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Edge splitting and connectivity augmentation in directed hypergraphs

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Abstract

We prove theorems on edge splittings and edge-connectivity augmentation in directed hypergraphs, extending earlier results of Mader and Frank, respectively, on directed graphs.

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1 Introduction

A *directed hypergraph* (or *dypergraph*, for short) is a pair $D = (V, E)$, where V is a finite set (the set of *vertices* of D) and E is a finite collection of *hyperedges*. Each hyperedge e is a set $Z \subseteq V$, with $|Z| \geq 2$, and with a specified *head vertex* $v \in Z$. We also use (Z, v) to denote a hyperedge on set Z and with head v. The vertices in $Z - v$ are the *tail vertices* of Z. The *size* of e is $|Z|$. If the size of e is two, that is, $e = (Z, v)$ for some $Z = \{u, v\}$, then e is called a *graph edge* and can simply be denoted by uv. Thus a directed graph (without loops) is a dypergraph with graph edges only.

In a recent paper Frank, Király, and Király [4] investigated several connectivity properties of (directed) hypergraphs. They showed that, using an appropriate definition of edge-connectivity, a number of classical results (Menger's theorem, Edmonds' branching theorem, Nash-Williams' orientation theorem) can be extended to hypergraphs.

A *path* (from vertex v_1 to vertex v_{k+1}) in a dypergraph is a sequence $v_1, e_1, v_2, e_2, ..., e_k, v_{k+1}$ of vertices and edges such that v_i is a tail of e_i and v_{i+1} is the head of e_i for $1 \leq i \leq k$. We say that an edge (Z, v) *enters* a set $X \subset V$ if $v \in X$ and $Z - X \neq \emptyset$. Let $\rho(X)$ denote the number of edges entering X. With this notation it is not difficult to show the following version of Menger's theorem, see [4]: in a dypergraph D there

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exist k edge-disjoint paths from s to t if and only if $\rho(X) \geq k$ for every set $X \subset V$ with $s \notin X, t \in X$. Hence it is natural to call a dypergraph $D = (V, E)$ *k-edge-connected* if $\rho(X) \geq k$ for every $\emptyset \neq X \subset V$. (We use $\subset$ to denote proper inclusion, while $\subseteq$ means $\subset$ or $=$.)

In this paper we focus on a different group of edge-connectivity questions. We prove theorems on edge splittings and edge-connectivity augmentation in dypergraphs, extending earlier results of Mader [8] and Frank [2] on directed graphs.

2 Preliminaries

In this section we introduce further notation and prove some basic facts on dypergraphs. Let $D = (V, E)$ be a dypergraph. For $X \subset V$ we have already defined the *in-degree* $\rho(X)$ of X. Let $\delta(X) = \rho(V - X)$ denote the *out-degree* of X. For a single vertex v we simply use $\rho(v)$ and $\delta(v)$. Furthermore, we use $d_2(X, Y)$ to denote the number of edges of size two between $X - Y$ and $Y - X$ in both directions, and put $\overline{d}(X, Y) = d_2(X \cap Y, V - (X \cup Y))$.

Two subsets $X, Y \subseteq V$ are *intersecting* if none of $X - Y$, $Y - X$, and $X \cap Y$ is empty. If, in addition, $X \cup Y \neq V$, then an intersecting pair X, Y is called *crossing.* A family of pairwise disjoint subsets of V is a *subpartition* of V. The equalities in the next three lemmas are easy to prove by counting the contribution of an edge to the two sides.

Lemma 2.1. *[4] Let $D = (V, E)$ be a dypergraph and let $X, Y \subseteq V$. Then*

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y) + d_2(X, Y). \tag{1}$$

Lemma 2.2. *Let $D = (V, E)$ be a dypergraph and let $X, Y \subseteq V$. Then*

$$\delta(X) + \delta(Y) \geq \delta(X \cup Y) + \delta(X \cap Y) + d_2(X, Y). \tag{2}$$

Lemma 2.3. *Let $D = (V, E)$ be a dypergraph and suppose that $X \cap Y$ is incident with edges of size two only for some sets $X, Y \subseteq V$. Then*

$$\rho(X) + \rho(Y) = \rho(X - Y) + \rho(Y - X) + \rho(X \cap Y) - \delta(X \cap Y) + \overline{d}(X, Y). \tag{3}$$

We shall often consider a dypergraph $D = (V + s, E)$ with a designated vertex s. In this case we shall always assume that the designated vertex s is incident to graph edges (that is, edges of size two) only. Let $N^-(s) = \{v \in V : vs \in E\}$ and $N^+(s) = \{u \in V : su \in E\}$. We say that $D = (V + s, E)$ is *(k, s)-edge-connected* if

$$\rho(X) \geq k \quad \text{for all } \emptyset \neq X \subset V, \tag{4}$$

and

$$\delta(X) \geq k \quad \text{for all } \emptyset \neq X \subset V. \tag{5}$$

Given a dypergraph $D' = (V, E')$, a *k-extension* of D' is a (k, s)-edge-connected dypergraph $D = (V + s, E)$, obtained from D' by adding a new vertex s and some edges of size two incident to s in such a way that (4) and (5) hold.

