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Detachment of vertices of graphs preserving edge-connectivity

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Abstract

The detachment of vertex is the inverse operation of merging vertices $s_1, \ldots, s_t$ into s. We speak about $\{d_1, \ldots, d_t\}$-detachment if for the detached graph G' the new degrees are specified as $d_{G'}(s_1) = d_1, \ldots, d_{G'}(s_t) = d_t$. We call a detachment k-feasible if $d_{G'}(X) \geq k$ whenever X separates two vertices of $V(G)-s$. In our main theorem, we give a necessary and sufficient condition for the existence of a k-feasible $\{d_1, \ldots, d_t\}$-detachment of vertex s. This theorem also holds for graphs containing 3-vertex hyperedges disjoint from s. From special cases of the theorem, we get a characterization of those graphs whose edge-connectivity can be augmented to k by adding γ edges and p 3-vertex hyperedges. We give a new proof for the theorem of Nash-Williams that characterizes the existence of a simultaneous detachment of the vertices of a given graph such that the resulted graph is k-edge-connected.

Keywords: detachment, graph, edge-connectivity

1 Introduction

Throughout this paper on a graph we mean an undirected, not necessarily simple graph (loops and multiple edges are allowed).

We denote the set of vertices and edges of hypergraph G by $V(G)$ and $E(G)$, respectively. The notation "$\subset$" means proper inclusion and instead of $X \cup \{s\}$ or $X \setminus \{s\}$ we simply write $X + s$ or $X - s$. $\Gamma(s)$ denotes the set of neighbours of s where each neighbour v is represented with multiplicity $d(s, v)$. Thus when we speak about some neighbours of s we do not necessarily mean that they are different. For $X \subset V(G)$ let $G[X]$ be the graph induced by X and define $G - s := G[V(G) - s]$ ($s \in V$). $X \subseteq V(G)$ *covers* vertex v if $v \in X$ and *separates* vertices a and b if it covers

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exactly one of them. A *cut* of G a proper subset X of V, and $d : 2^{V(G)} \to \mathbb{N}$ is the degree function, where $d(X)$ is the number of edges whose endvertices are separated by X.

$$\lambda(a,b) := \min\{d(X) \ : \ X \text{ separates } a \text{ and } b\}$$
$$\lambda(G) := \min\{d(X) \ : \ \emptyset \neq X \subset V(G)\}$$

A cut X with $d(X) = \lambda(G)$ is a *mincut* of G.

Two cuts X and Y on the ground set V are *intersecting* if none of the sets $X - Y$, $Y - X$ and $X \cap Y$ are empty. If, in addition $V - (X \cup Y) \neq \emptyset$ then they are called *crossing*.

Definition 1.1. Suppose that $d_1, \dots, d_t$ are positive integers, there is no loop incident to s and $d(s) = \sum_{i=1}^{t} d_i$. A $\{d_1, \dots, d_t\}$-detachment of $s \in V(G)$ is the following operation:
we delete vertex s and the edges incident to s, and connect new vertices $s_1, \dots, s_t$ to the neighbours of s by new edges such that for the resulted graph G' $d_{G'}(s_i) = d_i$ and $d_{G'}(z) = d_G(z)$ if $z \in \Gamma(s)$. In other words, a detachment is the inverse operation of merging vertices s_i with degree d_i into vertex s.

Obviously, there is a natural one to one correspondence between the edges of G and of G'. We call $\{d_1, \dots, d_t\}$ the *degree specification* of the detachment. If there is a $k \in \mathbb{N}$ such that $\lambda_{G'}(x, y) \geq k$ for $x, y \in V(G) - s$ then the detachment is *k-feasible*. A graph H is the detachment of G if H can be obtained by detaching certain vertices of G, one after the other. If $g : V(G) \to \mathbb{N}$ is a function and we detach each vertex v into $g(v)$ vertices then we obtain a *g-detachment* of G.

Lovász' edge-splitting theorem [3] asserts that a k-feasible $\{2, d(s) - 2\}$ detachment exists whenever $d(s)$ is even, $k \geq 2$ and g is k-edge-connected in $V - s$. Using this result, Frank [1] gave a short proof for the theorem of Watanabe and Nakamura that characterizes those graphs that may become k-edge-connected after adding a certain number of new edges.

If an arbitrary degree specification is imposed on s then making a k-feasible detachment at s is not always possible. In Section 3, our main theorem characterizes the existence of a k-feasible detachment for a given degree specification, generalizing the above mentioned theorem of Lovász. It turns out that the general case can be reduced to the case where $\{d_1, \dots, d_{t-1}, d_t\} = \{3, \dots, 3, d(s) - 3(t - 1)\}$.

In the proof of our main theorem, we have to introduce hyperedges. That is why our theorem is about *2-3-graphs*, i.e. hypergraphs with edges of size two or three. It is straightforward to generalize the above graph theoretic definitions to 2-3-graphs. By definition, a 3-edge contributes to the degree of a cut X by 1 if two of its vertices are separated by X, otherwise by 0. In the definition of the detachment we demand that no 3-edge is incident to the vertex s.

In Section 4, we apply our main theorem to generalize the theorem of Watanabe and Nakamura. We also give a new proof for the theorem of Nash-Williams [5] that characterizes those graphs for which the k-edge-connectivity property can be preserved by a g-detachment for a given g.

Before stating our main theorem, we summarize some properties of the degree function of 2-3-graphs.

