Egerváry Research Group
on Combinatorial Optimization

Technical reportS

TR-2001-14. Published by the Egrerváry Research Group, Pázmány P. sétány 1/C, H–1117, Budapest, Hungary. Web site: www.cs.elte.hu/egres . ISSN 1587–4451.

Highly edge-connected detachments of graphs and digraphs

Alex R. Berg, Bill Jackson, and Tibor Jordán

June 7, 2001

Highly edge-connected detachments of graphs and digraphs

Alex R. Berg*, Bill Jackson**, and Tibor Jordán***

Dedicated to the memory of Crispin Nash-Williams.

Abstract

Let $G = (V, E)$ be a graph or digraph and $r : V \to Z_+$. An r-detachment of G is a graph H obtained by 'splitting' each vertex $v \in V$ into $r(v)$ vertices. The vertices $v_1, ..., v_{r(v)}$ obtained by splitting v are called the *pieces* of v in H. Every edge $uv \in E$ corresponds to an edge of H connecting some piece of u to some piece of v. Crispin Nash-Williams [9] gave necessary and sufficient conditions for a graph to have a k-edge-connected r-detachment. He also solved the version where the degrees of all the pieces are specified. In this paper we solve the same problems for directed graphs. We also give a simple and self-contained new proof for the undirected result.

1 Introduction

All graphs and digraphs considered are finite and may contain loops and multiple edges. Let $G = (V, E)$ be a graph and $r : V \to Z_+$. An *r-detachment* of G is a graph H obtained by 'splitting' each vertex $v \in V$ into $r(v)$ vertices. The vertices $v_1, ..., v_{r(v)}$ obtained by splitting v are called the *pieces* of v in H. Every edge $uv \in E$ corresponds to an edge of H connecting some piece of u to some piece of v. An *r-degree specification* is a function f on V, such that, for each vertex $v \in V$, $f(v)$ is a partition of $d(v)$ into $r(v)$ positive integers. An *f-detachment* of G is an r-detachment in which the degrees of the pieces of each $v \in V$ are given by $f(v)$.

Crispin Nash-Williams [9] obtained the following necessary and sufficient conditions for a graph to have a k-edge-connected r-detachment or f-detachment. For X, Y disjoint subsets of $V(G)$, let $d(X, Y)$ be the number of edges of G from X to Y, and let $d(X) = d(X, V - X)$. A graph $G = (V, E)$ is *k-edge-connected* if $d(X) \geq k$ for

*BRICS, Department of Computer Science, University of Aarhus, Ny Munkegade building 540, 8000 Aarhus C, Denmark. e-mail: `aberg@brics.dk`.

**Department of Mathematical and Computing Sciences, Goldsmiths College, London SE14 6NW, England. e-mail: `b.jackson@gold.ac.uk`.

***Department of Operations Research, Eötvös University, Kecskeméti utca 10-12, 1053 Budapest, Hungary. e-mail: `jordan@cs.elte.hu`. Supported by the Hungarian Scientific Research Fund, grant no. T29772 and F34930.

every proper subset $X \subset V$. Let $e(X)$ be the number of edges between the vertices of X, $b(X)$ the number of components of $G - X$ and $r(X) = \sum_{x \in X} r(x)$. For $v \in V$, we use $deg(v)$ to denote the degree of v. Thus $e(v)$ is the number of loops incident to v and $deg(v) = d(v) + 2e(v)$.

Theorem 1.1 (Nash-Williams). *Let $G = (V, E)$ be a graph and $r : V \to Z_+$. Then G has a connected r-detachment if and only if $r(X) + b(X) \leq e(X) + e(X, V - X) + 1$ for every $X \subseteq V$.*
Furthermore, if G has a connected r-detachment then G has a connected f-detachment for every r-degree specification f.

Theorem 1.2 (Nash-Williams). *Let $G = (V, E)$ be a graph, $r : V \to Z_+$, and $k \geq 2$ be an integer. Then G has a k-edge-connected r-detachment if and only if*
(a) G is k-edge-connected,
(b) $d(v) \geq kr(v)$ for each $v \in V$,
and neither of the following statements is true:
(c) k is odd and G has a cut-vertex v such that $d(v) = 2k$, $e(v) = 0$ and $r(v) = 2$,
(d) k is odd, $|V| = 2$, $|E| = 2k$, and $r(v) = 2$ and $e(v) = 0$ for each vertex $v \in V$.
Furthermore, if G has a k-edge-connected r-detachment then G has a k-edge-connected f-detachment for any r-degree specification f for which each term d_i^v is at least k for every $v \in V$ and every $1 \leq i \leq r(v)$.