Lemma 2.4. *Let $D = (V+s, E)$ be a (k, s)-edge-connected dypergraph and let $A, B \subset V$ be intersecting sets. Then:*
(a) if $\delta(A) = k = \delta(B)$ and $A \cup B \neq V$, then $\delta(A \cup B) = k$;
(b) if $\rho(A) = k = \rho(B)$ and $A \cup B \neq V$, then $\rho(A \cup B) = k$;
(c) if $\delta(A) = k = \delta(B)$, $A \cup B = V$, and $\rho(s) \geq \delta(s)$ then $\overline{d}(A, B) = 0$;
(d) if $\delta(A) = k = \rho(B)$ then $d_2(V + s - A, B) = 0$.

Proof. (a) follows from Lemma 2.2, (b) and (d) follow from Lemma 2.1 and (c) follows from Lemma 2.3. □

A set $X \subset V$ is called *in-critical* if $\rho(X) = k$ and *out-critical* if $\delta(X) = k$. A set is *critical* if it is in-critical or out-critical (or both).

Lemma 2.5. *Let $D = (V + s, E)$ be a (k, s)-edge-connected dypergraph with $\rho(s) \geq \delta(s)$ and let A, B be intersecting maximal critical sets such that $d_2(s, A \cap B) \geq 1$. Then A and B are both in-critical and $A \cup B = V$.*

Proof. By Lemma 2.4(d) A and B must be either both in-critical or both out-critical. Lemma 2.4(a,b) implies $A \cup B = V$. Then, since $d_2(s, A \cap B) \geq 1$, we may use Lemma 2.4(c) to deduce that A and B are both in-critical. □

For two disjoint sets $X, Y \subset V$ let $\delta(X, Y)$ denote the number of graph edges with tail in X and head in Y.

Lemma 2.6. *Let $D = (V+s, E)$ be a (k, s)-edge-connected dypergraph and let $R \subset V$ be an in-critical set. Then $\delta(V - R, s) \geq \delta(s, R)$.*

Proof. The lemma follows since $k \leq \delta(V - R) = \rho(R + s) = \rho(R) - \delta(s, R) + \delta(V - R, s) = k - \delta(s, R) + \delta(V - R, s)$ □

3 Splitting off edges

Let $H = (V + s, E)$ be a dypergraph with a designated vertex $s \in V$. The operation *splitting off* replaces an edge su and a set of edges $\{v_1 s, v_2 s, ..., v_t s\}$ by a new hyperedge (Z, u), where $Z = \{u, v_1, v_2, ..., v_t\}$. This operation is also called a *t-splitting at s (on $\{su, v_1 s, v_2 s, ..., v_t s\}$)*. If $u = v_i$ for some $1 \leq i \leq t$ then v_i is not present as a tail vertex in Z. If $u = v_i$ for all i then no new hyperedge is added. A 1-splitting on edges su, vs corresponds to the well-known operation of "splitting off" in digraphs, which replaces su and vs by a new edge vu. A *complete splitting* at s is a sequence of splittings which isolates s.

Given a (k, s)-edge-connected dypergraph $D = (V + s, E)$, a t-splitting at s on $su, v_1 s, v_2 s, \ldots, v_t s$ (or the pair $(su, \{v_1 s, v_2 s, \ldots, v_t s\})$) is *admissible* if the dypergraph obtained by splitting off these edges is also (k, s)-edge-connected. An *admissible complete splitting* is a complete spitting in which each splitting is admissible i.e. a complete splitting which results in a k-edge-connected dypergraph on vertex-set V.

Lemma 3.1. *Let $D = (V + s, E)$ be a (k, s)-edge-connected dypergraph and let su, $v_1 s, v_2 s, \ldots, v_t s \in E$. The pair $(su, \{v_1 s, v_2 s, \ldots, v_t s\})$ is not admissible if and only if there exists $X \subset V$ such that one of the following sets of conditions holds (possibly after permuting the indices of the v_i's):*
(i) for some $2 \le r \le t$ we have $\delta(X) \le k+r-2$, $u \notin X$, and $v_i \in X$ for all $1 \le i \le r$,
(ii) for some $1 \le r \le t$ we have $\delta(X) \le k+r-1$, $u \in X$, and $v_i \in X$ for all $1 \le i \le r$,
(iii) $\rho(X) = k$, $u \in X$, and $v_i \in X$ for all $1 \le i \le t$.

