F' by a three-length path from x_i to y_j inside H_1. So – by induction – if the reduced graph satisfies (*) then we are done. Thus we may assume that the reduced graph had a set violating (*). Let Z'_1 be a maximal violating set.

Claim 2.2. *Z'_1 must contain both contracted vertices.*

We postpone the proof of Claim 2.2. If $t > 1$, call procedure C_4–reduction on the vertices of $H_2, \dots, H_t$, each time starting from G. If in any case this results in a graph with a C_4–free f'–factor, then, as before, we are done. So let us assume this is not the case and we have maximal violating sets $Z'_1, \dots, Z'_t$. By our claim, we may assume each Z'_k contains both contracted vertices of G_k.

Now we call procedure Edge–reduction with vertices u and v, again starting from G. If the reduced graph G' satisfies (*) then – by induction – we have a C_4–free f'–factor F' in it. In this case $F = F' + uv$ is a C_4–free f–factor of G. Indeed, F is clearly an f–factor, we must only check that there is no C_4 containing uv in F. But if there is, it must be H_k for some k. This is impossible because in this case F minus the edges of H_k would be a C_4–free factor of $G_k - x - y$ (where x and y are the contracted vertices), but $Z'_k - x - y$ is a violating set in this graph: $f'(Z'_k - x - y) = f'(Z'_k) - 2$ and Z'_k was violating.

So we may assume that there is a maximal violating set Z' in G'. First observe that from the maximality of Z' each component of $G' - Z'$ that is not an isolated node must be either a weight-4 K_2 or a weight-8 C_4. For any other component C wlog. assume that $f(C \cap X) \le f(C \cap Y)$, and now $Z' \cup (C \cap X)$ is a larger violating set. Observe further that $f'(Z') \le q'(Z') - 2$. Indeed, as $f'(X') = f'(Y') = f(X) - 1$, $f'(Z')$ and $q'(Z')$ have the same parity. We claim that $Z = Z'$ is a tight non-trivial set in G. Replacing the edge uv and increasing $f(u)$ and $f(v)$ to two can increase $f(Z) - q(Z)$ only if $u, v \in Z$ and even in this case it is increased by two. If $u \notin Z$ then u is a weight-1 isolated vertex of $G' - Z'$ so $q(Z)$ is also increased if $v \in Z$ and neither $f(Z)$ nor $q(Z)$ is changed if $v \notin Z$, so Z would be violating in G which is not possible.

Proof of Claim 2.2 Let x and y be the contracted vertices of G_1 and suppose that eg. $y \notin Z_1'$. Again, from the maximality of Z_1' each component of $G_1 - Z_1'$ that is not an isolated node must be either a weight-4 K_2 or a weight-8 C_4. We construct set Z_1 : take Z_1' and if it contains one of the contracted vertices then replace it by its ancestors. We claim that Z_1 is a violating set in G which is impossible by our assumption. Indeed, resettling the original graph and increasing the f–values on the four vertices of H_1 cannot increase $f(Z_1) - q(Z_1)$, because when the contracted vertices were outside of Z_1' then neither $q(Z_1)$ nor $f(Z_1)$ was changed (both contracted vertices were isolated in G_1), and if x was inside and y was outside then $f(Z_1)$ is increased by three and $q(Z_1)$ is also increased by three – we noted that y was an isolated vertex of $G_1 - Z_1'$. □

We can summarize the previous paragraphs as follow. Either we found a maximal non-trivial tight set in G, or we found a good f–factor using induction.

Now let Z be a maximal non-trivial tight set. By the maximality of Z we have again that each non-isolated component of $G - Z$ must be either a weight-4 K_2 or a weight-8 C_4. Call procedure Edge–reduction for all K_2 components of $G - Z$ and procedure C_4–reduction for all C_4 components of $G - Z$ one after the other, and delete all edges inside Z. The resulting graph G' can be broken down into two parts: G_1 that is spanned by $(Z \cap X') \cup (Y' \setminus Z)$ and G_2 that is spanned by $(X' \setminus Z) \cup (Z \cap Y')$. Clearly no edge connects the two parts and, by the non-triviality of Z, both have smaller total weight than G. Observe that Z is tight in G' because reductions did not change $q(Z)$. Furthermore, as $f(X) = f(Y)$, after reductions $f'(X) = f'(Y)$. By tightness $f'(Z) = f'(V' \setminus Z)$ and so $f'(Z \cap X') = f'(Y' \setminus Z)$ and $f'(X' \setminus Z) = f'(Z \cap Y')$.