In this paper we give necessary and sufficient conditions for a digraph to have a k-edge-connected r-detachment or f-detachment. Let $D = (V, E)$ be a digraph. For two disjoint subsets X, Y of V let $\rho(X, Y)$ denote the number of edges from Y to X and let $\rho(X) = \rho(X, V - X)$. Let $\delta(X, Y) = \rho(Y, X)$ and $\delta(X) = \rho(V - X)$. A digraph $D = (V, E)$ is *k-edge-connected* if $\rho(X) \geq k$ for every proper subset $X \subset V$. Let $d(X, Y) = \rho(X, Y) + \delta(X, Y)$. We use $e(v)$ to denote the number of loops incident to a vertex $v \in V$ and we let $\rho^*(v) = \rho(v) + e(v)$ and $\delta^*(v) = \delta(v) + e(v)$ denote the *in-degree* and the *out-degree* of a vertex $v \in V$, respectively.

The definition of an r-detachment H of a digraph D is similar to the undirected case. An r-degree specification of D is a function f on V, such that for each vertex $v \in V$, $f(v)$ is a sequence of ordered pairs (ρ_i^v, δ_i^v), $1 \leq i \leq r(v)$ of positive integers so that $\sum_{i=1}^{r(v)} \rho_i^v = \rho^*(v)$ and $\sum_{i=1}^{r(v)} \delta_i^v = \delta^*(v)$. An f-detachment of D is an r-detachment in which the in- and out-degrees of the pieces of each $v \in V$ are given by the pairs of $f(v)$.

Our main result is as follows.

Theorem 1.3. *Let $D = (V, E)$ be a digraph and let $r : V \to Z_+$. Then D has a k-edge-connected r-detachment if and only if*
(a) D is k-edge-connected, and
(b) $\rho^*(v) \geq kr(v)$ and $\delta^*(v) \geq kr(v)$ for all $v \in V$.
Furthermore, if D has a k-edge-connected r-detachment then D has a k-edge-connected f-detachment for any r-degree specification f for which each term ρ_i^v and δ_i^v is at least k for all $1 \leq i \leq r(v)$, $v \in V$.

In Section 2 we prove Theorem 1.3 by using 'edge-splittings' and 'edge-flippings'. This approach leads to a simple and self-contained new proof of Theorem 1.2 that we present in Section 3.

In the rest of this section we mention some related results and define the edge-splitting operation. Nash-Williams' above mentioned results and Theorem 1.3 give a complete characterization of graphs and digraphs with highly edge-connected detachments. The similar question for vertex-connectivity seems to be much more complicated. A recent result of Jackson and Jordán [3] solved the 2-vertex-connected case.

Detachments are closely related to 'edge-splittings'. By *splitting off* a pair us, sv of edges from a vertex s in a graph or digraph we mean the operation of deleting the edges us, sv and adding (a new copy of) the edge uv. The resulting graph or digraph will be denoted by $G_{u,v}$, where s will always be clear from the context. Well-known results by Lovász [5] and Mader [6], [7] give sufficient conditions for the existence of a pair of edges us, sv that can be split off preserving the edge-connectivity in $V - s$. We shall not use these results but we shall use the splitting off operation in our proofs.

In some sense splitting off a pair us, sv from a vertex s in a graph is equivalent to detaching s into two pieces of degree 2 and $deg(s) - 2$, respectively. Extending the splitting off theorem of Lovász, Fleiner [2] gave necessary and sufficient conditions for the existence of a detachment of s into $r(s)$ pieces of given degrees which preserves the edge-connectivity in $V - s$. Jordán and Szigeti [4] obtained an even more general result on detachments of s that preserve local edge-connectivities in $V - s$. This result implies Fleiner's theorem and Mader's splitting off theorem.

2 Detachments in digraphs

We shall use the following well known equalities.