Proof. It is easy to see that if any of (i),(ii), or (iii) holds then X will violate (4) or (5) after splitting off the pair $(su, \{v_1 s, v_2 s, \ldots, v_t s\})$. Conversely, suppose that after splitting off this pair there exists a set $X \subset V$ in the resulting dypergraph D' which violates (4) or (5). First suppose $\delta'(X) < k$. If $u \notin X$ then $\delta(X) - \delta'(X) = |\{v_i : v_i \in X, 1 \le i \le t\}| - 1$. Thus, for a suitable permutation of the indices and choice of r with $2 \le r \le t$, we must have $v_i \in X$ for $1 \le i \le r$, and $\delta(X) \le k+r-2$. That is, (i) holds. If $u \in X$ then $\delta(X) - \delta'(X) = |\{v_i : v_i \in X, 1 \le i \le t\}|$. Thus, for a suitable permutation of the indices and choice of r with $1 \le r \le t$, we must have $v_i \in X$ for $1 \le i \le r$, and $\delta(X) \le k + r - 1$. That is, (ii) holds. Next suppose $\rho'(X) < k$. Then we must have $u \in X$. Since $\rho(X) - \rho'(X) \le 1$, and equality holds if and only if $v_i \in X, 1 \le i \le t$, it follows that condition (iii) holds. □

Note that (i) and (ii) imply that (su, W) is not admissible for any $\{v_1 s, v_2 s, \ldots, v_r s\} \subseteq W$, while (iii) implies that (su, W) is not admissible for any $W \subseteq \{v_1 s, v_2 s, \ldots, v_t s\}$.

Lemma 3.2. *Let $D = (V + s, E)$ be a (k, s)-edge-connected dypergraph with $\rho(s) \ge \delta(s)$ and let $su \in E$. Then there is no admissible 1-splitting at s containing su if and only if there exist two maximal in-critical sets R_1, R_2 such that $R_1 \cup R_2 = V$ and $u \in R_1 \cap R_2$.*

Proof. Since there is no admissible 1-splitting containing the edge su, it follows from Lemma 3.1 that there exists a family of maximal critical sets $R_1, R_2, \ldots, R_t$ with $N^-(s) \subseteq \bigcup_{i=1}^t R_i$ and $u \in R_i$ for all $1 \le i \le t$. First suppose $t = 1$. Then $\delta(V - R_1, s) = 0$ and $\delta(s, R_1) \ge 1$, so Lemma 2.6 implies that R_1 is not in-critical. Thus R_1 is out-critical. Since $\delta(s) \le \rho(s)$, we have $\rho(V - R_1) = \delta(R_1) - \delta(R_1, s) + \delta(s, V - R_1) \le k - \rho(s) + \delta(s) - 1 < k$, contradicting (4). Hence $t \ge 2$. Then by Lemma 2.5 it follows that R_1 and R_2 are both in-critical sets and $R_1 \cup R_2 = V$. □

A simple but useful corollary of this lemma is a sufficient condition for the existence of a 1-splitting, in terms of the in- and out-degree of s. (See [1] for an application.)

Lemma 3.3. *Let $D = (V + s, E)$ be a (k, s)-edge-connected dypergraph with $\rho(s) \ge \delta(s) \ge k + 1$. Then for every edge $su \in E$ there exists an admissible 1-splitting at s containing su.*

Proof. Consider a fixed edge $su \in E$. If there is no admissible 1-splitting at s containing su then by Lemma 3.2 there exist two maximal in-critical sets R_1, R_2 such that $R_1 \cup R_2 = V$ and $u \in R_1 \cap R_2$. By Lemma 2.1 we get $2k = \rho(R_1) + \rho(R_2) \ge \rho(R_1 \cup R_2) + \rho(R_1 \cap R_2) \ge \delta(s) + k \ge (k + 1) + k$, a contradiction. □

A *flower* $\mathcal{F} = \{R_1, R_2, ..., R_t\}$ in a (k, s)-edge-connected dypergraph $D = (V + s, E)$ is a collection of maximal in-critical sets R_i, $1 \le i \le t$, such that $R_i \cup R_j = V$ for all $1 \le i < j \le t$. We call $\bigcap_{i=1}^{t} R_i$ the *core* of $\mathcal{F}$ and the sets $P_i = V - R_i$ the *petals* of $\mathcal{F}$. The *size* of the flower is equal to the number of petals t. If $u \in N^+(s)$ and u is in the core of $\mathcal{F}$ then we say the flower $\mathcal{F}$ is *centered on u*.

Theorem 3.4. *Let $D = (V + s, E)$ be a (k, s)-edge-connected dypergraph with $\rho(s) \ge \delta(s)$ and let $su \in E$. Then there is no admissible r-splitting containing su for all $1 \le r \le t$ if and only if D has a flower of size $t + 1$ centered on u.*

Proof. If D has a flower $\mathcal{F}$ of size $t + 1$ centered on u, then, since the petals of $\mathcal{F}$ are pairwise disjoint, for any $1 \le r \le t$ and any r-splitting $(su, \{v_1, v_2, ..., v_r\})$, there exists a petal P_j with $v_i \notin P_j$ for all $1 \le i \le r$. This implies that R_j satisfies Theorem 3.1(iii), and hence the splitting is not admissible.

To see the necessity, suppose that there is no admissible r-splitting containing su for all $1 \le r \le t$. Since, in particular, there is no admissible 1-splitting containing su, it follows from Lemma 3.2 that D has a flower of size 2 centered on u. Let $F = \{R_1, R_2, \dots, R_m\}$ be a flower of maximum size ($m \ge 2$) in D, centered on u, and suppose that $m \le t$. Since $\delta(V - R_i, s) \ge \delta(s, R_i) \ge 1$ by Lemma 2.6, we can choose $v_i \in P_i \cap N^-(s)$ for all i, $1 \le i \le m$. Since D has no m-splitting containing su, it follows from Lemma 3.1 that (for a suitable permutation of the indices) there exists $X \subset V$ such that either Lemma 3.1(i), (ii), or (iii) holds.

