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On a lemma of Scarf

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Abstract

The aim of this note is to point out some combinatorial applications of a lemma of Scarf, proved first in the context of game theory. The usefulness of the lemma in combinatorics has already been demonstrated in [1], where it was used to prove the existence of fractional kernels in digraphs not containing cyclic triangles. We indicate some links of the lemma to other combinatorial results, both in terms of its statement (being a relative of the Gale-Shapley theorem) and its proof (in which respect it is a kin of Sperner's lemma). We use the lemma to prove a fractional version of the Gale-Shapley theorem for hypergraphs, which in turn directly implies an extension of this theorem to general (not necessarily bipartite) graphs due to Tan [12]. We also prove the following result, related to a theorem of Sands, Sauer and Woodrow [10]: given a family of partial orders on the same ground set, there exists a system of weights on the vertices, which is (fractionally) independent in all orders, and each vertex is dominated by them in one of the orders.

Keywords: stable matching, partial order, matroid, simplicial complex

1 Introduction

A famous theorem of Gale and Shapley [5] states that given a bipartite graph and, for each vertex v, a linear order $\leq_v$ on the set of edges incident with v, there exists a stable matching. Here, a matching M is called *stable* if for every edge $e \notin M$ there exists an edge in M meeting e and beating it in the linear order of the vertex at which they are incident. (The origin of the name "stable" is that in such a matching no non-matching edge poses a reason for breaking marriages: for every non-matching edge, at least one of its endpoints prefers its present spouse to the potential spouse

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provided by the edge.) Alternatively, a stable matching is a kernel in the line graph of the bipartite graph, where the edge connecting two vertices (edges of the original graph) is directed from the larger to the smaller, in the order of the vertex of the original graph at which they meet.

It is well known that the theorem fails for general graphs, as shown by the following simple example: let G be an undirected triangle on the vertices u, v, w, and define: $(u, v) >_u (w, u)$, $(v, w) >_v (u, v)$, $(w, u) >_w (v, w)$. But the theorem is true for general graphs if one allows fractional matchings, as follows easily from a result of Tan [12] (see Theorem 2.2 below). For example, in the example of the triangle one could take the fractional matching assigning each edge the weight $\frac{1}{2}$: each edge is then dominated at some vertex by edges whose sum of weights is 1 (for example, the edge (u, v) is dominated in this way at v).

The notions of stable matchings and fractional stable matchings can be extended to hypergraphs. A *hypergraphic preference system* is a pair $(H, \mathcal{O})$, where $H = (V, E)$ is a hypergraph, and $\mathcal{O} = \{\leq_v : v \in V\}$ is a family of linear orders, $\leq_v$ being an order on the set $D(v)$ of edges containing the vertex v. If H is a graph we call the system a *graphic* preference system.

A set M of edges is called a *stable matching* with respect to the preference system if it is a matching (that is, its edges are disjoint) and for every edge e there exists a vertex $v \in e$ and an edge $m \in M$ containing v such that $e \leq_v m$.

Recall that a function w assigning non-negative weights to edges in H is called a *fractional matching* if $\sum_{v \in h} w(h) \leq 1$ for every vertex v. A fractional matching w is called *stable* if every edge e contains a vertex v such that $\sum_{v \in h, e \leq_v h} w(h) = 1$.

As noted, by a result of Tan every graphic preference system has a fractional stable matching. Does this hold also for general hypergraphs? The answer is yes, and it follows quite easily from a result of Scarf [11]. This result is the starting point of the present paper. It was originally used in the proof of a better known theorem in game theory, and hence gained the name "lemma". Its importance in combinatorics has already been demonstrated in [1], where it was used to prove the existence of a fractional kernel in any digraph not containing a cyclic triangle.

Scarf's lemma is intriguing in that it seems unrelated to any other body of knowledge in combinatorics. In accord, its proof appears to be of a new type. The aim of this paper is to bring it closer to the center of the combinatorial scene. First, by classifying it as belonging to the Gale-Shapley family of results. Second, by pointing out its similarity (in particular, similarity in proofs) to results related to Brouwer's fixed point theorem.

In [4], it was noted that the Gale-Shapley theorem is a special case of a result of Sands, Sauer and Woodrow [10] on monochromatic paths in edge two-coloured digraphs. This result can also be formulated in terms of dominating antichains in two partial orders (see Theorem 3.1 below). We shall use Scarf's lemma to prove a fractional generlisation of this "biorder" theorem to an arbitrary number of partial orders.

In [4], a matroidal version of the Gale-Shapley theorem was proved, for two matroids on the same ground set. Using Scarf's lemma, we prove a fractional version of this result to arbitrarily many matroids on the same ground set.