Claim 2.3. *Both G_1 and G_2 satisfy (*).*

Proof. Suppose for example that Z_1 is a violating set for G_1. Let $Z' = Z_1 \cup (Z \cap Y')$. Using the observation above and the fact that $G' - Z$ consists of isolated vertices Z' is a violating set in G'.

Now let $Z^* \subseteq V$ be the set we get from Z' if all contracted vertices in Z' are replaced by their ancestors. As Z' can contain contracted vertices in Y' only, it is easy to see that Z is violating in G (in every reversed reduction step we increase $f(Z)$ and $q(Z)$ by the same amount). This is a contradiction. □

By induction now we have a good f'-factor F_1 in G_1 and a good f'-factor F_2 in G_2. Clearly $F' = F_1 \cup F_2$ is a C_4–free f'–factor of G'.

Define F as follows. Take F', add all reduced edges, and for every reduced C_4 add the appropriate 3-length path between the two vertices having non-zero degree in F'. Clearly F is an f–factor, it remains to check that it is C_4–free. When we add edge xy to F' then x was matched to $Z \cap Y$ and y was matched to $Z \cap X$ and F' does not contain any edge connecting these two sets. When we add a length-3 path then no edge of F' connects the two endvertices. □ □

If $f \equiv 2$ then we can divide both sides of (*) by two:

Corollary 2.4. *A bipartite graph G has a C_4–free 2–factor iff*

$$\text{for all } Z \subseteq V: \quad |Z| \geq |I(G-Z)| + |K(G-Z)| + |C(G-Z)|. \tag{**}$$

The Ore-type deficiency version can be obtained also easily:

Corollary 2.5. *Let G be a bipartite graph with bipartition $V = X \cup Y$. Let $def_U(G)$ denote the minimum of $f(U) - f'(U)$ where the minimum is taken over all functions f' such that $0 \leq f'(v) \leq f(v)$ for all $v \in V$ and G has a C_4–free f'–factor. Then*

$$def_X(G) = \frac{\max_{Z\subseteq V}\big(q(Z) - f(Z)\big) - f(Y) + f(X)}{2}, \text{ and} \tag{1}$$

$$def_V(G) = \max_{Z\subseteq V}\big(q(Z) - f(Z)\big). \tag{2}$$

Proof. Let $k = \frac{\max_{Z\subseteq V}\left(q(Z)-f(Z)\right)+f(Y)-f(X)}{2}$, $l = \frac{\max_{Z\subseteq V}\left(q(Z)-f(Z)\right)-f(Y)+f(X)}{2}$. Observe that k and l are non-negative integers (if Z' is the set where the maximum is taken, each component D of $G-Z'$ with at least two vertices has $f(D\cap X) = f(D\cap Y)$). Construct graph G^* and function f^* as follows. Add k new vertices to X obtaining X^* and l new vertices to Y obtaining Y^*, let V' denote the set of all new vertices. Connect all new vertices to every vertex of the other class. For an old vertex v let $f^*(v) = f(v)$ and for a new one let $f^*(v) = 1$. Clearly $f^*(X^*) = f^*(Y^*)$ and moreover G^* satisfies (*), because if Z^* is a maximal violating set then it contains all new vertices and so for $Z = Z^* \setminus V'$ clearly $q(Z) - f(Z)$ is bigger than the maximum. By the Theorem we have a C_4–free f^*–factor in G^*. After deleting the new vertices we get the desired f'–factor of G where $f'(v) = f(v)$ minus the number of edges incident with v deleted. □

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