(i) There exists a set $X \subset V$ with $v_1, v_2, \dots, v_r \in X, u \notin X$ and $\delta(X) \le k + r - 2$ for some $2 \le r \le m$.

Let $Y = P_1 + s$. By Lemma 2.2 we have

$$\begin{aligned} k + (k + r - 2) &\ge \delta(Y) + \delta(X) \\ &\ge \delta(Y \cap X) + \delta(Y \cup X) + d_2(Y, X) \\ &\ge k + k + (r - 1), \end{aligned}$$

since $u \in V - (X \cup Y), v_2, v_3, \dots, v_r \in X - Y$ and $s \in Y - X$ and $r \ge 2$. This contradiction shows (i) cannot occur.

(ii) There exists $X \subset V$ with $u, v_1, v_2, \dots, v_r \in X$ and $\delta(X) \le k + r - 1$ for some $1 \le r \le m$.

First suppose $r = 1$. Then X is out-critical and, by applying Lemma 2.4(d) to X and R_1, we deduce that $d_2(V + s - X, R_1) = 0$, contradicting the fact that $su \in E(D), s \in (V + s - X) - R_1, u \in R_1 - (V + s - X)$. Thus $r \ge 2$. Choose a petal P_i such that $1 \le i \le r$, $P_i \cup X \ne V$ and $P_i \cap X \ne \emptyset$. Such a petal P_i exists since: if $\bigcap_{i=1}^{m} R_i - X \ne \emptyset$ then we can choose any P_i, $1 \le i \le r$; if $\bigcap_{i=1}^{m} R_i \subseteq X$ then, since $X \ne V$, there exists P_j, $1 \le j \le m$ such that $P_j - X \ne \emptyset$ and so we can choose $P_i \ne P_j$ with $1 \le i \le r$, using the fact that $r \ge 2$. In both cases we have $v_i \in P_i \cap X$.

Let $Y = P_i + s$. By Lemma 2.2

$$\begin{aligned} k + (k + r - 1) &\ge \delta(Y) + \delta(X) \\ &\ge \delta(Y \cap X) + \delta(Y \cup X) + d_2(Y, X) \\ &\ge k + k + r, \end{aligned}$$

since $(P_i+s)\cup X \neq V+s,\ v_i \in (P_i+s)\cap X,\ u \in X-Y,\ \{v_1,v_2,\dots,v_r\}-v_i \subseteq X-Y$ and $s \in Y-X$. This contradiction shows (ii) cannot occur.

(iii) There exists $X \subset V$ with $\rho(X)=k$, $u \in X$ and $v_1,v_2,\dots,v_m \in X$.

We can assume X is a maximal in-critical set. By Lemma 2.5, $X \cup R_i = V$ for all $1 \le i \le m$. Therefore $\mathcal{F}' = \{R_1,R_2,\dots,R_m,X\}$ is a flower of size $m+1$ centered on u, contradicting the maximality of m. This completes the proof of the theorem. □

Lemma 3.5. *Let $D=(V+s,E)$ be a (k,s)-edge-connected dypergraph with $\rho(s) \ge \delta(s)$ and let $\mathcal{F}$ be a flower of size m in D, centered on $u \in N^+(s)$. Then $m \le \lfloor \frac{\rho(s)-1}{\delta(s)} \rfloor + 1$.*

Proof. Let $\mathcal{F}=\{R_1,R_2,\dots,R_m\}$. Let $\delta(s,\bigcap_{i=1}^m R_i)=a$ and $\delta(s,P_i)=b_i$ for $1 \le i \le m$. By Lemma 2.6,

$$\delta(P_i,s)=\delta(V-R_i,s) \ge \delta(s,R_i) = a - b_i + \sum_{1\le j\le m} b_j$$

for all i, $1 \le i \le m$. Hence

$$\rho(s) \ge \sum_{i=1}^m \delta(P_i,s) \ge \sum_{i=1}^m \delta(s,R_i) = (m-1)\delta(s)+a.$$

Since $a \ge 1$ because $u \in \bigcap_{1\le i\le m} R_i$, we have $\rho(s) \ge (m-1)\delta(s)+a$ and hence $m \le \lfloor \frac{\rho(s)-1}{\delta(s)} \rfloor + 1$. □

Theorem 3.6. *Let $D=(V+s,E)$ be a (k,s)-edge-connected dypergraph with $\rho(s) \ge \delta(s)$. Then there exists an admissible complete splitting at s.*

Proof. Choose $u \in N^+(s)$. Suppose there is no admissible ρ_1-splitting containing su for all $\rho_1, 1 \le \rho_1 \le \rho(s)-\delta(s)+1$. By Theorem 3.4, D has a flower of size $\rho(s)-\delta(s)+2$ centered on u. This contradicts Lemma 3.5, since $\rho(s)-\delta(s)+2 > \lfloor \frac{\rho(s)-1}{\delta(s)} \rfloor + 1$. Hence D has a ρ_1-splitting D' containing su for some $\rho_1, 1 \le \rho_1 \le \rho(s)-\delta(s)+1$. Since in D' we have $\delta'(s) \le \rho'(s)$, we may complete the proof by applying induction to D'. □

If D is a (k,s)-edge-connected directed graph and $\rho(s)=\delta(s)$ then we obtain Mader's edge splitting theorem as a corollary.