Proposition 2.1. *Let $H = (V, E)$ be a digraph. For arbitrary subsets $X, Y \subseteq V$,*

$$\rho(X) + \rho(Y) = \rho(X \cap Y) + \rho(X \cup Y) + d(X - Y, Y - X), \text{ and} \quad (1)$$

$$\delta(X) + \delta(Y) = \delta(X \cap Y) + \delta(X \cup Y) + d(X - Y, Y - X).$$

Let $D = (V, E)$ be a k-edge-connected digraph and $s \in V$. For a pair us, sv of edges let us denote by $D_{u,v}$ the digraph obtained from D by splitting off us, sv. The new copy of uv obtained by the splitting will be called the *split edge*. A pair us, sv of edges is called *admissible* in D if $D_{u,v}$ is k-edge-connected. A subset $X \subseteq V - s$ is *in-critical* if $\rho(X) = k$ and *out-critical* if $\delta(X) = k$. A set X which is either in-critical or out-critical (or both) is called *critical*. It is easy to see that the pair us, sv is not admissible if and only if some critical set contains both u and v.

Note that splitting off a loop ss with another edge sv results in deleting the loop and keeping the edge sv. In this case the edge sv will also be called a split edge.

Lemma 2.2. *Let $D = (V, E)$ be a k-edge-connected digraph and let $s \in V$ be a vertex with $\rho^*(s) \geq k + 1$ and $\delta^*(s) \geq k + 1$. Then there is an admissible pair us, sv at s for any given edge sv.*

Proof. If there is a loop on s then the statement is trivial. Thus we can assume that there are no loops incident with s and hence $\rho(s) = \rho^*(s)$. Suppose that for any edge us the pair us, sv is not admissible. Let $R(s) = \{x \in V - s : xs \in E\}$. Then there exists a family of critical sets $\mathcal{F} = \{X_1, X_2, ..., X_t\}$ such that $R(s) \subseteq \cup_1^t X_i$ holds and $v \in X_i$ for $1 \leq i \leq t$. Choose $\mathcal{F}$ so that t is as small as possible. Suppose $t \geq 2$ and consider the pair X_1, X_2. If $\rho(X_1) = \rho(X_2) = k$ then by (1) and since D is k-edge-connected we have $k+k = \rho(X_1)+\rho(X_2) \geq \rho(X_1 \cap X_2)+\rho(X_1 \cup X_2) \geq k+k$, which implies that $\rho(X_1 \cup X_2) = k$ holds. Thus we could replace X_1 and X_2 by $X_1 \cup X_2$ in $\mathcal{F}$, contradicting the minimiality of t. A similar argument applies if $\delta(X_1) = \delta(X_2) = k$. So we may assume, without loss of generality, that $\rho(X_1) = \delta(X_2) = k$. Then $\rho(V - X_2) = k$, and by applying (1) to X_1 and $V - X_2$ we obtain that $d((V - X_2) - X_1, X_1 - (V - X_2)) = 0$. Since $s \in (V - X_2) - X_1$ and $v \in X_1 - (V - X_2)$ and $sv \in E$, this gives a contradiction. Thus $t = 1$ follows. This implies $R(s) \subseteq X_1$ and hence, since $\rho(s) = \rho^*(s) \geq k+1$ and $s \notin X_1$, we have $\delta(X_1) \geq k+1$. Since X_1 is critical, this gives $\rho(X_1) = k$. Therefore, since $\delta(s) = \delta^*(s) \geq k+1$, we must have $V - (X_1 + s) \neq \emptyset$. Now, since $R(s) \subseteq X_1$ and $v \in X_1$, we obtain $\rho(X_1 + s) = \rho(X_1) - \delta(s, X_1) \leq k - 1$, contradicting the fact that D is k-edge-connected. This proves the lemma. □

The next lemma shows that if the in-degree of x is large then we can 'flip' the head of an edge from x to another vertex y preserving k-edge-connectivity.

Lemma 2.3. *Let $D = (V, E)$ be a k-edge-connected digraph and let $s, y \in V$ with $\rho^*(s) \geq k+1$. Then there exists an edge zs such that $D - zs + zy$ is k-edge-connected.*

Proof. It is easy to see that $D - zs + zy$ is not k-edge-connected for some edge zs if and only if there is an out-critical set $X \subseteq V - s$ with $z, y \in X$. Thus we may assume that $e(s) = 0$ and hence $\rho(s) = \rho^*(s)$. Suppose that for every edge zs the digraph $D - zs + zy$ is not k-edge-connected. Then there is a family $\mathcal{F} = \{X_1, X_2, ..., X_t\}$ of out-critical sets with $R(s) \subseteq \cup_1^t X_i$. Choose $\mathcal{F}$ such that t is as small as possible. First suppose $t \geq 2$. Then, since $y \in X_1 \cap X_2$ and $s \notin X_1 \cup X_2$, (1) implies that $\delta(X_1 \cup X_2) = k$. Then we could replace X_1 and X_2 in $\mathcal{F}$ by $X_1 \cup X_2$, contradicting the minimality of t. Thus $t = 1$. Then we have $R(s) \subseteq X_1$ and hence $\delta(X_1) \geq \rho(s) = \rho^*(s) \geq k+1$, contradicting the fact that $\delta(X_1) = k$. This proves the lemma. □