We finish the introduction with stating Scarf's lemma. (Apart from the original paper, a proof can also be found in [1]. The basic ideas of the proof are mentioned in the last section of the present paper).

Theorem 1.1 (Scarf [11]). *Let $n < m$ be positive integers, b a vector in $\mathbb{R}^n_+$. Also let $B = (b_{i,j}), C = (c_{i,j})$ be matrices of dimensions $n \times m$, satisfying the following three properties: the first n columns of B form an $n \times n$ identity matrix (i.e. $b_{i,j} = \delta_{i,j}$ for $i, j \in [n]$), the set $\{x \in \mathbb{R}^n_+ : Bx = b\}$ is bounded, and $c_{i,i} < c_{i,k} < c_{i,j}$ for any $i \in [n]$, $i \neq j \in [n]$ and $k \in [m] \setminus [n]$.*

Then there is a nonnegative vector x of $\mathbb{R}^m_+$ such that $Bx = b$ and the columns of C that correspond to $\mathrm{supp}(x)$ *form a dominating set, that is, for any column $i \in [m]$ there is a row $k \in [n]$ of C such that $c_{k,i} \leq c_{k,j}$ for any $j \in \mathrm{supp}(x)$.*

As we shall see in the last section, under the assumption that the columns of B are in general position and the entries in each row of C are different the proof of the lemma yields a stronger fact, namely that there exists an odd number of vectors x as in the lemma.

2 Some applications

In this section we study some extensions of the stable marriage theorem.

Theorem 2.1 (Gale-Shapley [5]). *If $(G, \mathcal{O})$ is a graphic preference model and graph $G = (V, E)$ is bipartite then there exists a stable matching.* □

As we mentioned in the introduction, in nonbipartite graphic preference models Theorem 2.1 is not true. The first algorithm to decide the existence of a stable matching in this case is due to Irving [6]. Later on, based on Irving's proof, Tan gave a compact characterization of those models that contain a stable matching [12]. In what follows, we formulate Tan's theorem.

In a graphic preference model $(G, \mathcal{O})$, a subset $C = \{c_1, c_2, \ldots, c_k\}$ of E is a *preference cycle* if $c_1 <_{v_1} c_2 <_{v_2} c_3 <_{v_3} \ldots <_{v_{k-1}} c_k <_{v_k} c_1$ for different vertices $v_1, v_2, \ldots, v_k$ of V. A preference cycle C is *odd* if $|C|$ is odd, otherwise C is *even.* A *stable partition* of model $(G, \mathcal{O})$ is a subset S of E with the following properties.

1. Any component of S is either a cycle or an edge, and
2. each cycle component of S is a preference cycle, and
3. for any edge e of $E \setminus S$ there is a vertex v covered by S and incident with e such that $e <_v s$ for any edge s of S incident with v.

It is easy to see that a stable partition with no cycle component is a stable matching.

Let us define a *stable half-matching* as a stable fractional matching x so that $2x$ is an integral vector. Clearly, for a graphic preference model, the support of a stable half-matching is a stable partition. Also, if S is a stable partition then x_S is a stable half matching, where

$$x_S(e) = \begin{cases} 0 & \text{if } e \notin S \\ 1 & \text{if } e \text{ is an edge-component of } S \\ \frac{1}{2} & \text{if } e \text{ belongs to a cycle-component of } S \end{cases}$$

Theorem 2.2 (Tan, [12]). *Any graphic preference model has a stable partition.*

Note that a weaker version of Theorem 2.2 can be proved the following way: Define model $\mathcal{M}' = (G', \mathcal{O}')$ by $G' = (V', E')$, $V' := \{v_m, v_w : v \in V\}$, $E' := \{u_m v_w, u_w v_m : uv \in E\}$ and $u_m v_w <_{u_m} u_m v'_w$ iff $v_m u_w <_{u_w} v'_m u_w$ iff $uv <_u uv'$. That is, we introduce a bipartite preference model by duplicating the original one. According to Theorem 2.1, there is a stable matching M in $\mathcal{M}'$. Define $S := \{uv : u_m v_w \in M \text{ or } v_m u_w \in M\}$. It is straightforward to check that S satisfies the first two requirements in the definition of a stable partition, but instead of 3., we have the weaker property

3'. For any edge e of $E \setminus S$ there are two edges s_1 and s_2 of S such that $s_1 \leq_v e$ and $s_2 \leq_u e$ for some $u, v \in V$. If s_1 is an edge-component of S then $s_1 = s_2$ is allowed, otherwise $s_1 \neq s_2$.

However, the above weak version of Theorem 2.2 does not help us to decide the existence of a stable matching in a model. In particular, the following fact is not true with the weaker notion of stable partition.

Observation 2.3. If S is a stable partition for graphic preference model $\mathcal{M} = (G, \mathcal{O})$ and there are no odd preference cycles in S then there exists a stable matching of $\mathcal{M}$.