Corollary 3.7. *[8] Let $D=(V+s,E)$ be a (k,s)-edge-connected directed graph with $\rho(s)=\delta(s)$. Then there is an admissible complete splitting at s (consisting of a sequence of 1-splittings).*

The proof of the next characterization is very similar to the proof of Theorem 3.4.

Theorem 3.8. *Let $D=(V+s,E)$ be a (k,s)-edge-connected dypergraph with $\rho(s) \ge \delta(s)$. Then D has no admissible r-splitting for all r, $1 \le r \le t$, if and only if one of the following holds*
(i) there exists a flower $\mathcal{F}=\{R_1,R_2,\dots,R_{t+1}\}$ such that $N^+(s) \subseteq \bigcap_{i=1}^{t+1} R_i$,
(ii) there exists a flower $\mathcal{F}$ of size $t+2$.

Proof. If (i) or (ii) holds then for all $1 \leq r \leq t$ and any r-splitting on $(su, \{v_1, v_2, ..., v_r\})$, there exists a petal P_j of $\mathcal{F}$ such that $u \notin P_j$ and $v_i \notin P_j$, for all $1 \leq i \leq r$. This implies that R_j satisfies the conditions of Lemma 3.1(iii), and hence the splitting is not admissible.

To see the necessity, let us choose a vertex $u' \in N^+(s)$. Since there is no admissible t-splitting containing su', it follows from Theorem 3.4 that there is a flower $\mathcal{F} = \{R_1, R_2, \ldots, R_{t+1}\}$ in D, centered on u'. If $N^+(s) \subseteq \bigcap_{i=1}^{t+1} R_i$ then we conclude that (i) holds. Otherwise there is a vertex $u \in N^+(s)$ in one of the petals, say P_{t+1}, of $\mathcal{F}$. By Lemma 2.6, $\delta(V - R_i, s) \geq \delta(s, R_i) \geq 1$ for all $2 \leq i \leq t+1$. Thus we can choose $v_i \in P_i \cap N^-(s)$ for all $2 \leq i \leq t+1$. Since the splitting on $(su, \{v_1, v_2, ..., v_t\})$ is not admissible, it follows from Lemma 3.1 that (for a suitable permutation of the indices) there exists $X \subset V$ such that either Lemma 3.1(i), (ii), or (iii) holds.

If (i) or (ii) holds then, by exactly the same argument that we used in the proof of Theorem 3.4 (i) and (ii), we get a contradiction.

So suppose that (iii) holds, that is, there exists a set $X \subset V$ with $\rho(X) = k$, $u \in X$ and $v_1, v_2, \ldots, v_t \in X$. We can assume X is a maximal in-critical set. By Lemma 2.5, $X \cup R_i = V$ for all $1 \leq i \leq t+1$. Therefore $\mathcal{F}' = \{R_1, R_2, \ldots, R_{t+1}, X\}$ is a flower of size $t+2$ in D. Thus (ii) holds. This completes the proof of the theorem. □

The next result was stated by Frank [3] without proof.

Theorem 3.9. *Let $D = (V + s, E)$ be a (k, s)-edge-connected directed graph with $2\delta > \rho(s) \geq \delta(s)$. Then D has an admissible 1-splitting.*

Proof. Suppose that there is no admissible 1-splitting in D. Then by Theorem 3.8 either (i) there exists a flower $\mathcal{F} = \{R_1, R_2\}$ such that $N^+(s) \subseteq R_1 \cap R_2$, or (ii) there exists a flower $\mathcal{F} = \{R_1, R_2, R_3\}$ of size 3. First suppose (i) holds. Then by Lemma 2.6 we have $\rho(s) \geq \delta(V - R_1, s) + \delta(V - R_2, s) \geq \delta(s, R_1) + \delta(s, R_2) = 2\delta(s)$, a contradiction.

Next suppose (ii) holds. Then by Lemma 2.6 we have $\rho(s) \geq \delta(V - R_1, s) + \delta(V - R_2, s) + \delta(V - R_3, s) \geq \delta(s, R_1) + \delta(s, R_2) + \delta(s, R_3) = 2(\delta(s, P_1) + \delta(s, P_2) + \delta(s, P_3)) + 3\delta(s, R_1 \cap R_2 \cap R_3) \geq 2\delta(s)$, a contradiction. □

Lemma 3.10. *Let $D = (V + s, E)$ be a (k, s)-edge-connected dypergraph with $\rho(s) \geq \delta(s)$ and let $su \in E$. Suppose that there exists an admissible i-splitting and an admissible j-splitting containing su for some $1 \leq i < j$. Then there exists an admissible l-splitting containing su for all $i < l < j$.*