Given two positive integers ρ, δ, a (ρ, δ)-*detachment* at some vertex $s \in V$ is obtained by splitting s into two pieces s', s'' of in- and out-degrees $(\rho^*(s) - \rho, \delta^*(s) - \delta)$ and (ρ, δ), respectively. A (ρ, δ)-detachment is *admissible* in a k-edge-connected digraph if the resulting digraph is k-edge-connected.

Lemma 2.4. *Let $D = (V, E)$ be a k-edge-connected digraph and $s \in V$. Let ρ, δ be integers satisfying $k \leq \rho \leq \rho^*(s) - k$ and $k \leq \delta \leq \delta^*(s) - k$. Then D has an admissible (ρ, δ)-detachment at s.*

Proof. By symmetry we may suppose $\rho \leq \delta$. We use induction on $\delta - \rho$. If $\delta = \rho$ then, since $\delta^*(s) - \delta \geq k$ and $\rho^*(s) - \rho \geq k$, we can use Lemma 2.2 to deduce that D has a sequence of ρ admissible splittings. By subdividing each of the split edges by a new vertex and then contracting the subdividing vertices into a new vertex s'' we obtain a k-edge-connected digraph D'. Equivalently, D' arises from D by an (admissible) (ρ, δ)-detachment. Hence D has the required detachment in this case. Now suppose that $\delta \geq \rho+1$ and that D has an admissible $(\rho, \delta-1)$-detachment D''. Let s' and s'' be the vertices obtained by detaching s into two vertices of degrees $(\rho^*(s)-\rho, \delta^*(s)-\delta+1)$ and $(\rho, \delta-1)$ respectively. Since $\delta^*(s) - \delta + 1 \geq k+1$, we may apply Lemma 2.3 to find an edge zs' such that $D'' - s'z + s''z$ is k-edge-connected. This gives us an admissible (ρ, δ)-detachment of D. □

Proof of Theorem 1.3. The necessity of conditions (a) and (b) is obvious. To prove sufficiency (and the second part of the theorem) we shall show that if D is k-edge-connected and f is an r-degree-specification where each term is at least k then D has a k-edge-connected f-detachment. The proof is by induction on $\sum_{v\in V}(r(v)-1)$. If $r(v) = 1$ for all $v \in V$ then there is nothing to prove. So choose a vertex $v \in V$ with $r(v) \geq 2$. By Lemma 2.4, D has an admissible (ρ_1^v, δ_1^v)-detachment D' at v detaching v into two vertices v' and v'' with degrees $(\rho^*(v) - \rho_1^v, \delta^*(v) - \delta_1^v)$ and (ρ_1^v, δ_1^v), respectively. Now the theorem follows by applying induction to D' (where $r'(v') = r(v)-1, r'(v'') = 1, f'(v'') = ((\rho_1^v, \delta_1^v)), f'(v') = ((\rho_2^v, \delta_2^v), ..., (\rho_{r(v)}^v, \delta_{r(v)}^v)$, and for every other vertex u we have $r'(u) = r(u)$ and $f'(u) = f(u)$). □

3 Detachments in undirected graphs

In this section we give a relatively short self-contained proof for Theorem 1.2 by using the approach we developed in the directed case. This new proof, which is based on edge-splitting and edge-flipping operations, seems to be simpler than the original proof [9] or the proof given in [2]. We note that some parts of our proofs are similar to proofs from [1], [2], [5, 6.53], or [9], where the authors apply similar techniques.

We shall use the following well-known equalities for the degree function of a graph.