An immediate consequence of Observation 2.3 and Theorem 2.2 is that if a model $\mathcal{M}$ is free of odd preference cycles then there is a stable matching of $\mathcal{M}$.

Proof. Throw away each second edge of each cycle in S. By the definition of a stable partition, what is left from S after these deletions is exactly a stable matching of $\mathcal{M}$. $\square$

So if stable partition S does not contain an odd cycle then we immediately see a stable matching. On the other hand, an odd cycle in S means that no stable matching exists in $\mathcal{M}$. More specifically, there is the following theorem.

Theorem 2.4 (Tan [12]). *Let $\mathcal{M} = (G, \mathcal{O})$ be a graphic preference model and S be a stable partition of $\mathcal{M}$. If there is an odd cycle C in S then C is present in each stable partition of $\mathcal{M}$.*

To prove Theorem 2.2, we justify the promised fractional version of the Gale-Shapley theorem for hypergraphs.

Theorem 2.5. *Any hypergraphic preference system has a fractional stable matching.*

Proof. Let $(H, \mathcal{O})$ be a hypergraphic preference system, where $H = (V, E)$ and $\mathcal{O} = \{\leq_v : v \in V\}$. Let B be the incidence matrix of H, with the identity matrix adjoined to it at its left. Let C' be a $V \times E$ matrix satisfying the following two conditions:

(1) $c'_{v,e} < c'_{v,f}$ whenever $v \in e \cap f$ and $e <_v f$

(2) $c'_{v,f} < c'_{v,e}$ whenever $v \in f \setminus e$.

Let C be obtained from C' by adjoining to it on its left a matrix so that C satisfies the conditions of Theorem 1.1. Let x be a vector as in Theorem 1.1 for B and C, where b is taken as the all $1's$ vector $\mathbf{1}$. Define $x' = x|_E$, namely the restriction of x to E. Clearly, x' is a fractional matching. To see that it is dominating, let e be an edge of H. By the conditions on x, there exists a vertex v such that $c_{v,e} \leq c_{v,j}$ for all $j \in \text{supp}(x)$. Since $c_{v,v} < c_{v,e}$ it follows that $v \notin \text{supp}(x)$. Since $Bx = \mathbf{1}$ it follows that $\text{supp}(x)$ contains an edge f containing v (otherwise $(Bx)_v = 0$). Since $c_{v,f} \geq c_{v,e}$ it follows by condition (2) above that $v \in e$. The condition $(Bx)_v = 1$ now implies that e is dominated by x at v. □

In fact, the vector x' can be assumed to be a vertex of the fractional matching polytope of H. To see this, write $x' = \sum \alpha_i y_i$, where $\alpha_i > 0$ for all i, $\sum \alpha_i = 1$ and the y_i's are vertices of the fractional matching polytope. Then each y_i must be a fractional stable matching. It is well known (see e.g. [8]) that the vertices of the fractional matching polytope of a graph are half integral, that is, they have only $0, \frac{1}{2}, 1$ coordinates. This yields Theorem 2.2. Next we give a direct proof of this fact.

Proof of Theorem 2.2. Let $\mathcal{M} = (G, E)$ be a graphic preference model, x be a fractional stable matching for $\mathcal{M}$ that exists by Theorem 2.5 and define $S := \text{supp}(x)$. We shall prove that S is a stable partition. By the stability of x, we can orient each edge e of E so that the corresponding arc $\vec{e}$ points to a vertex v such that

$$\sum_{e \leq_v f} x(f) = 1 \ . \tag{1}$$

Let $D = (V, A)$ be the resulted digraph. From (1), it follows that if $e, f \in S$ then $\vec{e}$ and $\vec{f}$ have different endvertices. Also, if $\vec{e}, \vec{f}$ is a directed path for some $e, f \in S$ then $x(f) < 1$ as x is a fractional matching. Then (1) yields that there is an edge $g \in S$ such that $\vec{e}, \vec{f}, \vec{g}$ is a directed path. These two properties of S imply that the components of S correspond to disjoint edges and directed cycles in D. Condition 3. in the definition of a stable partition holds for S because if edge e of $E \setminus S$ is oriented as $\vec{e} = uv$ then $e <_v s$ for any $s \in S$ because of (1). So $S = \text{supp}(x)$ is indeed a stable partition of $\mathcal{M}$. □

For the sake of completeness we finish this section by proving Theorem 2.4.