Proof. Let $S = \{su, v_1s, v_2s, \ldots, v_js\}$ be an admissible j-splitting of D. Using induction on $j - i$, it suffices to show that there is an admissible $(j-1)$-splitting containing su. Suppose not. Let $S_t = \{su, v_1s, v_2s, \ldots, v_js\} - \{v_ts\}$ for all $t, 1 \leq t \leq j$. Since D has no $(j-1)$-splitting containing su, we can use the note following Lemma 3.1 and the fact that $\{su, v_1s, v_2s, \ldots, v_js\}$ is an admissible splitting, to deduce that there exists $X_t \subset V$ such that $\rho(X_t) = k$, and $\{u, v_1, v_2, \ldots, v_j\} - \{v_t\} \subseteq X_t$ for all $t, 1 \leq t \leq j$. We may assume that each X_t is a maximal in-critical set. Note that $v_t \notin X_t$ for all $t, 1 \leq t \leq j$, otherwise X_t would imply that S is not an admissible splitting. Thus the sets X_t are all distinct. Applying Lemma 2.5 we deduce that $X_r \cup X_t = V$ for all $r, 1 \leq r < t \leq j$. Since $u \in \bigcap_{t=1}^{j} X_t$ it follows that $F = \{X_1, X_2, \ldots, X_j\}$ is a flower of size j centered on u. Since $i \leq j-1$, the existence of F implies that there is no admissible i-splitting containing su by (the easy part of) Theorem 3.4. This contradicts the initial hypotheses of the lemma. □

Lemma 3.11. *Let $D = (V+s, E)$ be a (k,s)-edge-connected dypergraph with $\rho(s) \geq \delta(s)$ and $(t-1)\delta(s) < \rho(s) \leq t\delta(s)$. Then there exists an admissible t-splitting at s in D.*

Proof. Since $\rho(s) \leq t\delta(s)$, we have $\lfloor \frac{\rho(s)-1}{\delta(s)} \rfloor + 1 \leq (t-1)+1 = t$, and hence D has no flower of size more than t by Lemma 3.5. By Theorem 3.4 this implies that for each $su \in E$ there is an admissible i-splitting containing su for some $1 \leq i \leq t$. Suppose that there is no admissible t-splitting at s in D. By Theorem 3.6, D has an admissible complete splitting at s. Since $(t-1)\delta(s) < \rho(s)$, there is an admissible ρ_i-splitting in this complete splitting sequence with $\rho_i \geq t$. By our assumption this implies $\rho_i \geq t+1$. Let su' be the edge leaving s in this splitting. As we have seen, su' is contained in an admissible l-splitting for some $1 \leq l \leq t-1$. Now we can use Lemma 3.10 to deduce that su' is contained in an admissible t-splitting, a contradiction. □

Theorem 3.12. *Let $D = (V+s, E)$ be a (k,s)-edge-connected dypergraph with $\rho(s) \geq \delta(s)$ and $(t-1)\delta(s) < \rho(s) \leq t\delta(s)$. Then there exists an admissible complete splitting at s consisting of admissible ρ_i-splittings, $1 \leq i \leq \delta(s)$, such that $t-1 \leq \rho_i \leq t$ for all $i, 1 \leq i \leq \delta(s)$.*

Proof. By Lemma 3.11 we can perform admissible t-splittings at s as long as we maintain $(t-1)\delta(s) < \rho(s)$. Note that such a splitting will always maintain $\rho(s) \leq t\delta(s)$. It is also easy to see that when we get stuck (that is, when $(t-1)\delta(s) < \rho(s)$ fails in the current dypergraph D') then we have $(t-1)\delta'(s) = \rho'(s)$. By applying Lemma 3.11 to D' we can deduce that there is an admissible complete splitting at s in D' consisting of admissible $(t-1)$-splittings. Thus all the edges incident to s can be split off by admissible t- or $(t-1)$-splittings, as required. □

4 Connectivity augmentation

In this section we apply our results on admissible splittings in dypergraphs to extend earlier results on edge-connectivity augmentation of directed graphs. We start with the *k-edge-connectivity augmentation problem*: given a dypergraph $D = (V, E)$, find a smallest set F of hyperedges of size (at most) t for which $D' = (V, E \cup F)$ is k-edge-connected.

We shall characterize the minimum size of an augmenting set F with the help of Theorem 3.12 and the following result from the theory of submodular functions, due to Fujishige [6]. A function $p : 2^V \to R$ is *crossing supermodular* if $p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y)$ for every crossing pair $X, Y \subset V$.

Theorem 4.1. *[6] Let $p : 2^V \to Z$ be a crossing supermodular function with $p(V) = \gamma$. There exists a function $z : V \to Z$ satisfying $z(V) = \gamma$ and $z(A) \geq p(A)$ for all $\emptyset \neq A \subseteq V$ if and only if for every partition $\{Z_1, Z_2, ..., Z_{q+1}\}$ of V we have*

$$\gamma \geq \sum_{i=1}^{q+1} p(Z_i) \tag{6}$$

and

$$q\gamma \geq \sum_{1}^{q+1} p(V - Z_i). \tag{7}$$

Theorem 4.2. *Let $D = (V, E)$ be a dypergraph and γ be a non-negative integer. Then D can be made k-edge-connected by adding γ new hyperedges of size at most t if and only if*

$$\gamma \geq \sum_{i=1}^{r} k - \rho(X_i) \tag{8}$$

and

$$(t-1)\gamma \geq \sum_{i=1}^{r} k - \delta(X_i) \tag{9}$$

holds for every subpartition $\{X_1, X_2, ..., X_r\}$ of V.