Proposition 3.1. *Let $H = (V, E)$ be a graph. For arbitrary subsets $X, Y \subseteq V$:*

$$d(X) + d(Y) = d(X \cap Y) + d(X \cup Y) + 2d(X - Y, Y - X), \tag{2}$$

$$d(X) + d(Y) = d(X - Y) + d(Y - X) + 2d(X \cap Y, V - (X \cup Y)). \tag{3}$$

Given a k-edge-connected graph $G = (V, E)$ and $s \in V$, a non-empty subset $X \subset V - s$ is called *dangerous* if $d(X) \leq k+1$ and $d(s, X) \geq 2$. A set $X \subseteq V - s$ is *critical* if $d(X) = k$. We say that $X, Y \subset V$ are *crossing* if none of the sets $X-Y, Y-X, X\cap Y$ and $V-(X\cup Y)$ is empty. For some $w \in V$ let $N(w) = \{z \in V - w : wz \in E\}$. Recall the definition of splitting off (and the remark on splitting off loops). A pair us, sv of edges is *admissible* if $G_{u,v}$ is k-edge-connected. It is easy to see that the pair us and sv is non-admissible if and only if there exists a dangerous set $X \subseteq V - s$ such that $u, v \in X$.

Lemma 3.2. *Let $G = (V, E)$ be a k-edge-connected graph $(k \geq 2)$ and let $s \in V$ be a vertex with $deg(s) \geq k+2$. Then for any given edge us either there is an admissible pair us, sv or $deg(s) = k+2$, k is odd, and every edge sv with $v \neq u$ is in an admissible pair.*

Proof. If there is a loop on s then the lemma is trivial. Thus we may assume that there are no loops incident with s and hence $deg(s) = d(s)$. Suppose that $d(s) \geq k+2$ and the edge us is in no admissible pair. Then there is a family $\mathcal{F} = \{X_1, X_2, ..., X_t\}$ of (inclusionwise) maximal dangerous sets such that $N(s) \subseteq \cup_1^t X_i$ and $u \in X_i$ for all $1 \leq i \leq t$. Choose $\mathcal{F}$ so that t is as small as possible. If $t = 1$ then $d(X_1) \geq d(s) = deg(s) \geq k + 2$, a contradiction. If $t \geq 3$ then consider the triple $X = X_1$, $Y = X_2$, $Z = X_3$. The minimality of t implies that X, Y, Z are pairwise crossing. Without loss of generality we may assume that $|X \cap Y| \geq |X \cap Z|, |Y \cap Z|$. Let $M = X \cap Y$. Since X is maximal dangerous and G is k-edge-connected, (2) gives $k + 1 + k + 1 \geq d(X) + d(Y) \geq d(M) + d(X \cup Y) \geq k + k + 2$. Hence $d(M) = k$ holds. By applying (2) to M and Z, and using the fact that Z is maximal dangerous, we obtain that $M \subset Z$. Therefore, since $|X \cap Y| \geq |X \cap Z|, |Y \cap Z|$, we must have $M = X \cap Y = X \cap Z = Y \cap Z$. Applying (3) to each pair of sets from X, Y, Z gives that there is at most one edge from M to each of the sets $V - (X \cup Y), V - (X \cup Z), V - (Y \cup Z)$. Since $us \in E$ and $u \in M$, this implies $d(M) = 1$, contradicting the fact that G is k-edge-connected for some $k \geq 2$.

Hence we may assume that $t = 2$ and consider the pair $X = X_1$ and $Y = X_2$. By the minimality of t we have $X - Y \neq \emptyset \neq Y - X$. Thus, since $u \in X \cap Y$, applying (3) to X and Y implies that $d(X) = d(Y) = k + 1$, $d(X - Y) = d(Y - X) = k$ and $d(X \cap Y, V - (X \cup Y)) = 1$. Since X is maximal dangerous, (2) implies that $d(X \cap Y) = k$. Since $d(X) = d(X - Y) + d(X \cap Y) - 2d(X - Y, X \cap Y) = 2k - 2d(X - Y, X \cap Y)$, it follows that $d(X) = k+1$ is even and k is odd. Moreover, since $N(s) \subseteq X \cup Y$, we have $2k+2 = d(X) + d(Y) \geq d(s) - 1 + d(X \cap Y) + 1 \geq d(s) + k$. If $d(s) \geq k+3$ then this gives a contradiction and shows that su is in a splittable pair. Thus $d(s) = k + 2$. We have $(N(s) - u) \subseteq (X - Y) \cup (Y - X)$. Choose $v \in N(s) \cap (X - Y)$ and $w \in N(s) \cap (Y - X)$. These vertices exist by the minimality of t. We claim that sv, sw is an admissible pair. Suppose not, and let Z be a maximal dangerous set with $v, w \in Z$. By (2) and by the maximality of Z, and using the fact that $d(X - Y) = d(Y - X) = k$, we obtain that $(X - Y) \cup (Y - X) \subseteq Z$. Hence $d(s, Z) \geq d(s) - 1$ and $d(V - Z - s) = d(Z + s) = d(Z) - d(s, Z) + d(s, V - Z) \leq k + 1 - d(s) + 1 + 1 = k + 3 - (k + 2) = 1$, a contradiction, since G is k-edge-connected. This proves that sv, sw is admissible and hence every edge sv is in an admissible pair. □