2 The degree function

The degree function of a 2-3-graph is symmetric, and submodular, i.e.:

$$d(X) = d(V(G) - X) \text{ and}$$
$$d(X) + d(Y) \geq d(X \cap Y) + d(X \cup Y) \text{ (for all } X, Y \subset V(G)).$$

There is also a useful inequality for three sets for the degree function of a 2-3-graph:

$$\begin{aligned} d(X) + d(Y) + d(Z) \geq d(X \cap Y \cap Z) \quad + \quad & d(X - (Y \cup Z)) + \\ & d(Y - (Z \cup X)) + \\ & d(Z - (X \cup Y)) + \\ & 2\overline{d}(X, Y, Z), \end{aligned} \tag{1}$$

where $\overline{d}(X, Y, Z)$ denotes the number of edges (of size two) between $X \cap Y \cap Z$ and $V - (X \cup Y \cup Z)$. The following inequality is for two sets:

$$d(X) + d(Y) \geq d(X - Y) + d(Y - X) + 2\overline{d}(X, Y) \quad (X, Y \subseteq V(G)). \tag{2}$$

Here, $\overline{d}(X, Y)$ stands for the number of 2-edges connecting $X \cap Y$ to $V - (X \cup Y)$. These well known inequalities can be checked by enumerating the contribution of the different type of edges and 3-edges to the left and right sides of the inequality. The following lemma shows that vertices of degree 3 and 3-edges are interchangeable.

Lemma 2.1. *Let s be a vertex of degree 3 of a 2-3-graph G and a, b and c are the three neighbours of s. Let G' be the 2-3-graph obtained from G by deleting vertex s together with edges as, bs, cs and by adding the 3-edge abc. Then*

$$\lambda_G(x, y) = \lambda_{G'}(x, y)$$

for every $x, y \in G - s$.

Proof. It is enough to prove that for any $x - y$-cut C in G or in G' then there is an $x - y$-cut C' so that the degree of C' is not more than the degree of C in the other 2-3-graph. By symmetry, we may assume that $|C \cap \{a, b, c\}| \leq 1$. It is easy to check that $C' := C - s$ suffices. □

3 The main theorem

The main result of this paper is the following.

Theorem 3.1. *Given a 2-3-graph $G = (V, E)$ with a specified vertex $s \in V$ and a degree specification $\{d_1, \ldots, d_t\}$ ($d_i \geq 2$, $\sum_{i=1}^t d_i = d(s)$) for s. Assume $\lambda_G(x, y) \geq k \geq 2$ for every pair of vertices $x, y \in V - s$ and there is no loop or 3-edge incident to s. Then there exists a k-feasible $\{d_1, \ldots, d_t\}$-detachment of s if and only if*

$$\lambda(G - s) \geq k - \sum_{i=1}^{t} \left\lfloor \frac{d_i}{2} \right\rfloor. \tag{3}$$

Proof. We prove the necessity first. Assume that G' is obtained by a k-feasible detachment from G and that the vertex s is split into vertices s_i $(1 \leq i \leq t)$ with $d(s_i) = d_i$. Take an arbitrary cut X of $G-s$ and define the set S by $S := \{s_i \ : \ d_{G'}(X, s_i) > \frac{d_i}{2}\}$. Now

$$k \leq d_{G'}(X \cup S) \leq d_{G-s}(X) + \sum_{i=1}^{t} \left\lfloor \frac{d_i}{2} \right\rfloor .$$

Hence $\lambda(G-s) \geq k - \sum_{i=1}^{t} \left\lfloor \frac{d_i}{2} \right\rfloor$.

We prove the sufficiency in case of certain degree specifications. Then, we deduce the general theorem.

Case 1: $d(s) \geq 4$ and the degree specification is $\{2, d(s)-2\}$.

In this case, condition (3) is the consequence of the k-edge-connectivity assumption, hence it holds automatically. We apply the following theorem of Mader from [4].

Theorem 3.2. *Given a graph $G = (V, E)$ and a specified vertex $s \in V$ with $d(s) \neq 3$ and no cutting edge is incident to s. Then there exist a $\{2, d(s)-2\}$-detachment at s such that for the resulted graph G'*

$$\lambda_G(x, y) = \lambda_{G'}(x, y)$$

for every $x, y \in V - s$.

By Lemma 2.1, we can exchange each 3-edge $e = \alpha\beta\gamma$ into new edges $s_e\alpha$, $s_e\beta$ and $s_e\gamma$ where s_e is a new vertex. By Lemma 2.1, this does not change $\lambda(a, b)$ $(a, b \in V(G))$. Now take the $\{2, d(s)-2\}$-detachment provided by Theorem 3.2. Change back the 3-stars into 3-edges and we are done.

Let us point out that this Case 1. contains Lovász' edge-splitting theorem in [3]. In the Appendix, there is a self-contained proof for Case 1.

It is easy to check that we do not change the connectivity between vertices of $V-s$ if for vertex s_1 of degree two we replace edges s_1u and s_1v by edge uv. Then we can apply Case 1 again on s_2. Iterating this justifies

Case 2: $d(s) \geq 2q+2$ and the degree specification is $\{2, \ldots, 2, d(s)-2q\}$ where s is split into $q+1$ new vertices.

Case 3: $d(s) \geq 3p+2$ and the degree specification is $\{3, \ldots, 3, d(s)-3p\}$ where $p+1$ is the number of the new vertices.

We may assume that $d(s) \geq 6$ since $d(s) = 5$ is covered by to Case 1. We use induction on p. If $p = 0$ then there is nothing to prove. Assume that for $(p_0 - 1)$ the theorem is proved. We verify it for p_0. For brevity let $\Delta := \left\lfloor \frac{d(s)-p_0}{2} \right\rfloor$.

By the induction hypothesis, $\lambda(G-s) \geq k - \Delta$ holds. If there is a k-feasible $\{3, \ldots, 3, d(s)-3p_0\}$-detachment then we can carry out the detachment by making two successive detachments. The first one is a k-feasible $\{3, d(s)-3\}$-detachment of s. Next we apply the induction hypothesis for the resulted 2-3-graph and for the new vertex s_2 of degree $d(s)-3$.