Proof. Let F be a set of hyperedges of size t which makes D k-edge-connected and for every $v \in V$ let $z_F(v) = |\{e \in F : \text{ the head of } e \text{ is } v\}|$. Then we must have $z_F(X) \geq k - \rho(X)$ for all $X \subset V$. Since $|F| = z_F(V)$, the necessity of (8) follows. A similar argument shows that (9) is also necessary.

To see sufficiency suppose that (8) and (9) hold. Let us define a function $p : 2^V \to Z$ by $p(\emptyset) = 0$, $p(V) = \gamma$, $p(X) = k - \rho(X)$, for all $X \subset V$ with $|X| \geq 2$, and $p(x) = \max\{0, k - \rho(x)\}$ for all $x \in V$. Since the in-degree function ρ is submodular by Lemma 2.1, it is easy to see that p is crossing supermodular. We shall show that p satisfies conditions (6) and (7) of Theorem 4.1. Let $P = \{Z_1, Z_2, ..., Z_{q+1}\}$ be a partition of V and let $\{X_1, X_2, ..., X_r\}$ be the subpartition of V consisting of those elements $X_i \in P$ for which $p(X_i) > 0$. Then $\gamma \geq \sum_{i=1}^{r} k - \rho(X_i) \geq \sum_{i=1}^{q+1} p(Z_i)$ by (8) so (6) holds. Furthermore, since each edge of D has a unique head, we have $\sum_{i=1}^{q+1} \rho(Z_i) \leq \sum_{i=1}^{q+1} \delta(Z_i)$. Suppose $p(V - Z_i) = k - \rho(V - Z_i)$ for all $1 \leq i \leq q+1$. Then

$$\sum_{i=1}^{q+1} p(V - Z_i) = \sum_{i=1}^{q+1} k - \rho(V - Z_i) = \sum_{i=1}^{q+1} k - \delta(Z_i) \leq \sum_{i=1}^{q+1} k - \rho(Z_i) \leq \gamma \leq q\gamma$$

by (8), and (7) holds. Finally we consider the case when $p(V - Z_j) \neq k - \rho(V - Z_j)$ for some j, $1 \leq j \leq q+1$. Then we must have $|V - Z_j| = 1$, $q = 1$ and $p(V - Z_j) = 0$. Assuming without loss of generality that $j = 1$ and $V - Z_1 = \{v\}$, we have

$$\sum_{i=1}^{q+1} p(V - Z_i) = p(V - v) \leq \max\{0, k - \rho(V - v)\} \leq \gamma$$

by (8), and again (7) holds. It follows from Theorem 4.1 that there exists a function $z_{in} : V \to Z_+$ satisfying $z_{in}(V) = \gamma$ and $z_{in}(X) \geq k - \rho(X)$ for all $\emptyset \neq X \subset V$.

To finish the proof we construct an extension $D' = (V + s, E')$ of D by adding a new vertex s and $z_{in}(v)$ parallel graph edges from s to v for each $v \in V$, and a minimal set of graph edges from V to s to ensure that $\delta'(X) \geq k$ for all $\emptyset \neq X \subset V$. Then D' is (k, s)-connected and minimality implies that, for each edge $vs \in E'$, there exists an out-critical set $X \subset V$ with $v \in X$.

Claim 4.3. *$\delta'(s) = \gamma$ and $\rho'(s) \leq (t-1)\delta'(s)$.*

Proof. The fact that $\delta'(s) = \gamma$ follows since $z_{in}(V) = \gamma$. To prove the second inequality we apply the proof method of [2, Lemma 3.3]. Choose a family of out-critical sets $P = \{X_1, X_2, \ldots, X_r\}$ in D' which cover the set of in-neighbours of s and are such that r is as small as possible. If P is a subpartition of V then $\rho'(s) = \sum_{i=1}^{r} k - \delta(X_i) \leq (t-1)\gamma$ by (9). Hence we may assume that P is not a subpartition of V. Lemma 2.4(a) and the minimality of r now implies that $r = 2$ and $X_1 \cup X_2 = V$. Now

$$\rho'(s) \leq k - \delta(X_1) + k - \delta(X_2) = k - \rho(X_2 - X_1) + k - \rho(X_1 - X_2) \leq \gamma$$

by (8). □

Using the claim, we may modify D', if necessary, by adding more graph edges from V to s arbitrarily, until $\rho'(s) = (t-1)\delta'(s) = (t-1)\gamma$ holds in D'. By applying Theorem 3.12 to D' we obtain an admissible complete splitting at s, consisting of γ t-splittings. The hyperedges of size at most t obtained by these splittings form the required augmenting set of size γ for D. □

If D is a directed graph and $t = 2$ then we get Frank's theorem as a corollary.