We say that a graph $G = (V, E)$ is *k-edge-connected in $V - y$*, for some vertex $y \in V$, if every proper subset $X \subset V$ with $X \neq \{y\} \neq V - X$ satisfies $d(X) \geq k$.

Lemma 3.3. *Let $G = (V, E)$ be k-edge-connected ($k \geq 2$) in $V - y$ and let $d(y) \geq k - 1$, for some vertex $y \in V$. Then for any $x \in V - y$ with $deg(x) \geq k+1$ either there is an edge zx such that $G - zx + zy$ is k-edge-connected or k is odd, $d(y) = k - 1$, $deg(x) = d(x) = k + 1$ and $G - \{x, y\}$ is disconnected.*

Proof. It is easy to see that $G - zx + zy$ is not k-edge-connected for some edge zx with $z \neq y$ if and only if there is a set $X \subseteq V - x$ with $z, y \in X$ and $d(X) = k$. Thus we may assume that $e(x) = 0$ and hence $d(x) = deg(x)$. If $d(x, y) \geq k+1$ then $G - yx + yy$ is k-edge-connected. Thus we may assume that $d(x, y) \leq k$ and hence $N(x) - y \neq \emptyset$. Suppose that $G - zx + zy$ is not k-edge-connected for any edge zx with $z \neq y$. Then there is a family $\mathcal{F} = \{X_1, ..., X_t\}$ of sets with $d(X_i) = k$, $y \in X_i$ for all $1 \leq i \leq t$ and such that $N(x) \subseteq \cup_1^t X_i$. Choose $\mathcal{F}$ such that t is as small as possible. If $t = 1$ then we have $k = d(X_1) \geq d(x) = deg(x) \geq k+1$, a contradiction. Thus $t \geq 2$. Consider a pair $X = X_i$, $Y = X_j$ for some $1 \leq i < j \leq t$. Since G is k-edge-connected in $V-y$, we have $d(V-(X \cup Y)) = d(X \cup Y) \geq k$. Furthermore, if $d(X \cup Y) = k$, then we could replace X and Y by $X \cup Y$ in $\mathcal{F}$. Thus $d(X \cup Y) \geq k+1$. By (2) we have $2k = d(X)+d(Y) \geq d(X \cap Y) + d(X \cup Y) + 2d(X-Y, Y-X) \geq d(X \cap Y) + k + 1 + 2d(X-Y, Y-X)$. Since G is k-edge-connected in $V - y$, this gives $X \cap Y = \{y\}$, $d(y) = k - 1$, and $d(X-Y, Y-X) = 0$. Applying (3) to X and Y gives $d(y, V-(X \cup Y)) = 0$. Suppose that $t \geq 3$ and let $X, Y, Z \in \mathcal{F}$. Since the above properties hold for each pair in X, Y, Z, we have $X \cap Y = X \cap Z = Y \cap Z = \{y\}$. This yields $d(y) = 0$, contradicting $d(y) = k-1 \geq 1$. Thus $t = 2$. Since $k \leq d(X-y) = d(X) - d(y, Y-y) + d(y, X-y) = k - d(y, Y-y) + d(y, X-y)$, we have $d(y, X-y) \geq d(y, Y-y)$. Similarly, $d(y, Y-y) \geq d(y, X-y)$. Since $d(y) = d(y, X-y) + d(y, Y-y)$, this implies that k is odd and $d(y, X-y) = d(y, Y-y) = (k-1)/2$. Hence $2k = d(X) + d(Y) \geq d(y) + d(x) + d(V-x-(X \cup Y), X \cup Y) \geq 2k + d(V-x-(X \cup Y), X \cup Y)$. From this it follows that $d(x) = k+1$, $X \cup Y = V - x$, and $G - \{x, y\}$ is disconnected. □

Let $G = (V, E)$ be a graph with a designated vertex $s \in V$ and let $d \leq deg(s)$ be a positive integer. A *d-detachment* of G at s is obtained by detaching s into two pieces s' and s'' with degrees $deg(s') = deg(s) - d$ and $deg(s'') = d$, respectively. A d-detachment G' of a k-edge-connected graph is called *admissible* if G' is also k-edge-connected.