Let us change the vertex s_1 and the three edges incident to it to the 3-edge defined by $\Gamma(s_1)$. By Lemma 2.1, the k-edge-connectivity between the vertices of $G-s$ is maintained after this operation.

Now we make a k-feasible $\{3,\ldots,3,d(s)-3p_0\}$-detachment of s_2 in the resulted 2-3-graph G' into p_0 vertices. Changing back the previously introduced 3-edge, the obtained detachment is k-feasible by Lemma 2.1.

By the induction hypothesis, the latter k-feasible detachment exists if and only if

$$\lambda(G'-s_2) \geq k - \left\lfloor \frac{(d(s)-3)-(p_0-1)}{2} \right\rfloor = k - \Delta + 1 \tag{4}$$

Let us suppose that $\lambda(G-s) = k-\Delta$. Then the first detachment must satisfies two conditions:

- it is k-feasible and
- the addition of the 3-edge induced by the three neighbours of s_1 increases the edge-connectivity of $G-s$.

From now on, any three neighbours of s will be referred to as a *triad*. The *vertices of the triad* are the corresponding neighbours of s.

Consider the following family of the inclusionwise minimal mincuts of $G-s$.

$$\mathcal{B} = \{B : B \subset V-s,\ d_{G-s}(B) = k-\Delta,\ \nexists A \subset B :\ d_{G-s}(A) = k-\Delta\}.$$

Proposition 3.3. *The elements of $\mathcal{B}$ are disjoint and $|\mathcal{B}|=2$ or $|\mathcal{B}|=3$.*

Proof. The disjointness of the elements of $\mathcal{B}$ and $|\mathcal{B}|\geq 2$ follows from submodularity. From the disjointness we get $d(s) \geq \sum_{B_i\in\mathcal{B}} d(B_i,s) \geq |\mathcal{B}|\Delta$ and by

$$4\Delta \geq 4\left(\frac{d(s)-p_0}{2}\right) - 2 = d(s) + (d(s)-2p_0-2) \geq d(s)+p_0 > d(s),$$

we conclude that $|\mathcal{B}|\leq 3$. □

Obviously, the $\{3,d(s)-3\}$-detachment satisfies (4) if and only if the 3-edge induced by the neighbours of the new vertex s_2 contributes to the degree of each mincut X of $G-s$. In other words, this means that both X and $\overline{X}$ contain some neighbour of s_2. Since both a mincut X and its complement contain an element of $\mathcal{B}$, the important triads are those that have a vertex in every element of $\mathcal{B}$. We say that such a triads are *transversal.*

If $\lambda(G-s) > k-\Delta$ then $|\mathcal{B}| = 0$, thus there exists no transversal triad. For the sake of the unified approach, we choose two arbitrary disjoint sets B_1 and B_2 such that $d(B_i,s)\geq 1$ $(i=1,2)$ and $d(B_1\cup B_2,s)\geq 2\Delta$ and define $\mathcal{B} := \{B_1,B_2\}$. This can always be done unless $|\Gamma(s)|=1$. But then $\lambda(G-s)\geq k$ holds trivially and any detachment with the given degree specification is k-feasible.

To finish the proof, we have to find a transversal triad that induces a k-feasible detachment. First, we study k-feasible $\{3,d(s)-3\}$-detachments.

Lemma 3.4. *Suppose that the $\{3,d(s)-3\}$-detachment induced by triad T is not feasible. Then either*

- *at least two vertices of T are covered by a set $Y \subset V - s$ with $d_G(Y) = k$ or*
- *all the three vertices of T are covered by a set $Y \subset V - s$ with $k+1 \le d_G(Y) \le k+2$.*

Proof. By infeasibility, there is a cut X of G' separating two vertices of $V - s$ with $d_{G'}(X) < k$. By taking the complement if necessary, we can assume that $s_1 \in X$. Then $s_2 \notin X$, otherwise $d_G(X - s_1 - s_2 + s) = d_{G'}(X) < k$ is a contradiction. If no vertex of T is in $X - s_1$ then it is easy to see that $d_{G'}(X) = d_{G'}(X - s_1) + 3$, so $k > d_{G'}(X) = d_{G'}(X - s_1) + 3 = d_G(X - s_1) + 3 \ge k + 3$, a contradiction. Similarly, if $|(X - s_1) \cap T| = 1$ then $k > d_{G'}(X) = d_{G'}(X - s_1) + 1 = d_G(X - s_1) + 1 \ge k + 1$.

If $X - s_1$ contains exactly two neighbours of s_1 in $X - s_1$ then from $k > d_{G'}(X) = d_{G'}(X - s_1) - 1 = d_G(X - s_1) - 1 \ge k - 1$ we get $k = d_G(X - s_1)$.

The remaining case is that all the 3 neighbours are in $X - s_1$. It means that $k > d_{G'}(X) = d_{G'}(X - s_1) - 3 = d_G(X - s_1) - 3 \ge k - 3$ thus $k + 2 \ge d_G(X - s_1) \ge k$. □

Define
$$\mathcal{K} = \{X : \ \ X \subset V - s, \ \ d_G(X) = k\}\,.$$
We call a triad *legal* if no pair of its vertices are covered by a member of $\mathcal{K}$. If no k-feasible detachment exists then from Lemma 3.4 it follows that each transversal legal triad is covered by a set L with $k + 2 \ge d_G(L) \ge k + 1$.

In what follows, We focus on transversal legal triads. Let $\mathcal{L}$ be a family of different sets on the ground-set $V - s$ such that for every transversal legal triad T there is a set X of $\mathcal{L}$ with $T \subseteq X$ and $k+1 \le d_G(X) \le k+2$. Choose $\mathcal{L}$ so that $|\mathcal{L}|$ is minimal.