Proof of Theorem 2.4. Introduce preference model $\mathcal{M}' = (G', \mathcal{O}')$ by $G' = (V, E')$, $E' := \{e^u, e^v : e = uv \in E, e^u \text{ and } e^v \text{ are parallel to } e\}$, $\mathcal{O}' := \{<'_v : v \in V\}$, where

$$e^u <'_v f^w \text{ iff } (e <_v f \text{ or } (e = f, u = v, w \neq v))$$

(We duplicate all edges, and extend the order to the duplicates in a natural way, so that we only have to take extra care for the relation of the two copies of the same edge.)

Observe that if $C = \{e_1, e_2, \dots, e_k\}$ is a preference cycle of model $\mathcal{M}$ such that $e_{i+1} <_{v_i} e_i$ (i is modulo k) then $C' := \{e_1^{v_1}, e_2^{v_2}, \dots, e_k^{v_k}\}$ is a corresponding preference cycle of model $\mathcal{M}'$. So for any stable partition S of model $\mathcal{M}$ there is a corresponding subset S' of E' defined by

$$
\begin{aligned}
S' := \bigcup \{C' : C \text{ is a cycle component of } S\} \cup \\
\cup \{e^u, e^v : e = uv \text{ is an edge component of } S\} .
\end{aligned}
$$

Observe that S' is a stable partition of $\mathcal{M}'$ and each component of S' is a preference cycle of $\mathcal{M}'$.

Let S and T be stable partitions of model $\mathcal{M}$ so that C is an odd cycle component of S. For the corresponding stable partitions S' and T' of $\mathcal{M}'$ we get

$$
\begin{aligned}
|S'| + |T'| \leq |\{(s, t_1, t_2, v) : s \in S', t_1, t_2 \in T', v \in V, t_1 \neq t_2, s \leq'_v t_1, s \leq'_v t_2\}| + && (2)\\
+ |\{(t, s_1, s_2, v) : t \in T', s_1, s_2 \in S', v \in V, s_1 \neq s_2, t \leq'_v s_1, t \leq'_v s_2\}| \leq && (3)\\
\leq 2|V(S') \cap V(T')| \leq |V(S')| + |V(T')| = |S'| + |T'| && (4)
\end{aligned}
$$

The inequality in (2) is true because of property 3. of stable partitions S' and T' (note that both S' and T' are 2-regular subgraphs of (V, E')). The inequality between (3) and (4) holds because the contribution of each vertex that is covered by S' and T' is at most two. So there must be equality throughout (2) to (4). This means that $V(S) = V(S') = V(T') = V(T)$ (i.e. any two stable partitions cover the same set of vertices) and that every vertex v of $V(S')$ contributes exactly two at (2,3).

From this latter property, it follows that there is no vertex v of $V(S')$ that is incident with exactly three edges of $S' \cup T'$. This is because the contribution of v can only be two, if these three edges are e, f and g so that $e <'_v f$, $e <'_v g$ and $e \in S' \cap T'$. But if we follow the two preference cycle components of S' and T' starting at common edge e then we shall find a vertex $u \neq v$ of $V(S')$ so that u is incident with exactly three edges of $S' \cup T'$, and the common edge of S' and T' is the $<'_u$-maximal of the three. The contribution to (2,3) of vertex u is only one, hence the degrees in $S' \cup T'$ can only be 0, 2 and 4. It also follows that if S' and T' share an edge e then the cycle component containing e is the the same for S' and T'.

So assume that the odd cycle component C' of S' is not a component of T'. This means that each vertex v of C' is incident with exactly four edges of $S' \cup T'$. As the contribution of v in (2,3) is exactly two, we have only two possibilities for vertex v. Either the two $<'_v$-smaller edges of these four edges belong to S' and the two $<'_v$-bigger edges belong to T' (in which case we say that v is an S'-vertex) or vice versa, when v is a T'-vertex. From property 3. of S' it follows that for no edge $e = uv$ of C' it can happen that both u and v are S'-vertices. As C' is an odd cycle, it means that there must be an edge $e = uv$ so that both u and v are T'-vertices. But then the inequality in (2) is strict at e.

The contribution shows that $C' \subseteq T'$, hence C is a component of T. $\square$

3 Dominating antichains and matroid-kernels

In [10], there was proved a generalisation of the Gale-Shapley theorem by Sands *et al.* Its original formulation was in terms of paths in digraphs whose edges are two-coloured. But at its core is a fact about pairs of partial orders.

Let V be a finite ground set and $\leq_1$ and $\leq_2$ be two partial orders on V. A *dominating common antichain of* $\leq_1$ *and* $\leq_2$ is a subset A of V such that A is an antichain in both partial orders and for any element v of V there is an element a in A with $v \leq_1 a$ or $v \leq_2 a$.

Theorem 3.1 (see [4, 3]). *For any two partial orders $\leq_1$ and $\leq_2$ on the same finite ground set V, there exists a dominating antichain of $\leq_1$ and $\leq_2$.* □

The Gale-Shapley theorem is obtained by applying this theorem to the two orders on the edge set of the bipartite graph, each being obtained by taking the (disjoint) union of the linear orders induced by the vertices in one side of the graph.