Lemma 3.4. *Let $G = (V, E)$ be a k-edge-connected graph ($k \geq 2$) and $s \in V$. Let d_1, d_2 be integers with $k \leq d_1 \leq d_2$ and $d_1 + d_2 = deg(s)$. Then either (i) G has an admissible d_1-detachment at s or (ii) k is odd, s is a cutvertex, $d(s) = deg(s) = 2k$, and $d_1 = d_2 = k$.*

Proof. Suppose that (ii) does not occur. We show that there is an admissible d_1-detachment at s by induction on d_1. If $d_1 = k$ then by Lemma 3.2 there is a sequence of $\lceil (k-1)/2 \rceil$ admissible splittings at s. By subdividing each split edge by a new vertex and then contracting the subdividing vertices into a new vertex y we obtain a graph $G' = (V', E')$ which is either k-edge-connected or k is odd, G' is k-edge-connected in $V' - y$, and $d(y) = k-1$. In the former case we are done. In the latter case we have $d_{G'}(y) = k-1$ and $deg_{G'}(s) = deg_G(s) - (k-1) \geq k+1$. Since (ii) does not hold, we can use Lemma 3.3 to construct a k-edge-connected k-detachment at s by 'flipping' an edge zs to zy.

Now suppose $d_1 \geq k+1$. By induction, G has an admissible $(d_1 - 1)$-detachment G' at s. Since $d_2 + 1 \geq d_1 + 1 \geq k + 2$, we can use Lemma 3.3 to flip an edge in G' and obtain an admissible d_1-detachment of G. □

Proof of Theorem 1.2. Necessity is trivial. To see sufficiency suppose that G is k-edge-connected, r satisfies (b), and neither (c) nor (d) hold. First we show the existence of a k-edge-connected r-detachment by induction on $\sum_{v \in V} r(v) - 1$. If $r(v) = 1$ for all $v \in V$ then there is nothing to prove. So let us choose $v \in V$ with $r(v) \geq 2$.

Since (c) does not hold, there is an admissible k-detachment $G' = (V', E')$ at v by Lemma 3.4, where the two pieces of v are x and y with $deg'(x) = deg(v) - k$ and $d'(y) = deg'(y) = k$. Here d' and deg' denote the corresponding functions in G'. Let $r'(x) = r(v) - 1$, $r'(y) = 1$ and $r'(u) = r(u)$ for every $u \in V' - \{x, y\}$. Clearly, (a) and (b) hold in G' with respect to r'. Moreover, (d) cannot hold, since $r'(y) = 1$. If (c) does not hold either, then we are done by induction. Thus we may assume that k is odd and G' has a cutvertex s with $d'(s) = deg'(s) = 2k$ and $r'(s) = 2$. Let us call such a vertex s a *bad cutvertex*. Since (c) does not hold in G, for each bad cutvertex s we have that either s separates x and y in G', or $s = x$. We shall prove that by 'switching' two edges in G' we can create another k-edge-connected k-detachment of G at v where both (c) and (d) do not hold, and both (a) and (b) do hold. This will complete the proof by induction.