Remark. At this point it is not obvious that a transversal legal triad exists but the existence will follow from the proof.

Lemma 3.5. *If $L \in \mathcal{L}$ and $K \in \mathcal{K}$ and $d(L \cap K, s) \ge 1$ then $K \subset L$.*

Proof. Obviously, $L \not\subseteq K$ from the definition of $\mathcal{L}$. If $K \not\subset L$ then from inequality 2 we get:

$$(k+2) + k \ge d_G(L) + d_G(K) \ge d_G(L - K) + d_G(K - L) + 2d(K \cap L, s) \ge k + k + 2.$$

Thus $d_G(K) = d_G(L - K) = k$. This contradicts the legality of the transversal triad inside L because two of the vertices of this triad must be covered by K or by $L - K$. □

If $|\mathcal{B}| = 3$ then

$$d(s) \ge \sum_{B_i \in \mathcal{B}} d(B_i, s) \ge 3\left\lfloor \frac{d(s) - p_0}{2} \right\rfloor \ge 3\left(\frac{d(s) - p_0}{2}\right) - \frac{3}{2} \ge d(s) - \frac{1}{2}$$

and since $\left\lfloor \frac{d(s)-p_0}{2} \right\rfloor$ is integer, we get $d(B_i, s) = \Delta$. Therefore $d_G(B_i) = d_{G-s}(B_i) + d(B_i, s) = (k - \Delta) + \Delta = k$ thus $B_i \in \mathcal{K}$ ($\forall B_i \in \mathcal{B}$). It follows from submodularity that the maximal elements of $\mathcal{K}$ are disjoint. Each of them contains at most Δ neighbours of s therefore the elements of $\mathcal{B}$ are contained in different maximal elements of $\mathcal{K}$. Thus every transversal triad is legal. By Lemma 3.5, there is a set $X \in \mathcal{L}$ that contains all the elements of $\mathcal{B}$. I.e. $d(X,s) \geq \sum_{i=1}^{3} d(B_i,s) = 3\Delta > \Delta + 2$, a is a contradiction.

The remaining case is $\mathcal{B} = \{B_1, B_2\}$.

Lemma 3.6. *If $X, Y \in \mathcal{L}$ then $d(X \cap Y, s) \leq 2$.*

Proof. From $d_G(X) \leq k{+}2$ and $d_G(Y) \leq k{+}2$, inequality (2) gives that $2 \geq \overline{d}(X,Y) \geq d(X \cap Y, s)$. (Here we used that $X \not\subset Y$ since $|\mathcal{L}|$ is minimal.) □

We shall construct legal transversal triads by choosing the corresponding vertices one by one. Let $a \in B_1$ be a neighbour of s. Since the maximal elements of $\mathcal{K}$ are disjoint and each of them contains at most Δ neighbours of s, there must exist a neighbour $b \in B_2$ of s outside the element of $\mathcal{K}$ that might cover vertex a. Two maximal elements of $\mathcal{K}$ can contain at most 2Δ neighbours of s. Hence there is a vertex c such that $\{a,b,c\}$ is a legal triad and it is also transversal due to a and b.

If we cannot choose c from $B_1 \cup B_2$ then there exist two sets $a \in K_1 \in \mathcal{K}$ and $b \in K_2 \in \mathcal{K}$ such that $(B_1 \cup B_2) \cap \Gamma(s) \subseteq K_1 \cup K_2$. But then, set X of $\mathcal{L}$ covering legal transversal triad $\{a,b,c\}$ contains the sets K_1 and K_2 by Lemma 3.5. Thus there are too many edges from s to X because $d(X,s) \geq d(K_1 \cup K_2 \cup \{c\}, s) \geq d(B_1 \cup B_2 \cup \{c\}, s) \geq 2\Delta + 1 > \Delta + 2$, a contradiction.

By interchanging the notation if necessary, we may assume that $a \in B_1$ and $b, c \in B_2$. Since $d(X,s) \leq \Delta + 2$ and $d(s) - (\Delta + 2) \geq 2$ there exist $d, e \in \Gamma(s)$ such that $d, e \notin X$.

We claim that $d \neq e$. Otherwise $d(d,s) \geq 2$. Moreover, $\{a,b,d\}$ and $\{a,c,d\}$ are legal transversal triads by the usual argument. Consider the sets of $\mathcal{L}$ that cover them and the set X. By Lemma 3.6, $\{a,b,d\}$ and $\{a,c,d\}$ have no common covering set. So we have two sets Y_1 and Y_2 with $d(Y_1 \cap Y_2, s) \geq d(a,s) + d(d,s) \geq 1 + 2 = 3$. This contradicts Lemma 3.6.

Let us choose the members X, Y and Z of $\mathcal{L}$ that correspond to legal transversal triads $\{a,b,c\}$, $\{a,b,d\}$ and $\{a,b,e\}$, respectively.

(If these three sets are not different, say $Y = Z$, then instead of X, Y and Z we choose members X, U, W of $\mathcal{L}$ that correspond to legal transversal triads $\{a,b,c\}$, $\{a,c,d\}$ and $\{a,c,e\}$, respectively. Obviously, $X \neq U$ and $X \neq W$. Moreover, $U \neq W$, since $U = W$ would contradict Lemma 3.6 by $d(U \cap Y, s) \geq 3$ since $\{a,d,e\} \subseteq U \cap Y$.)

It is clear from Lemma 3.6 that $c \in X - (Y \cup Z)$, $d \in Y - (Z \cup X)$ and $e \in Z - (X \cup Y)$. $d_G(X \cap Y \cap Z) \neq k$ since $\{a,b,c\}$ is legal.