The theorem is false for more than two partial orders. But a fractional version is true. For given partial orders $\leq_1, \leq_2, \ldots, \leq_k$ on a ground set V, a nonnegative vector x of $\mathbb{R}_+^V$ is called a *fractional dominating antichain* if x is a *fractional antichain* (i.e. $\sum_{c \in C} x(c) \leq 1$ for any chain C of any of the partial orders $\leq_i$) and x is a *fractional upper bound* for any element of V, that is for each element v of V there is a chain $v = v_0 \leq_i v_1 \leq_i v_2 \leq_i \ldots \leq_i v_l$ of some partial order $\leq_i$ with $\sum_{j=0}^{l} x(v_j) = 1$. Note that if a fractional dominating antichain x happens to be integral then it is the characteristic vector of a dominating antichain.

Theorem 3.2. *Any finite set $\leq_1, \leq_2, \ldots, \leq_k$ of partial orders on the same ground set V has a fractional dominating antichain.*

Proof. For each $i \leq k$ let $\mathcal{D}_i$ be the set of maximal chains in the partial order $\leq_i$. Let $\mathcal{J} = \bigcup_{i \leq k} \{i\} \times \mathcal{D}_i$ (that is, $\mathcal{J}$ is the union, with repetition, of the families $\mathcal{D}_i$).

Let B' be the $V \times \mathcal{J}$ incidence matrix of the chains of $\mathcal{J}$ (that is, for $v \in V$ and a maximal chain D in $\leq_i$, the $(v, (i, D))$ entry of B' is 1 if $v \in D$, otherwise it is 0). Let $B := [I_n, B']$ be obtained by adding an $n \times n$ identity matrix I_n in front of B'.

Next we define a $V \times \mathcal{J}$ matrix C'. For $v \in V$ and $j = (i, D) \in \mathcal{J}$ define $C'_{v,j}$ as $|D| + 1$ if $v \notin D$, and as the height of v in D in the order $\leq_i$ if $v \in D$. Append now on the left of C' a matrix so that the resulting matrix C satisfies the conditions of Theorem 1.1.

Applying Theorem 1.1 to the above matrices B, C and the all 1's vector $b = \mathbf{1}_n$, we get a nonnegative vector $x \in \mathbb{R}^{\mathcal{J} \cup V}$. Let x' be the restriction of x to $\mathbb{R}^V$. As $B \cdot x = b = \mathbf{1}$, we have $B' \cdot x' \leq \mathbf{1}$, meaning that x' is a fractional antichain. The domination property of x implies that for any element v of V there is a chain D of some partial order $\leq_i$ such that for any element u from $D \cap \mathrm{supp}(x)$ we have $v \leq_i u$. Since $c_{(i,D),(i,D)}$ is smallest in row (i, D) of C, it follows that the column (i, D) of C does not belong to $\mathrm{supp}(x)$. The equality $(Bx)_{(i,D)} = 1$ thus means that $\sum_{d \in D} x(d) = 1$, showing that $\sum_{d \in D, d \geq_i v} x(d) = 1$. This proves the fractional upper bound property of x. □

Our last application is a generalisation of a matroid version of the Gale-Shapley theorem.

An *ordered matroid* is a triple $\mathcal{M} = (E, \mathcal{C}, \leq)$ such that $(E, \mathcal{C})$ is a matroid and $\leq$ is a linear order on E. For two ordered matroids $\mathcal{M}_1 = (E, \mathcal{C}_1, \leq_1)$ and $\mathcal{M}_2 = (E, \mathcal{C}_2, \leq_2)$ on the same ground set, a subset K of E is an *$\mathcal{M}_1\mathcal{M}_2$-kernel*, if K is independent in both matroids $(E, \mathcal{C}_1)$ and $(E, \mathcal{C}_2)$, and for any element e in $E \setminus K$ there is a subset C_e of K and an index $i = 1, 2$ so that

$$\{e\} \cup C_e \in \mathcal{C}_i \text{ and } e \leq_i c \text{ for any } c \in C_e.$$

Theorem 3.3 (see [4, 3]). *For any pair $\mathcal{M}_1$, $\mathcal{M}_2$ of ordered matroids there exists an $\mathcal{M}_1\mathcal{M}_2$-kernel.* $\square$

Let $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k$ be ordered matroids on the same ground set E, where $\mathcal{M}_i = (E, \mathcal{C}_i, \leq_i)$. A vector $x \in \mathbb{R}^E_+$ is called a *fractional kernel for matroids $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_k$* if it satisfies the following two properties:

(1) x is *fractionally independent*, namely $\sum_{e \in E'} x(e) \leq r_i(E')$ for any subset E' of E, where r_i is the rank function of the matroid $\mathcal{M}_i$.