Corollary 4.4. *[2] A directed graph $D = (V, E)$ can be made k-edge-connected by adding at most γ new edges if and only if*

$$\sum_i (k - \varrho(X_i)) \leq \gamma \text{ and } \sum_i (k - \delta(X_i)) \leq \gamma$$

hold for every sub-partition $\{X_1, \ldots, X_t\}$ of V.

5 Open problems

Theorem 3.8 characterised when there is no admissible 1-split in a dypergraph. A more general question is the maximum length of a sequence of admissible 1-splits. For this problem we offer the following conjecture.

Conjecture 5.1. *Let $D = (V + s, E)$ be a (k, s)-edge-connected dypergraph with $\rho(s) \geq \delta(s)$. Then D does not have a sequence of l admissible 1-splits at s if and* *only if there is a family* $\mathcal{F} = \{R_1, R_2, \ldots, R_r\}$ *of subsets of* V*, where* $2 \le r \le 2l+1$ *and* $R_i \cup R_j = V$ *for* $1 \le i < j \le r$*, such that*

$$\sum_{i=1}^{r} \rho(R_i) \le rk + (r-1)l - q - 1$$

where $q = \min\{l, \delta(s, P_1 \cup P_2 \cup \ldots \cup P_r)\}$*, and* $P_i = V - R_i$ *for all* $1 \le i \le r$*.*

Suppose that $\mathcal{F} = \{R_1, R_2, \ldots, R_r\}$ is a family with the above properties in D and let vs, su be a 1-split. If $u \in \cap_{i=1}^{r} R_i$ then splitting off vs, su reduces the in-degree of at least $r-1$ sets in $\mathcal{F}$ by one. Otherwise the splitting reduces the in-degree of at least $r-2$ sets in $\mathcal{F}$ by one. Hence for a dypergraph D' obtained from D by a sequence of l 1-splittings we have

$$\sum_{i=1}^{r} \rho'(R_i) \le \sum_{i=1}^{r} \rho(R_i) - (r-2)l - (l-q) \le rk + (r-1)l - q - 1 - (r-2)l - (l-q) = rk - 1,$$

which implies that $\rho'(R_i) \le k-1$ for some $1 \le i \le r$. Thus the sequence of 1-splittings is not admissible. This proves sufficiency.

The case $l = 1$ of the conjecture follows by choosing $t = 1$ in Theorem 3.8.

The longest splitting sequence problem is a special case of the following question. Let $D = (V + s, E)$ be a (k, s)-edge-connected dypergraph and let $\mathcal{S} = \{(p_1, p_2, ..., p_w), (q_1, q_2, ..., q_w)\}$ be a pair of sequences of positive integers with $\sum_1^w p_i = \delta(s)$ and $\sum_1^w q_i = \rho(s)$. An $\mathcal{S}$*-detachment* at s is obtained by replacing s by w vertices $s_1, s_2, ..., s_w$ and replacing every edge su (vs) by a new edge $s_i u$ (vs_i, respectively) for some $1 \le i \le w$, so that $\rho(s_i) = p_i$ and $\delta(s_i) = q_i$ hold for all $1 \le i \le w$. An $\mathcal{S}$-detachment is *admissible* if the resulting dypergraph also satisfies (4) and (5). It can be seen that if $p_i = q_i = 1$ for all $1 \le i \le w$ then an $\mathcal{S}$-detachment corresponds to a complete splitting consisting of 1-splits. An interesting problem is to characterise when there is an admissible $\mathcal{S}$-detachment in a directed (hyper)graph D.

Note that the corresponding problem for undirected graphs has been solved by Fleiner [5], see also [1, 7].

References

[1] A. Berg, B. Jackson, T. Jordán, Highly edge-connected detachments of graphs and digraphs, EGRES Report Series 2001-14, 2001, submitted. `http://www.cs.elte.hu/egres/`

[2] A. Frank, Augmenting graphs to meet edge-connectivity requirements, SIAM J. Disc. Math. 5, 25-53, 1992.

[3] A. Frank, Connectivity augmentation problems in network design, in: Mathematical Programming: State of the Art 1994, (Eds: J.R. Birge and K.G. Murty), The University of Michigan, Ann Arbor, MI, 34-63, 1994.

[4] A. Frank, T. Király, Z. Király, On the orientation of graphs and hypergraphs, EGRES Report Series 2001-06, 2001, submitted to Discrete Appl. Math. `http://www.cs.elte.hu/egres/`

[5] B. Fleiner, Detachments of vertices of graphs preserving edge-connectivity, 1997, submitted.

[6] S. Fujishige, Structures of polyhedra determined by submodular functions on crossing families, Math. Programming 29 (1984), no. 2, 125-141.

[7] T. Jordán, Z. Szigeti, Detachments preserving local edge-connectivity of graphs, BRICS Report Series 99-35, 1999, submitted. `http://www.brics.dk/`

[8] W. Mader, Konstruktion aller n-fach kantenzusammenhängenden Digraphen, European J. Combin., 3 (1982) pp 63-67.