From 1 we get

$$(k+2) + (k+2) + (k+2) \geq d_G(X) + d_G(Y) + d_G(Z) \geq$$

$$d_G(\overbrace{X \cap Y \cap Z}^{a,b\in}) + d_G(\overbrace{X - (Y \cup Z)}^{c\in}) + d_G(\overbrace{Y - (Z \cup X)}^{d\in}) + d_G(\overbrace{Z - (X \cup Y)}^{e\in}) +$$

$$2\overline{d}(X,Y,Z) \geq (k+1) + k + k + k + 4,$$

since $a, b \in X \cap Y \cap Z$ and $s \in V - (X \cup Y \cup Z)$. From this follows that $k \leq 1$.

The general case: The degree specification is $\{d_1, \ldots, d_t\}$ and d_i is odd for $1 \leq i \leq p$ and d_i is even for $p + 1 \leq i \leq t$.

The condition is $\lambda(G-s) \geq k - \left\lfloor \frac{d(s)-p}{2} \right\rfloor$. By Case 3., there exists a k-feasible $\{3,\ldots,3,d(s)-3p\}$-detachment of s into $p+1$ vertices. Change the new vertices of degree 3 into 3-edges. Perform a $\{2,\ldots,2\}$-detachment of the new vertex of degree $d(s)-3p$ and change back the 3-edges to 3-stars. By this, we get a k-feasible $\{2,\ldots,2,3,\ldots,3\}$-detachment of the original 2-3-graph where the number of new vertices of degree 3 is p. Merge at most one vertex of degree 3 with some others of degree 2 to get a vertex of degree d_1. By repeating this operation, we construct a k-feasible $\{d_1,\ldots,d_t\}$-detachment of the original 2-3-graph. □

One may ask for a necessary and sufficient condition for the existence of a detachment which preserves also the local edge-connectivities.

Conjecture 3.7. *Given a graph $G=(V,E)$ with a specified vertex $s \in V$ and a degree specification $\{d_1,\ldots,d_t\}$ ($d_i \geq 2$, $\sum_{i=1}^t d_i = d(s)$) for s. Assume there is no loop or cut-edge incident to s. Then there exists a $\{d_1,\ldots,d_t\}$-detachment of s such that $\lambda_G(x,y)=\lambda_{G'}(x,y)$ ($x,y \in V-s$) if and only if*

$$\lambda_{G-s}(x,y) \geq \lambda_G(x,y) - \sum_i \left\lfloor \frac{d_i}{2} \right\rfloor \quad (x,y \in V-s).$$

Remark. Conjecture 3.7 is true. Motivated by this paper, a generalized form of this conjecture was proved by Jordán and Szigeti in [2].

4 Applications

In this section, we apply Theorem 3.1 to deduce some well known theorems whose standard proofs are based on Lovász' edge splitting theorem [3]. The equivalent form of Lovász' theorem is the following :

Theorem 4.1. *$G=(V,E)$ is a given multigraph, $k \geq 2$ and $m: V \to \mathbb{N}$ is a function. There is a graph $H=(V,F)$ such that $d_H(v)=m(v)$ ($\forall v \in V$) and $G+H=(V,E\cup F)$ is k-edge-connected if and only if*
(i) $m(V)$ is even
(ii) $m(X) \geq k - d_G(X)$ for any proper subset X of V, where $m(X) := \sum_{x\in X} m(x)$

A generalization of Theorem 4.1 is the following.

Theorem 4.2. *$G=(V,E)$ is 2-3-graph, $k \geq 2$ and $m: V \to \mathbb{N}$ is a function. There is a 2-3-graph $H=(V,F)$ such that $d_H(v)=m(v)$ ($\forall v \in V$), F contains exactly p 3-edges and $G+H=(V,E\cup F)$ is k-edge-connected if and only if*
(i) $3p \leq m(V)$
(ii) $m(V)-3p$ is even
(iii) $m(X) \geq k - d_G(X)$ for any nonempty proper subset of V
(iv) $\lambda(G) \geq k - \left\lfloor \frac{m(V)-p}{2} \right\rfloor$

Proof. The conditions *(i)* and *(ii)* are necessary, because the total degree requirement of the 3-edges is not more than $m(V)$ and the requirement of the edges (of size two) is even. *(iii)* is also needed since the edges of H increase the degree of every set to k. Condition *(iv)* is equivalent to the inequality $|F| \geq k - \lambda(G)$.

To prove the sufficiency we add to the 2-3-graph an extra vertex s ($s \notin V$) and $m(v)$ new edges between s and v for every vertex $v \in V$. By *(iii)* and *(iv)*, Theorem 3.1 can be applied i.e. there is a k-feasible $\{2, \ldots, 2, 3, \ldots, 3\}$-detachment of s such that the number of new vertices of degree 3 is exactly p. Now the neighbours of each s_i are the endvertices of an edge of H. □

Watanabe and Nakamura [6] gave a characterization of the graphs that can be made k-edge-connected by adding γ edges. We prove the extension of this result along the lines of Frank's proof in [1].

A family of sets $\{X_1, \ldots, X_r\}$ is a *subpartition* of V if $\emptyset \neq X_i \subset V \ (1 \leq i \leq r)$ and $X_i \cap X_j = \emptyset \ (i \neq j)$.

Theorem 4.3. *The 2-3-graph $G = (V, E)$ can be made k-edge-connected by adding γ edges and p 3-edges if and only if*

(i) $\lambda(G) \geq k - (\gamma + p)$ and

(ii) $2\gamma + 3p \geq \sum_i (k - d(X_i))$ holds for every subpartition $\{X_1, \ldots, X_r\}$ of V.

Proof. If the required augmentation exists then condition *(i)* and *(ii)* follow from the facts that the addition of an edge can increase the edge-connectivity at most by one and a edge (of size two) or a 3-edge can contribute to the degree of at most 2 or 3 disjoint sets having degree less than k.