(2) every element e of E is *fractionally optimally spanned* in one of the matroids, namely there exists a subset E' of E and a matroid $\mathcal{M}_i$, such that $e \leq_i e'$ for any $e' \in E'$, and $\sum_{e \in E'} x(e) = r_i(E' \cup \{e\})$.

Note that a fractional matroid-kernel for two matroids that happens to be integral is a matroid kernel.

Theorem 3.4. *Every family $\mathcal{M}_i = (E, \mathcal{C}_i, \leq_i)$ $(i = 1, 2, \ldots, k)$ of ordered matroids has a fractional kernel.*

Proof. Let B' be a matrix whose rows are indexed by pairs (i, F) , where $1 \leq i \leq k$ and $F \subseteq E$, and whose columns are indexed by E, the $((i, F), e)$ entry being 1 if $e \in F$, 0 otherwise. Let $B := [I, B']$.

Define matrix C' on the same row and column sets as those of B', by letting its $((i, F), e)$ entry be the height of e in $\leq_i$ if $e \in F$ and $|F| + 1$ otherwise. Append an appropriate matrix on the left of C', so as to get a matrix C as in Theorem 1.1. Let b be the vector on E defined by $b_{(i,F)} := r_i(F)$.

Apply Theorem 1.1 to B, C and b. Let x be the vector whose existence is guaranteed in the theorem and x' be the restriction of x to E. We claim that x' is a fractional kernel for our matroids. As $Bx = b$ and both B and x are nonnegative, we have $B'x' \leq b$. In other words, x' is fractionally independent. The domination property of $\mathrm{supp}(x)$ yields that for any element e of E there is a subset F and a matroid $\mathcal{M}_i$ such that we have

$$e \leq_i f \qquad \text{for any element } f \text{ of } F' := F \cap \mathrm{supp}(x). \tag{5}$$

Since $c_{(i,F),(i,F)}$ is smallest in row (i, F) of C, column (i, F) does not belong to $\mathrm{supp}(x)$. Thus $(Bx)_{(i,F)} = r_i(F)$ implies

$$r_i(F') \geq \sum_{f \in F'} x'(f) = \sum_{f \in F} x(f) = r_i(F) \geq r_i(F').$$

In particular, $F' \neq \emptyset$, hence (5) and the definition of $\leq_i$ shows that $e \in F$, and this proves the optimal spanning property of x'. $\square$

We finish this section by pointing out a difference between Theorem 3.2 and Theorem 3.4. Namely, we show that Theorem 3.4 (the fractional version of Theorem 3.3), together with the well-known fact about the integrality of the matroid intersection polytope implies Theorem 3.3. The proof is analogous to the method of Aharoni and Holzman in [1]. There, they proved the existence of an integral kernel for any normal orientation of any perfect graph from the existence of a so called strong fractional kernel and from the linear description of the independent set polytope of perfect graphs. On the other hand, there is no similar polyhedral argument to deduce Theorem 3.1 from Theorem 3.2.

The polyhedral result that we need here is the following theorem of Edmonds.

Theorem 3.5 (Edmonds [2]). *If $\mathcal{M}_1 = (E, \mathcal{C}_1)$ and $\mathcal{M}_2 = (E, \mathcal{C}_2)$ are matroids on the same ground set then*

$$\begin{aligned}&\operatorname{conv}\{\chi^I : I \textit{ is independent both in } \mathcal{M}_1 \textit{ and in } \mathcal{M}_2\} = \\ &\qquad \{x \in \mathbb{R}^E : \mathbf{0} \leq x, \sum_{f \in F} x(f) \leq r_i(F) \textit{ for any } i \in \{1,2\} \textit{ and } F \subseteq E\}. \quad \square\end{aligned}$$

Alternative proof of Theorem 3.3. By Theorem 3.4, there is a fractional kernel x for ordered matroids $\mathcal{M}_1$ and $\mathcal{M}_2$. As x is fractionally independent, Theorem 3.5 implies that $x = \sum_{j=1}^{l} \lambda_j \chi^{I_j}$ is a convex combination of the characteristic vectors of common independent sets of $\mathcal{M}_1$ and $\mathcal{M}_2$. We claim that any of the I_j-s is an $\mathcal{M}_1\mathcal{M}_2$-kernel.