We shall use slightly different arguments when (i) x is a bad cutvertex and when (ii) x is not a bad cutvertex. In case (i) $G' - x$ has two components, X and Y. We may assume $y \in Y$. Then all bad cutvertices other than x are in Y. Let us pick two vertices $w \in X \cap N'(x)$ and $z \in N'(y)$ ($x = z$ may hold). Observe that the subgraphs $G'[X + x]$ and $G'[Y + x]$, induced by $X + x$ and $Y + x$, are both k-edge-connected, and hence there is a path P_1 from w to x in $G'[X + x] - wx$ and there is a path P_2 from y to x in $G'[Y + x] - yz$. We claim that 'switching' the edges xw and yz (that is, replacing the edges xw, yz in G' by the edges xz, wy) preserves k-edge-connectivity and results in a graph H where both (a) and (b) hold and both (c) and (d) do not hold (with respect to r'). Clearly (b) holds for H and r', and (d) does not. Suppose that (a) does not hold. Then it can be seen that there is a set $Q \subset V'$ with $d'(Q) \leq k + 1$, $w, y \in Q$ and $x, z \notin Q$. It is also easy to verify by the k-edge-connectivity of G' that any set $T \subset V'$ with $d'(T) \leq k + 1$ induces a connected subgraph of G'. The subgraphs of G' induced by Q and $V - Q$ contain a wy-path and a xz-path, respectively. But this is impossible, since x separates w and y. Hence H is indeed k-edge-connected. To see that (c) does not hold in H we have to show that the bad cutvertices of G' are no longer cutvertices in H and that no vertex of G' has become a bad cutvertex in H. Since $P_1 \cup wy \cup P_2$ forms a cycle in H containing x, y and w (and all the bad cutvertices of G'), it follows that there is no bad cutvertex in H, except possibly x. To see that x is not a (bad) cutvertex in H either, observe that there exist $k - 1$ edge-disjoint paths from w to x in $G'[X + x] - wx$ and there exist $k - 1$ edge-disjoint paths from y to x in $G'[Y + x] - yz$. Thus y can reach x in H via at least $2k - 2 \geq k + 1$ different edges incident to x (recall that k is odd). Since $deg_H(x) = 2k$, and H is k-edge-connected, we deduce that x is not a cutvertex in H. This completes the proof in case (i).

Now consider case (ii). Let s be a bad cutvertex in G' and let X, Y denote the components of $G' - s$, where $y \in Y$. Since (d) does not hold in G, it can be seen that we may choose $w \in N'(x)$ and $z \in N'(y)$ such that $w \neq z$. We shall assume that $w \neq s$ (the case when $w = s$ and $z \neq s$ is similar). Since $G'[X + s]$ and $G'[Y + s]$ are k-edge-connected, there is a path P^* from y to z in $G'[Y + x] - yz$ and there exist $k - 1$ edge-disjoint paths $P_1, ..., P_{k-1}$ from x to w in $G'[X + x] - xw$. Since $w \neq s$ and $d_{G'[X+s]}(s) = k$ and $k \geq 3$, one of these paths, say P_1, avoids s. We claim that switching the edges xw and yz results in a k-edge-connected graph H for which both (a) and (b) hold and both (c) and (d) do not hold (with respect to r'). The argument used in case (i) shows that (a) and (b) hold for H and r', and (d) does not. To see that there are no bad cutvertices in H observe that $P_1 \cup xz \cup P^* \cup yw$ is a cycle in H containing x, y, z, w. Thus no vertex separates x and y in H and so H cannot contain a bad cutvertex distinct from x. Furthermore, x is not a bad cutvertex in H because it is not a bad cutvertex in G' and y, z are contained in a cycle of H. This completes the proof of case (ii) and proves the first part of the theorem on r-detachments. The second part on f-detachments follows easily from Lemma 3.3, since the degrees of the pieces in a k-edge-connected r-detachment can be modified by flipping edges so that it satisfies any given degree specification in which each term is at least k. □

References

[1] A. Frank, Augmenting graphs to meet edge-connectivity requirements, SIAM J. Disc. Math. 5, 25-53, 1992.

[2] B. Fleiner, Detachments of vertices of graphs preserving edge-connectivity, 1997, submitted.

[3] B. Jackson, T. Jordán, Non-separable detachments of graphs, EGRES Report Series 2001-12, 2001, submitted.

[4] T. Jordán, Z. Szigeti, Detachments preserving local edge-connectivity of graphs, BRICS Report Series 99-35, 1999, submitted.

[5] L. Lovász, Combinatorial Problems and Exercises, North-Holland, Amsterdam, 1979.

[6] W. Mader, A reduction method for edge-connectivity in graphs, Annals of Discrete Math. 3, 1978, pp. 145-164.

[7] W. Mader, Konstruktion aller n-fach kantenzusammenhängenden Digraphen, European J. Combin., 3 (1982) pp 63-67.

[8] C. St. J. A. Nash-Williams, Detachments of graphs and generalised Euler trails, Surveys in combinatorics 1985, 137–151, London Math. Soc. Lecture Note Ser., 103, Cambridge Univ. Press, 1985.

[9] C. St. J. A. Nash-Williams, Connected detachments of graphs and generalized Euler trails, J. London Math. Soc., Vol. 31, 1985, pp. 17-29.