Let $m : V \to \mathbb{N}$ be a function such that $m(V)$ is minimal and $k - d(X) \leq m(X)$ for every set $\emptyset \neq X \subset V$.

Lemma 4.4. $m(V) \leq 2\gamma + 3p$

Proof. We call a set $\emptyset \neq X \subset V$ *critical* if $k - d(X) = m(X)$. If $m(v) > 0$ then v is in a critical set. Let Y_v by the minimal critical set containing v. We claim that the maximal elements of $\{Y_v \ : \ v \in V\}$ are disjoint. Indirectly, let Y_v and Y_u be two maximal sets intersecting each other.

$m(Y_u) + m(Y_v) = k - d(Y_u) + k - d(Y_v) \leq k - d(Y_u - Y_v) + k - d(Y_v - Y_u) \leq m(Y_u - Y_v) + m(Y_v - Y_u) = m(Y_u) + m(Y_v) - 2m(Y_u \cap Y_v) \leq m(Y_u) + m(Y_v)$

The above inequalities are equalities, hence $m(Y_u \cap Y_v) = 0$. Further, the sets $Y_u - Y_v$ and $Y_v - Y_u$ are critical and they cannot be minimal covering sets of vertex u and v, respectively.

Let $\{X_i\}$ be the subpartition of the maximal members of $\{Y_v \ : \ v \in V\}$. From condition *(ii)* it follows that

$$m(V) = \sum_i m(X_i) = \sum_i (k - d(X_i)) \leq 2\gamma + 3p.$$

□

If $m(V) < 2\gamma + 3p$ then increase $m(v)$ at some vertex v so that $m(V)$ equals to $2\gamma + 3p$. Now $\gamma + p = \left\lfloor \frac{m(V)-p}{2} \right\rfloor$ thus the conditions of Theorem 4.2 are satisfied. □

The next theorem is valid for graphs that may contain loops. If there are loops at vertex s then each loop increases the degree of s by two. We also demand that after the detachment at s any edge that come from a loop must connect two new vertices or must remain loop.

Theorem 4.5. *(Nash-Williams [5]) Let $G = (V, E)$ be a multigraph, $|V| \geq 2$, $k \geq 2$, $g : V \to \mathbb{N}$ is a function and $\zeta_v = \{\zeta_v^1, \ldots, \zeta_v^{g(v)}\}$ is a degree specification for each $v \in V$. Then there is a k-edge-connected g-detachment of G if and only if*
(i) G is k-edge-connected,
(ii) $d(v) \geq k \cdot g(v)$,
(iii) if k is odd then none of the following conditions are true:
(a) there is a cut-vertex s in G such that $d(s) = 2k$, $g(s) = 2$.
(b) $|V| = 2$, $d(v) = 2k$ and $g(v) = 2$ ($\forall v \in V$) and there is no loop in G.
Moreover, there is a k-edge-connected g-detachment of G with degree specification ζ if besides the previous ones, condition (ii)' is satisfied:
(ii)' $\zeta_v^i \geq k \quad (v \in V,\ 1 \leq i \leq g(v))$.

Proof. If there is such a detachment then *(i)* and *(ii)-(ii)'* must hold since a detachment does not increase the edge-connectivity and every vertex has degree at least k in a k-edge-connected graph. By Theorem 3.1, there is no k-edge-connected g-detachment if *(iii)a* is true. If *(iii)b* holds then by detaching only one vertex, *(iii)a* holds for the resulted graph thus the detachment cannot be completed.

In order to prove the sufficiency, we use induction on the number of vertices v for which $g(v) \geq 2$. We detach the vertices one by one. Our purpose is to detach only one vertex and maintaining conditions *(i)-(iii)*.

Case 1: k is odd and there is a vertex $s \in V$ such that there is no loop at s and $d(s) = 2k$, $g(s) = 2$.

We show a $\{k, k\}$-detachment of s into vertices s_1, s_2 such that the resulted graph satisfies conditions *(i)-(iii)*.

Perform a k-feasible $\{k, k\}$-detachment of s that exists by Theorem 3.1. This implies that any cut of the resulted graph G' has degree at least k if it separates two vertices of $V(G') - s_1 - s_2$. The other cuts or their complements are the subsets of $\{s_1, s_2\}$ and since $d(s_1) = d(s_2) = k$, they have degree at least k as well, thus condition *(i)* is satisfied.

Assume that the detachment of s creates a cut-vertex s^* for which $d(s^*) = 2k$ and $g(s^*) = 2$. Since G' is k-edge-connected, $G' - s^*$ has exactly two components induced by subsets A and B of V. By *(iii)*, s^* was not a cut-vertex in G therefore s_1 and s_2 are in different components. We may assume that $s_1 \in A$ and $s_2 \in B$. The subgraphs $G'[A + s^*]$ and $G'[B + s^*]$ are k-edge-connected because all of their cuts that do contain s^* have the same degree as in G'. By *(iii)b*, we may assume that $|A| \geq 2$ and s_1 has a neighbour $a \in A$. (If both A and B consist of only one vertex then by condition *(iii)* there must exist a loop for example at s_1, but this contradicts the condition of Case

1.) By k-edge-connectivity, there are k edge disjoint paths in $G'[A+s^*]$ connecting s_1 and a. Since a path contains 0 or 2 edges incident to s^* and $d_{G'[A+s^*]}(s^*) = k$ and k is odd, there are at least $k - \left\lfloor \frac{k}{2} \right\rfloor \geq 2$ paths that are disjoint from s^*. We modify the detachment. Delete edge $e = s_1a$ and another one $f = s_2b$ and add new edges $e^* = s_2a$ and $f^* = s_1b$. We claim that the constructed graph G^*

(a) satisfies the degree specification at s_1 and s_2,

(b) k-edge-connected,

(c) and does not contain a cut-vertex z such that $d(z) = 2k$ and $g(z) = 2$.