As x is a fractionally optimally spanning set, for any element e of E there is an index $i \in \{1,2\}$ and a subset E' of E such that $w_i(e') \leq w_i(e)$ for any element e' of E and

$$\sum_{j=1}^{l} \lambda_j |I_j \cap E'| = \sum_{e \in E'} x(e) = r_i(E' \cup \{e\}) \geq \sum_{e \in E' \cup \{e\}} x(e) = \sum_{j=1}^{l} \lambda_j |I_j \cap (E' \cup \{e\})| \,.$$

This means that each common independent set I_j intersects E' in $r_i(E' \cup \{e\})$ elements, that is each I_j spans e. $\square$

In contrast to the above argument, there is no polyhedral proof for Theorem 3.1 from Theorem 3.2 along similar lines. Namely, it can happen that for two partial orders $<_1$ and $<_2$ on the same ground set, a fractional dominating antichain is not a convex combination of dominating antichains. Figure 1 shows the Hasse diagrams of two partial orders on four elements. As any two elements of the common ground set are comparable in one of the partial orders, a dominating antichain contains exactly one element. However, it is easy to check that the all-$\frac{1}{3}$ vector is a fractional dominating antichain of total weight $\frac{4}{3}$.

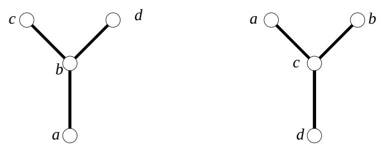

Figure 1: Hasse diagrams of the counterexample partial orders.

4 A link with a theorem of Shapley

A *simplicial complex* is a non-empty family $\mathcal{C}$ of subsets of a finite ground set such that $A \subset B \in \mathcal{C}$ implies $A \in \mathcal{C}$. Members of $\mathcal{C}$ are called *simplices* or *faces*. Let us call a simplicial complex *manifold-like* if, denoting its rank by n (that is, the maximum cardinality of a simplex in it is $n+1$), every face of cardinality n in it is contained in two faces of cardinality $n+1$. The *dual* $\mathcal{D}^*$ of a complex $\mathcal{D}$ is the set of complements of its simplices. Just like in the case of complexes, members of a dual complex are also called *faces*.

Lemma 4.1. *If $\mathcal{C}, \mathcal{D}$ are two manifold-like complexes on the same ground set X, then the number of maximum cardinality faces of $\mathcal{C}$ that are also minimum cardinality faces of $\mathcal{D}^*$ is even.*

Proof. Let $\mathcal{C}_{max}$ be the family of faces of $\mathcal{C}$ of maximum cardinality and $\mathcal{D}^*_{min}$ be the set of faces of $\mathcal{D}^*$ of minimum cardinality. We may clearly assume that these two cardinalities are equal, as otherwise the lemma claims the triviality that zero is an even number. Fix an element x of X, and define an auxiliary digraph $\vec{G}$ on $\mathcal{C}_{max} \cup \mathcal{D}^*_{min}$ by drawing an arc from $C \in \mathcal{C}_{max}$ to $D \in \mathcal{D}^*_{min}$ if $D \setminus C = \{x\}$.

Let $D \in \mathcal{D}^*_{min}$. If $x \notin D$ or $D \setminus \{x\} \notin \mathcal{C}$ then no arc enters D in $\vec{G}$. Otherwise, as $\mathcal{C}$ is manifold-like, there are exactly two different members C_1, C_2 of $\mathcal{C}_{max}$ of the form $C_i = D \setminus \{x\} \cup \{y_i\}$ for some different elements y_1, y_2 of X. If $y_1 \neq x \neq y_2$ then the in-degree of D in $\vec{G}$ is two and D is not a member of $\mathcal{C}_{max}$. Else D has in-degree exactly one, and $x \in D \in \mathcal{C}_{max} \cap \mathcal{D}^*_{min}$.

Similarly, let $C \in \mathcal{C}_{max}$. If $x \in C$ or $C \cup \{x\} \notin \mathcal{D}^*$ then no arc of $\vec{G}$ leaves C. Otherwise, $\mathcal{D}$ being manifold-like, there are exactly two members D_1, D_2 of $\mathcal{D}^*_{min}$ of the form $D_i = C \cup \{x\} \setminus \{y_i\}$ for some different elements y_1, y_2 of X. If $y_1 \neq x \neq y_2$ then the out-degree of C is exactly two and C is not a member of $\mathcal{D}^*_{min}$. Else the out-degree of C in $\vec{G}$ is exactly one and $x \notin C \in \mathcal{C}_{max} \cap \mathcal{D}^*_{min}$.