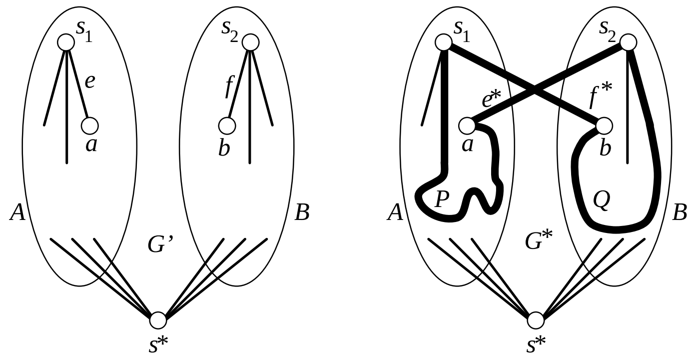

Figure 1: The graphs G' and G^*.

(a) follows directly from the construction.

We verify (b). By the above argument, there is a path P in $G^*[A]$ connecting s_1 and a. By the k-edge-connectivity of $G'[B+s^*]$, the subgraph $G'[B+s^*] - f = G^*[B+s^*]$ contains a path Q connecting s_2 and b that may contain vertex s^*.

Let X be an arbitrary cut of G^*. $A \cup B + s^* = V(G^*)$ thus X separates $A + s^*$ or $B + s^*$. Accordingly

$$d_{G^*}(X) \geq d_{G^*[A+s^*]\cup(e^*+Q+f^*)}(X \cap V(G^*[A+s^*] \cup (e^* + Q + f^*))) \geq$$

$$d_{G'[A+s^*]}(X \cap (A + s^*)) \geq k$$

or the analogous inequality with B and P instead of A and Q holds since $G'[A+s^*]$ and $G'[B+s^*]$ are k-edge-connected.

Suppose that there is a cut-vertex $z \in G^*$ such that $d_{G^*}(z) = 2k$ and $g(z) = 2$. The circle defined by the edges e^*, f^* and by the paths P and Q, shows that s_1 and s_2 are in the same component of $G^* - z$. (Here we used the fact that $s^* \notin P$.) By merging s_1 and s_2 into one vertex, it turns out that z is a cut-vertex of G and this is forbidden by *(iii)a*.

Case 2: k is even or no loopless vertex s exists with $d(s) = 2k$ and $g(s) = 2$.

Let z be a vertex with $g(z) \geq 2$. If no degree specification is imposed on z then choose one that satisfies *(ii)'*.

Suppose that z has no loops. Since we are not in Case 1, $k - \sum_{i=1}^{g(z)} \left\lfloor \frac{\zeta_z^i}{2} \right\rfloor \leq 0$ therefore there is a k-feasible $\{\zeta_z^1, \ldots, \zeta_z^{g(z)}\}$-detachment of z resulting the graph G' by Theorem 3.1. Then for every cut X separating two vertices of $V(G') - z_1 - \ldots - z_{g(z)}$, inequality $d_{G'}(X) \geq k$ holds. The other cuts or their complements are the subsets of $\{z_1, \ldots, z_{g(z)}\}$, thus, by *(ii)'* and by the absence of loops, these cuts have degree at least k as well. Consequently G' satisfies condition *(i)*. Conditions *(ii)* and *(ii)'* hold trivially for the induced degree specification. Property *(iii)* is also valid since there is no vertex $s \in V(G)$ with $d(s) = 2k$ and $g(s) = 2$.

In the remaining cases we suppose that z has h loops.

If $2h \geq k$ then we subdivide each loop with a vertex and merge these vertices into new vertex q. The resulted graph G^q is k-edge-connected. *(i)* implies that $d(V - z, z) \geq k$. From this and from *(ii)-(ii)'* and by a parity argument we get that $k - \sum_{i=1}^{g(z)} \left\lfloor \frac{\zeta_z^i}{2} \right\rfloor \leq 0$. We make a k-feasible $\{\zeta_z^1, \ldots, \zeta_z^{g(z)}\}$-detachment of G^q at z and after this we perform a $\{2, \ldots, 2\}$-detachment at vertex q. This can be done because the original and the resulted graphs are clearly k-edge-connected and we can apply Theorem 3.1 for them. Obviously, the final graph satisfies conditions *(i)–(iii)*.

If $2h < k$ then delete the loops incident to z and define new degree specification $\{\zeta_z^{*1}, \ldots, \zeta_z^{*g(z)}\}$ for the resulted graph G^* with $\zeta_z^{*i} := \zeta_z^i - h$ for $i = 1, 2$ and $\zeta_z^{*i} := \zeta_z^i$ for $i > 2$. It is enough to construct a k-feasible detachment with respect to the new specification because connecting the vertices z_1 and z_2 with h edges we get a k-feasible $\{\zeta_z^1, \ldots, \zeta_z^{g(z)}\}$-detachment at z.

If $g(z) \geq 4$ then $k - \sum_{i=1}^{g(z)} \left\lfloor \frac{\zeta_z^{*i}}{2} \right\rfloor \leq 0$ holds and there is a k-feasible $\{\zeta_z^{*1}, \ldots, \zeta_z^{*g(z)}\}$-detachment of z in G^* by Theorem 3.1.

If $g(z) = 3$ and $k - \sum_{i=1}^{3} \left\lfloor \frac{\zeta_z^{*i}}{2} \right\rfloor \leq 0$ then we have a k-feasible $\{\zeta_z^{*1}, \zeta_z^{*2}, \zeta_z^{*3}\}$-detachment of G^* by Theorem 3.1. If $k - \sum_{i=1}^{3} \left\lfloor \frac{\zeta_z^{*i}}{2} \right\rfloor > 0$ then from $d_{G^*}(V - z, z) \geq k$ and from *(ii)-(ii)'* k is odd, $\zeta_z^{*1} + \zeta_z^{*2} = k + 1$, $\zeta_z^{*3} = k$ and $2h = k - 1$. Since ζ_z^{*3} is even, we can make a k-feasible $\{\zeta_z^{*1} + \zeta_z^{*2}, \zeta_z^{*3}\}$-detachment of G^* that reduces this case to following:

$g(z) = 2$. If ζ_z^{*1} or ζ_z^{*2} is even then Theorem 3.1 gives us the corresponding detachment. If both ζ_z^{*1} and ζ_z^{*2} are odd then we add a loop to vertex z and divide the loop with vertex q. In the resulted graph there is a k-feasible $\{2, \ldots, 2\}$-detachment of z by the successive use of Theorem 3.2. Merging some new vertices we get a k-feasible $\{\zeta_z^{*1} + 1, \zeta_z^{*2} + 1\}$-detachment of z. Connecting z_1 and z_2 with the missing $h - 1$ edges, we obtain a k-feasible $\{\zeta_z^1, \zeta_z^2\}$-detachment of z. It is follows from the constructions that the above obtained graphs satisfies conditions *(i)–(iii)*. □

5 Appendix

We present a self contained proof for Case 1 of Theorem 3.1.

Assume that every $\{2, d(s) - 2\}$-detachment is infeasible. Hence for every $x, y \in \Gamma(s)$ there is a set $x, y \in U \subset V - s$ with $d_G(U) \leq k+1$. For $x \in \Gamma(s)$ let $\mathcal{H}^x = \{U_1^x, \ldots, U_{l_x}^x\}$

be a family of sets with minimum number of elements that $\Gamma(s) \subseteq U_1^x \cup \ldots \cup U_{l_x}^x$, $x \in U_i^x$ and $d_G(U_i^x) \leq k+1$ $(1 \leq i \leq l_x)$.

Suppose that a is a neighbour of s with $l_a \geq 3$. Fix different sets X, Y, Z of $\mathcal{H}^a$. X can not be omitted from $\mathcal{H}^a$ so there is a neighbour of s that is covered by X but not by Y or Z. Similarly we find that $Y - (Z \cup X) \neq \emptyset$ and $Z - (X \cup Y) \neq \emptyset$. We apply inequality (1) for 2-3-graph G:

$$\begin{aligned} d_G(X) + d_G(Y) + d_G(Z) \geq \; & d_G(X \cap Y \cap Z) + d_G(X - (Y \cup Z)) + \\ & d_G(Y - (Z \cup X)) + d_G(Z - (X \cup Y)) + \\ & 2\overline{d}(X, Y, Z). \end{aligned}$$

Each term on the left is at most $k+1$ and the first four terms on the right are at least k and the last one is at least 2 due to the edge sa. Hence $k \leq 1$, a contradiction.

Hence $l_x \leq 2$ for every $x \in \Gamma(s)$. Inequality

$$d_G(U_i^x) - 1 \leq k \leq d_G(U_i^x + s) = d_G(U_i^x) - 2d(U_i^x, s) + d(s)$$

implies that $d(U_i^x, s) \leq \left\lceil \frac{d(s)}{2} \right\rceil$. Thus one U_i^x cannot contain all neighbours of s, that is $l_x = 2$ for every $x \in \Gamma(s)$. Since $x \in U_1^x \cap U_2^x$, it follows that $d(s)$ is odd, $d(U_i^x, s) = \left\lceil \frac{d(s)}{2} \right\rceil$, $d_{G-s}(U_i^x) = k - \left\lfloor \frac{d(s)}{2} \right\rfloor$, $d_G(U_i^x) = k+1$ and $d(U_1^x \cap U_2^x, s) = 1$ $(\forall x \in \Gamma(s))$.

From these equalities and from inequality 2, we get that $d_G(U_1^x - U_2^x) = d_G(U_2^x - U_1^x) = k$. Consider an element X of $\mathcal{H}$ that covers a vertex $y \in \Gamma(s) \cup (U_1^x - U_2^x)$ and a vertex $z \in \Gamma(s) \cup (U_2^x - U_1^x)$. Since $d(X, s) = \left\lceil \frac{d(s)}{2} \right\rceil$, there exists a vertex $w \in \Gamma(s) \cap ((U_1^x - U_2^x) \cup (U_2^x - U_1^x) - X)$. We may assume that $w \in U_1^x - U_2^x$.

Apply inequality (2) to X (that has degree $k+1$) and to $U_1^x - U_2^x$ (that has degree k). Since edge sy connects $V - (X \cup (U_1^x - U_2^x))$ and $X \cap (U_1^x - U_2^x)$ we gain the contradiction $(k+1) + k \geq k + k + 2$.

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References

[1] A. Frank: *Augmenting graphs to meet edge-connectivity requirements* SIAM J. Discrete Math. 5, No. 1 (Feb. 1992), pp. 22-53.

[2] T. Jordán, Z. Szigeti: *Detachments Preserving Local Edge-Connectivity of Graphs* BRICS Report Series RS-99-35, submitted to SIAM Journal on Discrete Mathematics.

[3] L. Lovász: Combinatorial Problems and Exercises, North-Holland 1979, pp. 52.

[4] W. Mader: *A reduction method for edge-connectivity in graphs*, Ann. Discrete Math., Vol.3, 1978, pp. 145-164. Vol.12, 1960, pp. 555-567.

[5] C. St. J. A. Nash-Williams: *Connected detachments of graphs and generalized Euler trails*, J. London Math. Soc., Vol.31, 1985, pp. 17-29.

[6] T. Watanabe and A. Nakamura: *Edge-connectivity augmentation problems*, J. Comp. System Sci., Vol.35, 1987, pp. 96-144.