Let G be the underlying undirected graph of $\vec{G}$. The above argument shows that a vertex v of G has degree zero or two if $v \in \mathcal{C}_{max} \Delta \mathcal{D}^*_{min}$ and v has degree one if $v \in \mathcal{C}_{max} \cap \mathcal{D}^*_{min}$. As the number of odd degree vertices of a finite graph is even, the lemma follows. $\square$

What examples are there of manifold-like complexes? Of course, a triangulation of a closed manifold is of this sort. (We call this complex a *manifold-complex*.) Another well known example of a dual manifold-like complex is the *cone complex*: let X be a set of vectors in $\mathbb{R}^n$, and b a vector not lying in the positive cone spanned by any $n-1$

elements of X. Consider the set $\mathcal{C}^* := \{A \subseteq X : b \in \text{cone}(A)\}$. It is a well known fact from linear programming that if $b \in \text{cone}(A)$, where $A \subset X$, $|A| = n$ and $z \in X \setminus A$, then there exists a unique element $a \in A$ such that $b \in \text{cone}(A \cup \{z\} \setminus \{a\})$. That is, $\mathcal{C}^*$ is indeed a dual manifold-like complex.

A third example of a manifold-like complex is the *domination complex*. Let C be a matrix as in Theorem 1.1 with the additional property that in each row of C all entries are different. Then it is not difficult to check that the family of dominating column sets together with the extra member $[n]$ is a manifold-like complex. (For the details, see [1].)

Lemma 4.1 directly implies a generalisation of Sperner's lemma.

Lemma 4.2. *Let the vertices of a triangulation T of a closed n-dimensional manifold be labelled with vectors from $\mathbb{R}^{n+1}$. Let $b \in \mathbb{R}^{n+1}$ be a vector that does not belong to the cone spanned by fewer than $n+1$ labels. Then there are an even number of simplices S of the triangulation with the property that b is in the cone spanned by the vertex-labels of S.*

Proof. Let $\mathcal{C}$ be the manifold complex of T. Define family $\mathcal{D}^*$ by

$$\mathcal{D}^* := \{A \subseteq V(T) : b \in \text{cone}(L(A))\},$$

where $V(T)$ is the set of vertices of triangulation T and for subset A of $V(T)$, $L(A)$ denotes the set of labels on the vertices of A. By the condition on the vertex-labels, $\mathcal{D}^*$ is a cone complex. Clearly, the common members of $\mathcal{C}$ and $\mathcal{D}^*$ are minimum cardinality faces of $\mathcal{C}$ and maximum cardinality faces of $\mathcal{D}^*$. By Lemma 4.1, there is an even number of common members of $\mathcal{C}$ and $\mathcal{D}^*$, and these common members are exactly those simplices of T whose labels contain b in their cone. $\square$

Sperner's lemma is obtained from Lemma 4.2 by taking the n-sphere S^n as the closed manifold, by choosing the vertex-labels from the standard unit vectors $(0, 0, \ldots, 1, \ldots, 0)$ and by fixing $b = \mathbf{1}$ (the all 1's vector). This is not the standard way the lemma is stated, but is well known to be equivalent to it, see e.g. [7]. The more general Lemma 4.2 is undoubtedly known, but we do not know a reference to it. Shapley [9] proved it for the case that the labels are $0, 1$ vectors, but his proof works also for general vectors.

Next we apply Theorem 4.1 to prove Scarf's lemma. The proof is essentially the same as in [1].

Proof of Theorem 1.1. By slightly changing vector b and the entries of matrix C, we can construct vector b' and matrix C' with the following properties. No $n-1$ columns of B span a cone that contains b' and if n columns of B span a cone that contains b' then this cone also contains b. For C' we require that in each row of C' all entries are different, and if $c_{ij} < c_{ik}$ then for the corresponding C'-entires the same holds: $c'_{ij} < c'_{ik}$.

Define family $\mathcal{D}^*$ on $[m]$ by $X \in \mathcal{D}^*$ if and only if $\text{cone}_B(X)$ (the cone of those columns of B that are indexed by X) contain b'. Then $\mathcal{D}^*$ is a cone complex, by the choice of b'. Let $\mathcal{C}$ be the domination complex defined by C'. By the choice of B, b'

and C', any common member of $\mathcal{C}$ and $\mathcal{D}^*$ is a maximum cardinality face of $\mathcal{C}$ and a minimum cardinality face of $\mathcal{D}^*$. So Lemma 4.1 implies that there is an even number of such common faces. But a common face of $\mathcal{C}$ and $\mathcal{D}^*$ is either $[n]$ or it corresponds to a dominating set of C' (which is also a dominating set of C) and to a column set of B that contains b' (hence b as well) in its cone. As $[n]$ is indeed a common face of $\mathcal{C}$ and $\mathcal{D}^*$, we get that there exists a common face of the second type. $\square$

Shapley's theorem can be proved via Brouwer's fixed point theorem (which is also easily implied by it). This, and the similarity between its proof and the proof of Scarf's lemma, suggests that perhaps there is a fixed point theorem related to the latter. A supporting fact is that in [4] there was given a proof of Gale-Shapley's theorem using the Knaster-Tarski fixed point theorem for lattices.

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